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ON NEW GENERALIZATIONS OF HILBERT TYPE INTEGRAL INEQUALITIES

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Abstract: This paper deals with new generalizations of Hubert type integral inequalities, which were recently proved by Pachpatte [1]. Two new inverse inequalities are also proved.

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1. Introduction

Several authors including Pachpatte [1], Yang [2], [3], Zhao [4], [5] have given considerable attention to Hibert integral inequalities and Hibert type integral inequalities and their applications. Very recently, Pachpatte [6] proved two new integral inequalities similar to some extensions of Hubert inequality (see Hardy et al [7], p. 253), and these two inequalities can be stated in Theorem 1.1 and

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Theorem 1.2.

Theorem 1.1. *Let $p > 1$ and $q > 1$ be constants such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f(s)$ and $g(t)$ be real-valued continuous functions defined on $I_x = [0, x)$ and $I_y = [0, y)$, respectively, and $f(0) = g(0) = 0$. Then*

$$\int_0^x \int_0^y \frac{|f(s)||g(t)|}{qs^{p-1} + pt^{q-1}}\, dsdt \leq K(p,q,x,y) \left(\int_0^x (x-s)|f'(s)|^p\, ds \right)^{\frac{1}{p}} \left(\int_0^y (y-t)|g'(t)|^q \right)^{\frac{1}{p}}, \quad (1.1)$$

for $x, y \in I_0 = (0, \infty)$, where $K(p,q,x,y) = (pq)^{-1} x^{\frac{p-1}{p}} y^{\frac{q-1}{q}}$.

Theorem 1.2. *Let $p > 1$ and $q > 1$ be constants such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f(s,t)$ and $g(k,r)$ be real-valued continuous functions defined on $I_x \times I_y$ and $I_z \times I_w$, respectively, and let $f(0,t) = g(0,t) = 0$, $f(s,0) = g(s,0) = 0$. We denote the partial derivatives $\frac{\partial}{\partial s} u(s,t)$, $\frac{\partial}{\partial t} u(s,t)$ and $\frac{\partial^2}{\partial s \partial t} u(s,t)$ by $D_1 u(s,t)$, $D_2 u(s,t)$ and $D_2 D_1 u(s,t) = D_1 D_2 u(s,t)$, respectively. Then*

$$\int_0^x \int_0^y \left(\int_0^z \int_0^w \frac{|f(s,t)||g(k,r)|}{q(st)^{p-1} + p(kr)^{q-1}}\, dkdr \right) dsdt$$
$$\leq C(p,q,x,y,z,w) \left(\int_0^x \int_0^y (x-s)(y-t)|D_2 D_1 f(s,t)|^p\, dsdt \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^z \int_0^w (z-k)(w-r)|D_2 D_1 g(k,r)|^p\, dkdr \right)^{\frac{1}{p}}, \quad (1.2)$$

for $x, y, z, w \in I_0$, where I_0, I_x, I_y, I_z, I_w are as in Theorem 1.1, and

$$C(p,q,x,y,z,w) = \frac{1}{pq} (xy)^{\frac{p-1}{p}} (zw)^{\frac{q-1}{q}}, \quad (1.3)$$

for $x, y, z, w \in I_0$.

Clearly, Theorem 1.1 and Theorem 1.2 can be changed to the following theorems.

Theorem 1.3. *Under the hypotheses of Theorem 1.2, let $f(s)$ and $g(t)$ be two real-valued nonnegative increasing functions, then*

$$\int_0^x\int_0^y \frac{f(s)g(t)}{qs^{p-1}+pt^{q-1}}\,dsdt \le K(p,q,x,y)\left(\int_0^x (x-s)(f'(s))^p\,ds\right)^{\frac{1}{p}}\left(\int_0^y (y-t)(g'(t))^q\,dt\right)^{\frac{1}{p}}, \quad (1.4)$$

for $x,y\in I_0$, where $K(p,q,x,y)$ is given by (1.2).

