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# SECOND ORDER OPTIMALITY CONDITIONS FOR $C^1$ MULTIOBJECTIVE OPTIMIZATION PROBLEMS. SCALARIZATION

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**Abstract:** Multiobjective optimization is known as a useful mathematical model in order to investigate some real world problems with conflicting objectives, arising from economics, engineering and human decision making. In this work, we use a notion of approximate Hessian recently introduced by Jeyakumar and Luc [13] and some schalarization method to establish second order necessary and sufficient optimality conditions for constrained multiobjective optimization problems. Throughout the paper, the data are only assumed to be of class $C^1$ but not necessarily of class $C^{1.1}$.

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## 1. Introduction

A lot of research has been carried out in the realm of multiobjective optimization

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problems. Corley [4] has given optimality conditions for convex and nonconvex multiobjective problems in terms of Clarke derivative. Luc [16] also gives optimality conditions when the data are upper semidifferentiable. Luc and Malivert [18] extend the concept of invex functions to invex multifunctions and study optimality conditions for multiobjective optimization with invex data in terms of contingent derivative.

For many optimization problems, notably in mathematical programming, the characterization of optimal solutions with the help of second order conditions was always of a great interest in order to refine first order optimality conditions and obtain sufficient optimality conditions. The need of second order informations also appears in numerical algorithms.

In this paper, we are concerned with the multiobjective optimization problem

$$(P): \begin{cases} \min F(x) \\ \text{subject to } G(x) \cap -Z^{+} \neq \emptyset \end{cases}$$

where $X$, $Y$ and $Z$ are Banach spaces, $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ are $\mathrm{C}^1$-set valued mappings and $Y^+ \subset Y$ and $Z^+ \subset Z$ are closed convex cones.

Our aim is to establish second order optimality conditions for $(P)$ provided that the support functions of $F$ and $G$ are continuously differentiable or $\mathrm{C}^1$ for short. As in [11] and [17], the main tool we are going to exploit is an approximate Hessian of continuously differentiable functions and its recession matrices. The notion of approximate Jacobian and approximate Hessian have been introduced and studied by Jeyakumar and Luc [13]. It is important to notice that several known second order generalized derivatives of continuously differentiable functions are examples of approximate Hessian, including the generalized Hessian introduced in [12] by using the Clarke generalized Jacobian, Cominetti and Correa generalized Hessian [3] and Murdukhovich second order subdifferential [19]. Consequently, the results obtained by using approximate Hessian remain true when applied to the generalized second order subdifferentials above. Moreover, a $\mathrm{C}^{1.1}$ function with a locally Lipschitz gradient may admit an approximate Hessian which closed convex hull is strictly contained in the Clarke generalized Hessian or in the Mordukhovich second order coderivative. Therefore the optimality conditions we are going to establish by means of approximate Hessian are not only valid for $\mathrm{C}^{1.1}$ problems when the Clarke generalized Hessian or the Modukhovich second order coderivative is used, but sometimes also yield sharper results. For more details, see [14] and [21].

The rest of the paper is organized as follows: Sections 2 establish some scalarization results. Section 3 is devoted to the optimality conditions. Section 4 discusses an application to a mathematical programming problem.

## 2. Scalarization

As it was mentioned in the introduction, we are concerned with the multiobjective optimization problem

$$(P) : \begin{cases} \min F(x) \\ \text{subject to } G(x) \cap -Z^{+} \neq \emptyset, \end{cases}$$

where $X$, $Y$ and $Z$ are Banach spaces, $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ are $C^1$-set valued mappings and $Y^{+} \subset Y$ and $Z^{+} \subset Z$ are closed convex and pointed cones with nonempty interiors.

We denote the domain and the graph of $F$ respectively by

$$\text{dom }(F) := \{x \in X : F(x) \neq \emptyset\},$$

$$\text{gr }(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

If $V$ is a nonempty subset of $X$, then

$$F(V) = \underset{x \in V}{\cup} F(x).$$

Let $y^* \in Y^*$. The function $C_F(y^*, x) := \inf_{y \in F(x)} \langle y^*, y \rangle$ is called the support function of $F$, where $\langle ., . \rangle$ denotes the inner product. Assume that the barrier cone of $F(x)$, i.e., the set

$$Y_F := \left\{ y^* \in Y^* : \inf_{y \in F(x)} \langle y^*, y \rangle > -\infty \right\}$$

is closed and does not depend on $x$. This is the case, for example, when $F$ is locally Lipschitz [5]. Denoting this cone by $Y_F$, we say that $F$ is $C^1$-mapping if for any $y^* \in Y_F$, $C_F(y^*, .)$ is a $C^1$-function.