Theorem 1.4. *Under the hypotheses of Theorem 1.2, let $f(s,t)$ and $g(k,r)$ be two real-valued nonnegative continuous functions. Let $D_iD_ju(s,t)\ge 0$, $i,j=1,2$ $(i\neq j)$. Then*

$$\int_0^x\int_0^y \left(\int_0^z\int_0^w \frac{f(s,t)g(k,r)}{q(st)^{p-1}+p(kr)^{q-1}}\,dkdr\right)dsdt \le C(p,q,x,y,z,w)\left(\int_0^x\int_0^y (x-s)(y-t)(D_2D_1f(s,t))^p\,dsdt\right)^{\frac{1}{p}} \times\left(\int_0^z\int_0^w (z-k)(w-r)(D_2D_1g(k,r))^p\,dkdr\right)^{\frac{1}{p}}, \quad (1.5)$$

for $x,y,z,w\in I_0$, where $C(p,q,x,y,z,w)$ is given by (1.4).

The main purpose of this paper is to generalize inequalities (1.1), (1.2), (1.4), (1.5) in various ways. The inverse inequalities of (1.4) and (1.5) are also given.

2. Main Results

Our main results are given in the following theorems.

Theorem 2.1. *Let $h \geq 1$, $I \geq 1$ and $p > 1$ be constants and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f(s)$ and $g(t)$ be as in Theorem 1.1. Then*

$$\int_0^x \int_0^y \frac{|F(s,h)||G(t,l)|}{hl(qs^{p-1} + pt^{q-1})} \, dsdt \leq K(p,q,x,y) \times \left(\int_0^x (x-s)|f^{h-1}(s)f'(s)|^p \, ds \right)^{\frac{1}{p}} \left(\int_0^y (y-t)|g^{l-1}(t)g'(t)|^q \, dt \right)^{\frac{1}{q}}, \quad (2.1)$$

where

$$F(s,h) = f^h(s) - f^h(0), \quad G(t,l) = g^l(t) - g^l(0),$$

and

$$K(p,q,x,y) = \frac{1}{pq} x^{\frac{p-1}{p}} y^{\frac{q-1}{q}}. \quad (2.2)$$

Remark 1. Taking $h = l = 1$ and $f(0) = g(0) = 0$ in (2.1), then inequality (2.1) reduces to inequality (1.1).

Inequality (2.1) can also be changed to the following inequality

$$\int_0^x \int_0^y \frac{|F(s,h)||G(t,l)|}{hl(qs^{p-1} + pt^{q-1})} \, dsdt \leq K(p,q,x,y) \left(\frac{1}{p} \int_0^x (x-s)|f^{h-1}(s)f'(s)|^p \, ds + \frac{1}{q} \int_0^y (y-t)|g^{l-1}(t)g'(t)|^q \, dt \right). \quad (2.3)$$

This is just a generalization of the inequality (7) in Pachpatte [6]. We can give a well strengthened version of (2.1) in the form

$$\int_0^x \int_0^y \frac{|F(s,h)||G(t,l)|}{hl \cdot K'(s,t,p,q)} \, dsdt \leq K(x,y,p,q) \times \left(\int_0^x (x-s)|f^{h-1}(s)f'(s)|^p \, ds \right)^{\frac{1}{p}} \left(\int_0^y (y-t)|g^{l-1}(t)g'(t)|^q \, dt \right)^{\frac{1}{q}}, \quad (2.4)$$

where $K'(u,v,p,q) = u^{\frac{p-1}{p}} v^{\frac{q-1}{q}}$.

Theorem 2.2. *Let h, l, p and q be as in Theorem 2.1. Let $f(s,t)$, $g(k,r)$, $D_i u(s,t)$, $i=1,2$ and $D_iD_j u(s,t)$, $i,j=1,2$, $i\neq j$ be as in Theorem 1.2, then*

$$\int_0^x\int_0^y\left(\int_0^z\int_0^w \frac{|F(s,t,h)||G(k,r,l)|}{q(st)^{p-1}+p(kr)^{q-1}}\,dkdr\right)dsdt$$
$$\leq C(p,q,x,y,z,w)\left(\int_0^x\int_0^y (x-s)(y-t)|D_2^*D_1^*f(s,t,h)|^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^s\int_0^w (z-k)(w-r)|D_2^*D_1^*g(k,r,l)|^q\,dkdr\right)^{\frac{1}{q}}, \quad (2.5)$$