Let $A$ be a nonempty subset of $Y$. A point $\overline{y} \in A$ is said to be a Pareto (respectively, a weak Pareto) minimal point of $A$ with respect to $Y^{+}$ if

$$\begin{aligned} (A - \overline{y}) \cap \left(-Y^{+}\right) &= \emptyset, \\ (\text{ respectively. } (A - \overline{y}) \cap \left(-Y^{++}\right) &= \emptyset \text{ }), \end{aligned}$$

here $Y^{++}$ denotes the topological interior of $Y^{+}$.

Setting $\Omega := \{x \in X : G(x) \cap -Z^{+} \neq \emptyset\}$, a point $(\overline{x}, \overline{y}) \in \text{gr }(F)$, $\overline{x} \in \Omega$, is said to be a weak local Pareto minimal point with respect to $Y^{+}$ of the problem $(P)$ if there exists a neighborhood $V$ of $\overline{x}$ such that

$$F(V \cap \Omega) \subset \overline{y} + Y \setminus \left(-Y^{++}\right).$$

Let $C \subset X$ be a convex set. The set valued mapping $F$ from $C$ into $Y$ is said to be $Y^+$-convex on $C$, if $\forall x_1,\ x_2 \in C,\ \forall \lambda \in [0,1]$

$$\lambda F(x_1) + (1-\lambda) F(x_2) \subset F(\lambda x_1 + (1-\lambda) x_2) + Y^+.$$

It has been proven in Taa [20] that if $C$ is concex and if the set valued mapping $F$ is $Y^+$-convex on $C$ then $(\overline{x}, \overline{y}) \in \text{gr}\ (F)$ with $\overline{x} \in C$ is a local weak Pareto minimal point of $(P)$ if and only if it is a global weak Pareto minimal point of $(P)$.

**Definition 2.1.** [15] Let $C \subset X$ be a nonempty set. The set valued mapping $F$ from $C$ into $Y$ is said to be $Y^+$-convexlike on $C$, if $\forall x_1, x_2 \in C,\ \forall \lambda \in [0,1],\ \exists z \in C$ such that

$$\lambda F(x_1) + (1-\lambda) F(x_2) \subset F(z) + Y^+.$$

If $C$ is convex and if $F$ is $Y^+$-convex on $C$ then it is easy to see that $F$ is $Y^+$-convexlike on $C$. Theorem 2.1 establishes a scalarization result for a weak local Pareto minimal solution of $(P)$.

**Theorem 2.1.** *Let $\overline{x} \in \Omega$ and $\overline{y} \in F(\overline{x})$. Suppose that $(\overline{x}, \overline{y})$ is a weak local Pareto minimal point of $(P)$. If $F$ is $Y^+$-convexlike on $\Omega$, then there exists $y_0^* \in (-Y^+)^\circ \setminus \{0\}$ such that $C_F(y_0^*, \overline{x}) = \langle y_0^*, \overline{y} \rangle$ and $\overline{x}$ is a local solution of the constrained mathematical programming problem*

$$(P_2) \qquad \textit{Minimize}\ \ C_F(y_0^*, x) \quad \textit{subject to}\ \ x \in \Omega.$$

*Here $(Y^+)^\circ$ denotes the negative polar cone of $Y^+$ defined by*

$$\left(Y^+\right)^\circ = \left\{y^* \in Y^* : \langle y^*, y \rangle \leq 0 \ \textit{for all}\ y \in Y^+\right\}.$$

*Proof.* Since $(\overline{x}, \overline{y})$ is a weak local Pareto minimal point of $(P)$, there exists a neighborhood $V$ of $\overline{x}$ such that for all $x \in V \cap \Omega$

$$F(x) \subset \overline{y} + \left(Y \setminus \left(-Y^{++}\right)\right). \tag{2.1}$$

Let $\Delta := \{p \in Y : \exists x \in V \cap \Omega \text{ such that } (F(x) - p) \cap (\overline{y} - Y^{++}) \neq \emptyset\}$. From (2.1), one has $0 \notin \Delta$. Let us prove that $\Delta$ is a convex subset of $Y$. Let $\lambda \in [0,1],\ p_1 \in \Delta$ and $p_2 \in \Delta$. From definition, there exist $x_1 \in V \cap \Omega$ and $x_2 \in V \cap \Omega$ such that

$$\begin{cases} (F(x_1) - p_1) \cap (\overline{y} - Y^{++}) \neq \emptyset, \\ (F(x_2) - p_2) \cap (\overline{y} - Y^{++}) \neq \emptyset. \end{cases}$$

Consequently, there exist $z_1 \in F(x_1) - p_1$ and $z_2 \in F(x_2) - p_2$ such that

$$\begin{cases} z_1 \in \overline{y} - Y^{++}, \\ z_2 \in \overline{y} - Y^{++}. \end{cases} \tag{2.2}$$

Fix $z := \lambda z_1 + (1-\lambda) z_2$ and $p := \lambda p_1 + (1-\lambda) p_2$.

On the one hand, since $Y^{++}$ is a convex cone, one has $z \in \overline{y} - Y^{++}$. On the other hand,

$$z := \lambda z_1 + (1-\lambda) z_2 \in \lambda F(x_1) + (1-\lambda) F(x_2) - (\lambda p_1 + (1-\lambda) p_2).$$

From the convexity assumption of $F$, there exists $x \in V \cap \Omega$ such that $z \in F(x) - p + Y^+$. Thus, there exists $y_0 \in Y^+$ such that

$$z - y_0 \in F(x) - p. \tag{2.3}$$

To get the result, it suffices to prove that $z - y_0 \in \overline{y} - Y^{++}$. From (2.2), we have

$$\begin{cases} z_1 - y_0 \in \overline{y} - y_0 - Y^{++} \subset \overline{y} - Y^{+} - Y^{++} \subset \overline{y} - Y^{++}, \\ z_2 - y_0 \in \overline{y} - y_0 - Y^{++} \subset \overline{y} - Y^{+} - Y^{++} \subset \overline{y} - Y^{++}; \end{cases}$$

which implies that

$$z - y_0 \in \overline{y} - Y^{++}. \tag{2.4}$$

Combining (2.3) and (2.4), we conclude that $(F(x) - p) \cap (\overline{y} - Y^{++}) \neq \emptyset$, and that $\Delta$ is a convex subset of $Y$.

The set $\Delta$ is open. Indeed, consider $p \in \Delta$. From definition, there exists $x \in \Omega$ such that

$$(F(x) - p) \cap \left(\overline{y} - Y^{++}\right) \neq \emptyset.$$

Then, there exist $x \in V \cap \Omega$ and $z \in F(x) - p$ such that $z \in \overline{y} - Y^{++}$. Consequently, there exist $x \in V \cap \Omega$ and $u \in F(x)$ such that $\overline{y} + p - u \in Y^{++}$. Obviously, there exists $\delta > 0$ such that $\overline{y} + p - u + \mathbb{B}(0,\delta) \subset Y^{++}$. Then, for all $b \in \mathbb{B}(0,\delta)$,

$$q = u - (p+b) \in (F(x) - (p+b)) \cap \left(\overline{y} - Y^{++}\right),$$

which means $p + b \in \Delta$, i.e. $p + \mathbb{B}(0,\delta) \subset \Delta$. Thus, $\Delta$ is open.

Now, using a separation theorem, there exists $0 \neq y_0^* \in Y^*$ such that

$$\langle y_0^*, p \rangle \geq 0 \text{ for all } p \in \Delta.$$

Let $x \in V \cap \Omega$, $y \in F(x)$, $r \in Y^{++}$ and $\varepsilon > 0$. Taking $p = y - \overline{y} + \varepsilon r$, one has

$$\begin{cases} y - p \in F(x) - p, \\ y - p = \overline{y} - \varepsilon r \in \overline{y} - Y^{++}, \end{cases}$$

which implies $p \in \Delta$. Then

$$\langle y_0^*, y\rangle - \langle y_0^*, \overline{y}\rangle + \varepsilon \langle y_0^*, r\rangle \geq 0. \tag{2.5}$$

On the one hand, taking $x = \overline{x}$ and $y = \overline{y}$, one has

$$\langle y_0^*, r\rangle \geq 0 \text{ for all } r \in Y^{++}.$$

It follows that $y_0^* \in (-Y^+)^\circ$.

On the other hand, $\varepsilon > 0$ is arbitrary in (2.5)

$$\langle y_0^*, y\rangle - \langle y_0^*, \overline{y}\rangle \geq 0 \text{ for all } x \in V \cap \Omega \text{ and } y \in F(x). \tag{2.6}$$

In particular, for $x = \overline{x}$, one has

$$\langle y_0^*, y\rangle - \langle y_0^*, \overline{y}\rangle \geq 0 \text{ for all } y \in F(\overline{x}).$$

Thus, $C_F(y_0^*, \overline{x}) \geq \langle y_0^*, \overline{y}\rangle$ and then

$$C_F(y_0^*, \overline{x}) = \langle y_0^*, \overline{y}\rangle. \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$\langle y_0^*, y\rangle \geq C_F(y_0^*, \overline{x}) \text{ for all } x \in V \cap \Omega \text{ and } y \in F(x).$$

Finally,

$$C_F(y_0^*, x) = \inf_{y \in F(x)} \langle y_0^*, y\rangle \geq C_F(y_0^*, \overline{x}) \text{ for all } x \in V \cap \Omega,$$

and the proof is finished. □

The converse result is given by the following theorem.