where

$$F(s,t,h)=f^h(s,t)-f^h(0,t)-f^h(s,0)+f^h(0,0),$$

$$D_2^*D_1^*f(s,t,h)=h(h-l)f^{h-1}(s,t)\cdot D_1f(s,t)\cdot D_2f(s,t) + hf^{h-1}(s,t)\cdot D_2D_1f(s,t),$$

$$G(k,r,l)=g^l(k,r)-g^l(0,r)-g^l(k,0)+g^l(0,0),$$

$$D_2^*D_1^*G(k,r,l)=l(l-l)g^{l-1}(k,r)\cdot D_1g(k,r)\cdot D_2g(k,r) + lg^{l-1}(k,r)\cdot D_2D_1g(k,r),$$

and

$$C(p,q,x,y,z,w)=\frac{1}{pq}(xy)^{\frac{p-1}{p}}(zw)^{\frac{q-1}{q}}. \quad (2.6)$$

Remark 2. It is obvious that inequality (2.5) is a new Hubert type inequality.

We take $h=l=1$, $f(0,0)=f(0,t)+f(s,0)$ and $g(0,0)=g(0,r)+g(k,0)$ in (2.5). We obviously have $F(s,t,h)=f(s,t)$, $G(k,r,l)=g(k,r)$, $D_2^*D_1^*f(s,t,h)=D_2D_1f(s,t)$ and $D_2^*D_1^*g(k,r,l)=D_2D_1(k,r)$, then inequality (2.5) reduces to inequality (1.2).

Moreover, (2.5) can also be changed to the inequality

$$\int_0^x\int_0^y\left(\int_0^z\int_0^w \frac{|F(s,t,h)||G(k,r,l)|}{q(st)^{p-1}+p(kr)^{q-1}}\,dkdr\right) dsdt$$
$$\le C(p,q,x,y,z,w)\left(\frac{1}{p}\int_0^x\int_0^y (x-s)(y-t)|D_2^*D_1^*f(s,t,h)|^p\,dsdt\right.$$
$$\left.+\frac{1}{q}\int_0^z\int_0^w (z-k)(w-r)|D_2^*D_1^*g(k,r,l)|^q\,dkdr\right). \quad (2.7)$$

This is a generalization of the inequality (13), which was given by Pachpatte [6]. On the other hand, we can give a well strengthed version of (2.7) as

$$\int_0^x\int_0^y\left(\int_0^z\int_0^w \frac{|F(s,t,h)||G(k,r,l)|}{C'(p,q,k,r,s,t)}\,dkdr\right) dsdt$$
$$\le C'(p,q,x,y,z,w)\left(\int_0^x\int_0^y (x-s)(y-t)|D_2^*D_1^*f(s,t,h)|^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^z\int_0^w (z-k)(w-r)|D_2^*D_1^*g(k,r,l)|^q\,dkdr\right)^{\frac{1}{q}}, \quad (2.8)$$

where $C'(p,q,u,v,m,n)=(uv)^{\frac{p-1}{p}}(mn)^{\frac{q-1}{q}}$.

Theorem 2.3. *Let h, l, p and q be as in Theorem 2.1 and let $f(s)$ and $g(t)$ be as in Theorem 1.3 and $f(s)>f(0)$ and $g(t)>g(0)$. Let ϕ and ψ be two real-valued nonnegative continuous and convex functions defined on $\mathbb{R}^+$. Then*

$$\int_0^x\int_0^y \frac{F(s)G(t)\cdot\phi\left(\frac{F(s,h)}{hF(s)}\right)\cdot\psi\left(\frac{G(t,l)}{lG(t)}\right)}{qs^{p-1}+pt^{q-1}}\,dsdt$$

$$\leq K(p,q,x,y)\left(\int_0^x (x-s)\left(f'(s)\cdot\phi\left(f^{h-1}(s)\right)\right)^p ds\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^y (y-t)\left(g'(t)\cdot\psi\left(g^{l-1}(t)\right)\right)^q dt\right)^{\frac{1}{q}}, \quad (2.9)$$

where $K(p,q,x,y)$, is as in Theorem 1.1 and

$$F(s,h)=f^h(s)-f^h(0), \quad G(t,l)=g^l(s)-g^l(0),$$

$$F(s)=F(s,1), \quad G(t)=G(t,1).$$

The inequality is reversed if ϕ and ψ are two real-valued nonnegative continuous, monotone increasing and concave functions defined on $\mathbb{R}^+$ and $p<1$ ($p\neq 0$).

Remark 3. If we take $\phi(x)=x$, $\psi(y)=y$, $h=l=1$ and $f(0)=g(0)=0$ in (2.9), then inequality (2.9) reduces to inequality (1.4).

In inequality (2.9), if ϕ and ψ are changed to two real-valued nonnegative continuous, monotone increasing and concave functions defined on $\mathbb{R}^+$ and $p<1$ ($p\neq 0$), then the inverse inequality of (2.9) is valid. Hence, under the hypotheses of Theorem 1.3, $p>1$ is changed to $p<1$ ($p\neq 0$), then

$$\int_0^x\int_0^y \frac{f(s)g(t)}{qs^{p-1}+pt^{q-1}}\,dsdt \geq K(p,q,x,y)$$

$$\times\left(\int_0^x (x-s)(f'(s))^p\,ds\right)^{\frac{1}{p}}\left(\int_0^y (y-t)(g'(t))^q\,dt\right)^{\frac{1}{p}}, \quad (2.10)$$

where $K(p,q,x,y)$, is as in Theorem 1.1.

This is just the inverse inequality of (1.4). Moreover (2.9) can also be changed to the inequality

$$\int_0^x\int_0^y \frac{F(s)G(t)\cdot\phi\left(\frac{F(s,h)}{hF(s)}\right)\psi\left(\frac{G(t,l)}{lG(t)}\right)}{qs^{p-1}+pt^{q-1}}\,dsdt$$

$$\leq K(p,q,x,y)\left(\frac{1}{p}\int_0^x (x-s)\left(f'(s)\cdot\phi\left(f^{h-1}(s)\right)\right)^p ds\right.$$

$$\left.+\frac{1}{q}\int_0^y (y-t)\left(g'(t)\cdot\psi\left(g^{l-1}(t)\right)\right)^q dt\right), \quad (2.11)$$

Clearly, this is just another generalization of (7), which was given by Pachpatte [6]. On the other hand, we can give a well strengthened version of (2.11) as

$$\int_0^x\int_0^y \frac{F(s)G(t)\cdot\phi\left(\frac{F(s,h)}{hF(s)}\right)\psi\left(\frac{G(t,l)}{lG(t)}\right)}{K'(p,q,s,t)}dsdt$$

$$\leq K'(p,q,x,y)\left(\int_0^x (x-s)\left(f'(s)\cdot\phi\left(f^{h-1}(s)\right)\right)^p ds\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^y (y-t)\left(g'(t)\cdot\psi\left(g^{l-1}(t)\right)\right)^q dt\right)^{\frac{1}{q}}, \quad (2.12)$$

where $K'(p,q,u,v)=u^{\frac{p-1}{p}}v^{\frac{q-1}{q}}$.

Theorem 2.4. *Under the hypotheses of Theorem 1.4, if h,l,p and q are as in Theorem 2.1 and let $D_2^*D_1^*f(s,t,h)$ and $D_2^*D_1^*g(k,r,l)$ be as in Theorem 2.2, and $D_iu(s,t)>0$ $(i=1,2)$. Let ϕ and ψ be two real-valued nonnegative convex functions, then*

$$\int_0^x\int_0^y\left(\int_0^z\int_0^w \frac{stkr\left(\phi\left(\frac{F(s,t,h)}{st}\right)\psi\left(\frac{G(k,r,l)}{kr}\right)\right)}{q(st)^{p-1}+p(kr)^{q-1}}dkdr\right)dsdt$$

$$\leq C(p,q,x,y,z,w)\left(\int_0^x\int_0^y (x-s)(y-t)\left(\phi\left(D_2^*D_1^*f(s,t,h)\right)\right)^p dsdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^z\int_0^w (z-k)(w-r)\left(\psi\left(D_2^*D_1^*g(k,r,l)\right)\right)^q dkdr\right)^{\frac{1}{q}}, \quad (2.13)$$