**Theorem 2.2.** *Let $\overline{y} \in F(\overline{x})$. $(\overline{x}, \overline{y})$ is a local Pareto minimal point of $(P)$ if there exists $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that $\overline{x}$ is a local solution of $(P_2)$ and $C_F(y^*, \overline{x}) = \langle y^*, \overline{y}\rangle$.*

*Proof.* Suppose the contrary. Then there exist sequences $(x_n) \to \overline{x}$ and $(y_n) \subset Y$ such that for all $n \in \mathbb{N}$, $\overline{y} - y_n \in Y^+ \setminus \{0\}$, $0 \in G(x_n)$ and $y_n \in F(x_n)$.

Hence there exists $s_n \in Y^+ \setminus \{0\}$ such that $y_n = \overline{y} - s_n$. Thus, $\langle y^*, y_n\rangle = \langle y^*, \overline{y}\rangle - \langle y^*, s_n\rangle$.

Since $s_n \in Y^+ \setminus \{0\}$ and $y^* \in (-Y^+)^\circ / \{0\}$ we have $\langle y^*, s_n\rangle > 0$. Consequently,

$$\langle y^*, y_n\rangle < \langle y^*, \overline{y}\rangle, \text{ for all } n \in \mathbb{N},$$

and then $C_F(y^*, x_n) \leq \langle y^*, y_n\rangle < \langle y^*, \overline{y}\rangle = C_F(y^*, \overline{x})$. This is a contradiction. □

## 3. Second Order Optimality Conditions

For all the sequel, let $X = \mathbb{R}^n$, $Y = \mathbb{R}^p$ and $Z = \mathbb{R}^k$. We begin by introducing

**Definition 3.1.** Let $(\overline{x}, \overline{y}) \in$ gr $(F)$ and $\overline{z} \in G(\overline{x}) \cap -Z^+$. We say that $(y^*, z^*) \in Y_F \times Z_G$ satisfy the first order optimality condition for problem $(P)$ at $(\overline{x}, \overline{y})$ if

$$\begin{cases} \nabla C_F\left(y^*, \overline{x}\right) + \nabla C_G\left(z^*, \overline{x}\right) = 0, \\ C_G\left(z^*, \overline{x}\right) = \langle z^*, \overline{z}\rangle \text{ and } C_F\left(y^*, \overline{x}\right) = \langle y^*, \overline{y}\rangle, \\ y^* \in \left(-Y^+\right)^\circ, \ z^* \in \left(-Z^+\right)^\circ \text{ and } \left(y^*, z^*\right) \neq (0,0). \end{cases} \tag{3.1}$$

**Remark 3.1.** The existence of such vector $(y^*, z^*)$ follows from Theorem 4.2 of [9].

To progress we need the definitions of *approximate Jacobian* and *approximate Hessian.*

**Definition 3.2.** (see [13]) Let $f$ be a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$. A closed set of $(m \times n)$-matrices $\partial f(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^m)$ is said to be an *approximate Jacobian* of $f$ at $x$ if for every $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, one has

$$(vf)^+(x,u) \leq \sup_{M \in \partial f(x)} \langle v, M(u)\rangle,$$

where $vf$ is the real function $\sum_{i=1}^{m} v_i f_i$, here $v_1$, ..., $v_m$ are components of $v$ and $f_1$, ..., $f_m$ are components of $f$, and $(vf)^+(x,u)$ is the upper Dini directional derivative of the function $vf$ at $x$ in the direction $u$; that is

$$(vf)^+(x,u) := \limsup_{t \searrow 0} \frac{(vf)(x+tu) - (vf)(x)}{t}.$$

If for every $x \in \mathbb{R}^n$, $\partial f(x)$ is *approximate Jacobian* of $f$ at $x$, then the set valued mapping $x \longmapsto \partial f(x)$ from $\mathbb{R}^n$ to $L(\mathbb{R}^n, \mathbb{R}^m)$ is called an *approximate Jacobian* of $f$.

For the next concept, we consider a continuously differentiable function $f$ defined on $\mathbb{R}^n$. The gradient map $\nabla f$ is then a continuous vector function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

**Definition 3.3.** (see [17]) A closed set $\partial^2 f(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ is *approximate Hessian* of $f$ at $x$ if it is an *approximate Jacobian* of $\nabla f$ at $x$. If for each $x$ we have some subset $\partial^2 f(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ which is an *approximate Hessian* of $f$ at $x$, then the set valued mapping $x \longmapsto \partial^2 f(x)$ is called an *approximate Hessian* of $f$.

**Remark 3.2.** An approximate Hessian shares all properties of the approximate Jacobian.

For the reader convenience, we list some of them (see [13] for proofs).