where $C(p,q,x,y,z,w)$ is as in Theorem 1.2 and

$$F(s,t,h)=f^h(s,t)-f^h(0,t)-f^k(s,0)+f^k(0,0)\,,$$

$$G(k,r,l)=g^l(k,r)-g^l(0,r)-g^l(k,0)+g^l(0,0)\,.$$

The inequality is reversed if ϕ and ψ are two real-valued nonnegative continuous concave functions and $p<1$ ($p\neq 0$).

Remark 4. In (2.13), if $h=l=1$, $\phi(x)=x$, $\psi(y)=y$, $f(0,0)=f(0,t)+f(s,0)$ and $g(0,0)=g(0,r)+g(k,0)$, then (2.13) will be changed to (1.5).

Since, when ϕ and ψ are nonnegative real-valued continuous concave functions and $p<1$ ($p\neq 0$), the inverse inequality of (2.13) is valid. Hence, when $p<1$ ($p\neq 0$), the inverse inequality of (1.6) is also valid.

Similarly, from (2.13) we also can get another generalization of the inequality (13), which was given by Pachpatte [6].

Moreover, we can get a strengthened version of (2.13)

$$\int_0^x\int_0^y\left(\int_0^z\int_0^w\frac{stkr\phi\left(\dfrac{F(s,t,h)}{st}\right)\phi\left(\dfrac{G(k,r,l)}{kr}\right)}{C'(p,q,s,t,k,r)}\,dkdr\right)dsdt$$

$$\leq C'(p,q,x,y,z,w)\left(\int_0^x\int_0^y(x-s)(y-t)\left(\phi\left(D_2^*D_1^*f(s,t,h)\right)\right)^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^z\int_0^w(z-k)(w-r)\left(\psi\left(D_2^*D_1^*g(k,r,l)\right)\right)^q\,dsdt\right)^{\frac{1}{q}},\quad(2.14)$$

where $C'(p,q,u,v,m,n)=(uv)^{\frac{p-1}{p}}(mn)^{\frac{q-1}{q}}$.

3. Proofs of Theorems

Since the proofs resemble one another, we give the detail of the proofs of Theorem 2.2 and Theorem 2.3 only, the proofs of Theorem 2.1 and Theorem 2.4 can be similarly completed.

Proof of Theorem 2.2. From the hypotheses of Theorem 2.2, we note that

$$F(s,t,h) = \int_0^s \int_0^t D_2^* D_1^* f(\xi,\eta,h)\, d\xi d\eta\,, \tag{3.1}$$

where $(s,t) \in I_x \times I_y$. We have

$$\begin{aligned}
&\int_0^s \int_0^t D_2^* D_1^* f(\xi,\eta,h)\, d\xi d\eta \\
&= \int_0^s \int_0^t \Big(h(h-1) f^{h-1}(\xi,\eta) \cdot D_1 f(\xi,\eta) \cdot D_2 f(\xi,\eta) \\
&\quad + h f^{h-1}(\xi,\eta) D_2 D_1 f(\xi,\eta) \Big)\, d\xi d\eta \\
&= \int_0^s \left(\int_0^t D_2 \Big(h f^{h-1}(\xi,\eta) \cdot D_1 f(\xi,\eta) \Big)\, d\eta \right) d\xi \\
&= \int_0^s \Big(D_1 f^h(\xi,t) - D_1 f^h(\xi,0) \Big)\, d\xi
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&= \int_0^s D_1 f^h(\xi,t)\, d\xi - \int_0^s D_1 f^h(\xi,0) s\, d\xi \\
&= f^h(s,t) - f^h(0,t) - f^h(s,0) + f^h(0,0) = F(s,t,h)\,.
\end{aligned}$$

Applying the special case of the Hölder integral inequality to (3.1) gives

$$\begin{aligned}
|F(s,t,h)| &\le \int_0^s \int_0^t |D_2^* D_1^* f(\xi,\eta,h)|\, d\xi d\eta \\
&\le (st)^{\frac{p-1}{p}} \left(\int_0^s \int_0^t |D_2^* D_1^* f(\xi,\eta,h)|^p\, d\xi d\eta \right)^{\frac{1}{p}}.
\end{aligned} \tag{3.3}$$

Similarly,

$$|G(k,r,l)| \le (kr)^{\frac{q-1}{q}} \left(\int_0^k \int_0^r |D_2^* D_1^* g(\sigma,\tau,h)|^q\, d\sigma d\tau \right)^{\frac{1}{q}}. \tag{3.4}$$

Using the special case of Young inequality (see Beckenbach and Bellman [8], p. 151), it follows from (3.3) and (3.4) that

$$|F(s,t,h)||G(k,r,l)| \le \left(\frac{(st)^{p-1}}{p} + \frac{(kr)^{q-1}}{q}\right) \times \left(\int_0^s \int_0^t |D_2^* D_1^* f(\xi,\eta,h)|^p \, d\xi d\eta\right)^{\frac{1}{p}} \times \left(\int_0^k \int_0^r |D_2^* D_1^* g(\sigma,\tau,l)|^q \, d\sigma d\tau\right)^{\frac{1}{q}} . \quad (3.5)$$

We first divide both sides of (3.5) by $[q(st)^{p-1} + p(kr)^{q-1}]$, then we apply method similar to Pachpatte [6] to integrate both sides of the resulting inequality so that we obtain (2.5) from (3.5) due to the Hölder integral inequality and the Fubini Theorem. We can also get inequality (2.8) from the proof of Theorem 2.2. □

Proof of Theorem 2.3. From the hypotheses of Theorem 2.3, we have the following identities

$$\frac{F(s,h)}{h \cdot F(s)} = \frac{\int_0^s f'(\tau) f^{h-1}(\tau)\, d\tau}{\int_0^s f'(\tau)\, d\tau}, \quad s \in I_x .$$

Since ϕ is a nonnegative continuous and convex functions defined on $\mathbb{R}^+$ and in view of the Jensen inequality and the Hölder inequality, we have

$$\phi\left(\frac{F(s,h)}{h \cdot F(s)}\right) = \phi\left(\frac{\int_0^s f'(\tau) f^{h-1}(\tau)\, d\tau}{\int_0^s f'(\tau)\, d\tau}\right) \le \frac{1}{F(s)} \int_0^s f'(\tau) \cdot \phi(f^{h-1}(\tau))\, d\tau \le \frac{1}{F(s)} s^{\frac{p-1}{p}} \left(\int_0^s \left(f'(\tau) \cdot \phi(f^{h-1}(\tau))^p\right) d\tau\right)^{\frac{1}{p}} . \quad (3.6)$$

Similarly

$$\psi\left(\frac{G(t,h)}{lG(t)}\right) \leq \frac{1}{G(t)} t^{\frac{q-1}{q}} \left(\int_0^t \left(g'(\sigma) \cdot \psi(g^{l-1}(\sigma))^q \right) d\tau \right)^{\frac{1}{q}}. \quad (3.7)$$

In view of the special case of the Young inequality, it follows from (3.6) and (3.7) that

$$\frac{F(s)G(t) \cdot \phi\left(\frac{F(s,h)}{hF(s)} \cdot \frac{G(t,l)}{lG(t)}\right)}{qs^{p-1} + pt^{q-1}} \leq \frac{1}{pq} \left(\int_0^s \left(f'(\tau) \cdot \phi(f^{h-1}(\tau)) \right)^p d\tau \right)^{\frac{1}{p}} \times \left(\int_0^t \left(g'(\sigma) \cdot \psi(g^{l-1}(\sigma)) \right)^q d\tau \right)^{\frac{1}{q}}. \quad (3.8)$$

Integrating both sides of (3.8) first over t from 0 to y and integrating again both sides of the resulting inequality over s from 0 to x and using the Hölder integral inequality, we can also complete the proof of Theorem 2.3. □

Remark 5. We can easily get inequality (2.12) from the proof of Theorem 2.3.

Following the same steps as in the proof of Theorem 2.3 with suitable modifications we can prove the inverse inequality of (2.9).

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