**i)** If $\partial^2 f(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ is *approximate Hessian* of $f$ at $x$, then every closed subset of $L(\mathbb{R}^n, \mathbb{R}^n)$ which contains $\partial^2 f(x)$ is an *approximate Hessian* of $f$ at $x$;

**ii)** If $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiables and if $\partial^2 f(x)$ and $\partial^2 g(x)$ are *approximate Hessians* of $f$ at $x$ respectively, then the closure of the set $\partial^2 f(x) + \partial^2 g(x)$ is an *approximate Hessian* of $f + g$ at $x$.

**iii)** ( Generalized Taylor expansion ) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable; let $x, y \in \mathbb{R}^n$. Suppose that for each $z \in [x, y]$, $\partial^2 f(z)$ is an *approximate Hessian* of $f$ at $z$. Then there exists $\zeta \in (x, y)$ such that

$$f(y) \in f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2}\overline{co}\left\langle \partial^2 f(\zeta)(y - x), y - x \right\rangle. \qquad (3.2)$$

We shall need some more terminologies. Let $A \subset \mathbb{R}^n$ be a nonempty set. The recession cone of $A$, which is denoted by $A_\infty$, consists of all limits $\lim\limits_{i \to \infty} t_i a_i$, where $a_i \in A$ and $\{t_i\}$ is a sequence of positive numbers converging to 0. It is important to remark that a set is bounded if and only if its recession cone is trivial. Elements of the recession cone of $\partial^2 f(x)$ are called *recession Hessian matrices.*

Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set valued mapping. It is said to be upper semicontinuous at $\overline{x}$ if for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$H(\overline{x} + \delta \mathbb{B}_n) \subset H(\overline{x}) + \varepsilon \mathbb{B}_m,$$

where $\mathbb{B}_n$ and $\mathbb{B}_m$ denote the closed unit balls in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. The Euclidean norm in $\mathbb{R}^n$, as well as in $L(\mathbb{R}^n, \mathbb{R}^n)$ is denoted by $\|...\|$.

### 3.1. Necessary Optimality Conditions

Suppose that $F$ is $Y^+$-convexlike on $\Omega$ and that $(\overline{x}, \overline{y})$ is a weak local Pareto minimal point of $(P)$. From Theorem 2.1, there exists $y_0^* \in (-Y^+)^\circ \setminus \{0\}$ such that $\overline{x}$ minimize $C_F(y_0^*, x)$ over $\Omega$.

We denote by $\Lambda$, the set of all $z^* \in Z_G$ such that $(y_0^*, z^*)$ satisfy the first order optimality condition for problem $(P)$ at $(\overline{x}, \overline{y})$. Under a suitable regularity assumption such as the one discussed in [6] and [7], the non-emptiness of $\Lambda$ can be deduced from any of the followings : Theorem 2 of [1], Theorem 3.2 of [10] or Theorem 4.2 of [9].

**Definition 3.4.** (see [7]) The problem $(P)$ is said to be regular at $\overline{x} \in \Omega$ if the system

$$\begin{cases} \nabla C_G(z^*, \overline{x}) = 0, \\ C_G(z^*, \overline{x}) = 0, \end{cases}$$

has the unique solution $z^* = 0$.

Now, take an arbitrary vector $z^* \in Z_G$ such that $(y_0^*, z^*)$ satisfy the first order optimality condition for problem $(P)$ at $(\overline{x}, \overline{y})$. Then the set $\Omega^{**} := \{x \in \Omega : C_G(z^*, x) = \langle z^*, \overline{z} \rangle\}$ is nonempty, since $\overline{x} \in \Omega^{**}$. Setting

$$T(\Omega^{**}, \overline{x}) := \{v \in \mathbb{R}^n : v = \lim t_i (x_i - \overline{x}),\ x_i \in \Omega^{**},\ x_i \to \overline{x},\ t_i > 0\},$$

$$T_0(\Omega^{**}, \overline{x}) := \{v \in \mathbb{R}^n : \text{ there is some } \delta > 0 \text{ such that } \overline{x} + tv \in \Omega^{**} \text{ for } t \in [0, \delta]\},$$

$$L(y^*, z^*, .) := C_F(y^*, .) + C_G(z^*, .),$$

we give our first result where the argument is similar to that used by Luc in [17], but we give the proof in a more general situation.

**Theorem 3.1.** *Suppose that $(P)$ is regular at a point $\overline{x} \in \Omega$ and assume the following conditions hold:*

***i.*** *The set valued mapping $F$ is $Y^+$-convexlike on $\Omega$;*

***ii.*** *The support functions of $F$ and $G$ are continuously differentiable and $(\overline{x}, \overline{y})$ is a weak local Pareto minimal point of $(P)$;*

***iii.*** *The first order optimality condition is satisfied at $\overline{x}$, for some vector $z^* \in \Lambda$;*

***iv.*** *The set $\partial^2 L(y_0^*, z^*, \overline{x})$ is an* approximate Hessian *of $L$ at $\overline{x}$.*

*Then for each $d \in T_0(\Omega^{**}, \overline{x})$, there is*

$$M \in \partial^2 L(y_0^*, z^*, \overline{x}) \cup \left(\left[\partial^2 L(y_0^*, z^*, \overline{x})\right]_\infty \setminus \{0\}\right)$$

*such that $\langle d, M(d) \rangle \geq 0$.*

*Proof.* From Theorem 2.1, since $(\overline{x}, \overline{y})$ is a weak local Pareto minimal point of $(P)$, there exists $y_0^* \in (-Y^+)^\circ \setminus \{0\}$ such that $\overline{x}$ is a local solution of the constrained mathematical programming problem $(P_2)$:

$$\text{Minimize} \quad C_F(y_0^*, x) \quad \text{subject to} \quad x \in \Omega.$$

Let $d \in T_0(\Omega^{**}, \overline{x})$. There is $\delta > 0$ such that $[\overline{x}, \overline{x} + \delta u] \subset \Omega^{**}$. Since $\overline{x}$ is a local solution, there is $n_0 \geq 1$ such that $\delta > \dfrac{1}{n_0}$ and

$$L\left(y_0^*, z^*, \overline{x} + \frac{1}{n}d\right) - L(y_0^*, z^*, \overline{x}) = C_F\left(y_0^*, \overline{x} + \frac{1}{n}d\right) - C_F(y_0^*, \overline{x}) \geq 0,$$

for $n \geq n_0$.

In view of the Classic Mean Value Theorem, there is $t_n \in (0, \delta)$ such that

$$L\left(y_0^*, z^*, \overline{x} + \frac{1}{n}d\right) - L(y_0^*, z^*, \overline{x}) = \nabla L(y_0^*, z^*, \overline{x} + t_n d)\left(\frac{1}{n}d\right),$$

for $n \geq n_0$.

Then

$$\langle d, \nabla L(y_0^*, z^*, \overline{x} + t_n d)\rangle \geq 0,$$

for $n \geq n_0$, which together with (ii) implies

$$\limsup_{t \searrow 0} \frac{\langle d, \nabla L(y_0^*, z^*, \overline{x} + td) - \nabla L(y_0^*, z^*, \overline{x})\rangle}{t} \geq 0.$$

By the definition of approximate Hessian we derive

$$0 \leq (u \circ \nabla L)^+(\overline{x}, d) \leq \sup_{M \in \partial^2 L\left(y_0^*, z^*, \overline{x}\right)} \langle d, M(d)\rangle.$$

There exists then a sequence of approximate Hessian matrices $\{M_n\} \subset \partial^2 L(y_0^*, z^*, \overline{x})$ such that $\lim_{n \to \infty} \langle d, M_n(d)\rangle \geq 0$.

If the sequence $\{M_n\}$ is bounded, then we may assume that it converges to some $M \in \partial^2 L(y_0^*, z^*, \overline{x})$, because the letter set is closed, and obtain $\langle d, M(d)\rangle \geq 0$.

If the sequence $\{M_n\}$ is unbounded, we may assume that

$$\lim_{n \to \infty} \|M_n\| = \infty \text{ and } \lim_{n \to \infty} \frac{M_n}{\|M_n\|} = M_0 \in \left(\partial^2 L(y_0^*, z^*, \overline{x})\right)_\infty \setminus \{0\},$$

and obtain $\langle d, M_0(d)\rangle \geq 0$. □

**Remark 3.3.** Suppose that $L$ has an *approximate Hessian* map $\partial^2 L$ which is upper semicontinuous at $\overline{x}$. Then Theorem 3.1 remains true if we adopt $T(\Omega^{**}, \overline{x})$ instead of $T_0(\Omega^{**}, \overline{x})$. The argument is similar to that used by Luc [17]; the interested reader may consult this reference for the proof.

**Remark 3.4.** Theorem 3.1 above is a generalization of Theorem 3.1 of [7] and Theorem 3.2 of [12] for $C^{1.1}$ data. Moreover, it is an extension of [17] for multiobjective optimization problems under inclusion constraints.

### 3.2. Sufficient Optimality Conditions

As we can see, necessary optimality conditions given above are insufficient for the Pareto optimality of $(\overline{x}, \overline{y})$. To remedy, we shall impose stronger conditions.

**Theorem 3.2.** *Assume that the following conditions hold:*

*1. The first order optimality condition for problem $(P)$ is satisfied at $(\overline{x}, \overline{y})$, for some $(y^*, z^*) \in Y_F \times Z_G$ with $C_G(z^*, \overline{x}) = 0$.*

*2. The support functions of $F$ and $G$ are continuously differentiable.*

*3. There is an upper semicontinuous* approximate Hessian *of $L$ such that*

$$\forall u \in T(\Omega, \overline{x}) \setminus \{0\}, \quad \forall M \in \partial^2 L(y^*, z^*, \overline{x}) \cup \left( \left[ \partial^2 L(y^*, z^*, \overline{x}) \right]_\infty \setminus \{0\} \right), \quad \langle u, M(u) \rangle > 0.$$

*Then $(\overline{x}, \overline{y})$ is a weak local Pareto minimal point of $(P)$.*

*Proof.* By contrary, suppose that $(\overline{x}, \overline{y})$ is not a weak local Pareto minimal point of $(P)$. By Theorem 2.2, $\overline{x}$ is not a local solution of the constrained mathematical programming problem:

$$\text{Minimize} \quad C_F(y^*, x) \quad \text{subject to} \quad x \in \Omega.$$

Then, there exists $x_n \in \Omega \setminus \{\overline{x}\}$ such that $x_n \to \overline{x}$ and $C_F(y^*, x_n) < C_F(y^*, \overline{x})$ for all $n$. By taking a subsequence if necessary, we may assume that the sequence $u_n := \|x_n - \overline{x}\|^{-1} (x_n - \overline{x})$ is convergent. Since $\|u_n\| = 1$, there exists $d_1 \in \mathbb{R}^n$, $\|d_1\| = 1$ such that $\dfrac{x_n - \overline{x}}{\|x_n - \overline{x}\|} \to d_1$.

Remark that $d_1 \in T(\Omega, \overline{x}) \setminus \{0\}$. It follows that

$$\begin{aligned} &L(y^*, z^*, x_n) - L(y^*, z^*, \overline{x}) \\ &\qquad = C_F(y^*, x_n) - C_F(y^*, \overline{x}) + C_G(z^*, x_n) - C_G(z^*, \overline{x}) \le 0. \end{aligned}$$

Using the Taylor expansion, we have

$$\begin{aligned} &L(y^*, z^*, x_n) - L(y^*, z^*, \overline{x}) - \nabla L(y^*, z^*, x_n - \overline{x}) \\ &\qquad \in \frac{1}{2} \overline{co} \left\langle x_n - \overline{x}, \partial^2 L(y^*, z^*, \widehat{x}_n)(x_n - \overline{x}) \right\rangle, \end{aligned} \tag{3.3}$$

for some $\widehat{x}_n \in (\overline{x}, x_n)$. Combining (3.1) and (3.3), there exists a matrix $M_n \in \partial^2 L(y^*, z^*, \widehat{x}_n)$ such that $\langle x_n - \overline{x}, M_n(x_n - \overline{x}) \rangle \le \dfrac{\|x_n - \overline{x}\|^2}{n}$.

If the sequence $\{M_n\}$ is bounded, then we may assume that it converges to some $M \in \partial^2 L(y^*, z^*, \overline{x})$, due to the upper semicontinuity of the approximate Hessian mapping $\partial^2 L$.

The latter inequality implies

$$\langle d_1, M(d_1)\rangle = \lim_{n\to\infty} \left\langle \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|}, M_n\left(\frac{x_n - \overline{x}}{\|x_n - \overline{x}\|}\right)\right\rangle \leq 0,$$

which contradicts the hypothesis.

If the sequence $\{M_n\}$ is unbounded, then due to the upper semicontinuity of the approximate Hessian mapping $\partial^2 L$, we may assume that

$$\lim_{n\to\infty} \|M_n\| = \infty \text{ and } \lim_{n\to\infty} \frac{M_n}{\|M_n\|} = M_0 \in \left[\partial^2 L(y^*, z^*, \overline{x})\right]_\infty \setminus \{0\},$$

which again contradicts the hypothesis. A contradiction! □

As a special case, take the following optimization problem

$$(P_3) : \begin{cases} \min f(x) \\ \text{subject to } 0 \in G(x) \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and the support function of $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ are $\mathrm{C}^{1.1}$ functions.

Without any convexity assumption, Dien and Sach necessary and sufficient optimality conditions [7] can be derived from Theorem 3.1 and Theorem 3.2. It suffices to replace the *approximate Hessian* by the *generalized Hessian* and eliminate the recession parts of the formulas.

**Remark 3.5.** By using the Taylor expansion one can show that a function is $\mathrm{C}^{1.1}$ if and only if it admits a locally bounded approximate Hessian. The recession part in the inclusions above is a very characteristic feature of those problems that have $\mathrm{C}^1$, but not $\mathrm{C}^{1.1}$ data. Without this part the inclusions may fail. For examples and details, see [17].

## 4. Application

In this section we are concerned with the mathematical programming problem

$$(P^*) : \quad \begin{array}{c} \min\ f(x) \\ \begin{cases} g_i(x) \leq 0 & i = 1, 2, \ldots\ldots, m, \\ h_j(x) = 0 & j = 1, 2, \ldots\ldots, k, \end{cases} \end{array}$$

where $f$, $g_i$, and $h_j$ are C$^1$ functions. Denotes by $IR_+^m$ the nonnegative orthant of $IR^m$.

Setting $\Omega := \{x : g_i(x) \leq 0,\ h_j(x) = 0 \text{ for all } i,\ j\}$, $g(x) = (g_1(x), g_2(x), ..., g_m(x))$ and $h(x) = (h_1(x), h_2(x), ..., h_k(x))$, problem $(P^*)$ is reduced to the problem $(P)$, when the set valued mappings $F$ and $G$ from $IR^n$ into $IR$ and $Z = IR^m \times IR^k$ respectively are defined by

$$F(x) = f(x),\ G(x) := (g(x), h(x)) + IR_+^m \times \{0_{IR^k}\}$$

here $IR_+^m$ is the nonnegative orthant of $IR^m$.

Obviously in that case $Z_G = IR_+^m \times IR^k$ and for any $z^* = (\lambda,\ \mu) \in Z_G$ we have

$$C_G(z^*, x) = \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle$$

and

$$L(x) = f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle.$$

Take $\overline{x} \in \Omega$ and $z^* = (\lambda,\ \mu) = (\lambda_1,\ \lambda_2,\ .....,\ \lambda_m,\ \mu) \in IR_+ \times ... \times IR_+ \times IR^k$.

Setting $I = \{1,\ 2,\ ...,\ m\}$ and $q(\overline{x}) = \{i \in I : g_i(\overline{x}) = 0\}$ we get

$$T(\Omega^{**}, \overline{x}) = \left\{ d : \text{such that} \quad \begin{array}{c} \langle \nabla g_i(\overline{x}), d \rangle \leq 0,\ \forall i \in q(\overline{x}) \setminus p(\lambda), \\ \langle \nabla g_i(\overline{x}), d \rangle = 0,\ \forall i \in p(\lambda), \\ \langle \nabla h_j(\overline{x}), d \rangle = 0,\ j = 1,\ 2, ..., k \end{array} \right\},$$

where $p(\lambda) = \{i \in I : \lambda_i > 0\}$.

It can be verified that $C_G(z^*, \overline{x}) = 0$ if and only if $\lambda_i g_i(\overline{x}) = 0$ for all $i \in I$.

From Theorem 3.1, we deduce some results of Luc [17]. Theorem 4.1 gives necessary optimality conditions for the mathematical programming problem $(P^*)$.

**Theorem 4.1.** (see [17]) *Assume that the following conditions hold:*

***i.*** *The functions $f$, $g_i$ and $h_j$ are continuously differentiables and $\overline{x}$ is a local solution of the problem $(P^*)$;*

***ii.*** *There exists a vector $(\lambda,\ \mu) \in IR_+^m \times IR^k$ satisfying the first order optimality condition at $\overline{x} \in \Omega$;*

***ii.*** *The set $\partial^2 L(\overline{x})$ is an* approximate Hessian *of $L$ which is upper semicontinuous at $\overline{x}$.*

*Then for each $d \in T(\Omega^{**}, \overline{x})$, there is*

$$M \in \partial^2 L(\overline{x}) \cup \left( \left[ \partial^2 L(\overline{x}) \right]_\infty \setminus \{0\} \right)$$

*such that*

$$\langle d, M(d) \rangle \geq 0.$$

The following theorem provides sufficient optimality conditions for problem $(P^*)$.

**Theorem 4.2.** (see [17]) *Assume that the following conditions hold:*

***i.*** *The functions $f$, $g_i$ and $h_j$ are continuously differentiables;*

***ii.*** *There exists a vector $(\lambda,\ \mu) \in IR_+^m \times IR^k$ satisfying the first order optimality condition at $\overline{x} \in \Omega$;*

***iii.*** *There is an* approximate Hessian $\partial^2 L(\overline{x})$ *of $L$ which is upper semi-continuous at $\overline{x}$ such that for every $d \in T(\Omega^{**}, \overline{x}) \backslash \{0\}$ and $M \in \partial^2 L(\overline{x}) \cup \left( \left[\partial^2 L(\overline{x})\right]_\infty \setminus \{0\}\right)$, one has*

$$\langle d, M(d)\rangle > 0.$$

*Then $\overline{x}$ is a strict local solution to problem $(P^*)$.*

**Remark 4.1.** When the functions $f$, $g_i$ and $h_j$ are $C^2$ functions, we get from Theorem 4.2 a well-known result of [8].

**Remark 4.2.** The problem $(P^*)$ is regular at $\overline{x} \in \Omega$ if the vectors $\nabla g_i(\overline{x})$, $i \in q(\overline{x})$, $\nabla h_j(\overline{x})$, $j = 1, ..., k$ are lineary independent.


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