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# DERIVATIONS ON SEMIPRIME RINGS

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**Abstract:** In this paper we investigate some properties of derivations on prime and semiprime rings. Among other results we prove that if $R$ is a semiprime ring, $I$ is a nonzero two-sided ideal of $R$ and $f, g$ are derivations of $R$ satisfying $f(x)y+yg(x) = 0$ for all $x, y \in I$, then $f(u)[x, y] = [x, y]g(u) = 0$ for all $x, y \in I$; in particular, $f$ and $g$ map $I$ into the center of $R$. If $R$ is a noncommutative prime ring, then $f = g = 0$ on $R$, which may be regarded as an analog of Posner Lemma for a pair of derivations satisfying this identity.

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## 1. Introduction

Throughout, $R$ denotes an associative ring with center $Z(R)$. We write the commutator $[x, y] = xy - yx$ for $x, y \in R$. We shall frequently use the commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$. We recall that $R$ is *prime* if $aRb = (0)$ implies $a = 0$ or $b = 0$; it is *semiprime* if $aRa = (0)$ implies $a = 0$. An additive map $d : R \to R$ is called a

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*derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A mapping $f : R \to R$ is called *centralizing* if $[f(x), x] \in Z(R)$; in particular, if $[f(x), x] = 0$ for all $x \in R$, then it is called *commuting*. There has been considerable interest in commuting and centralizing derivations and related maps on prime and semiprime rings. A classical result in the theory of centralizing derivations is a Theorem of E. Posner [6, Theorem 2] which states that noncommutative prime rings do not admit nonzero centralizing derivations. A number of algebraists have extended Posner's result in several ways (see e.g. [2, 4, 5, 9], where further references are given).

Our aim in this paper is to present some results, which can be regarded as a contribution to the theory of derivations on prime and semiprime rings. Our work is inspired by Vukman [9]. For instance, Vukman [9] has proved that if $f$ is a derivation on a noncommutative prime ring $R$ of characteristic not two such that the mapping $x \to [d(x), x]$ is commuting on $R$, then $d = 0$. Alternatively, this result states that if $d$ is a derivation, which satisfies the identity $d(x)x^2 + x^2d(x) - 2xd(x)x = 0$ for all $x \in R$, then $d = 0$. This is, in fact, an analog of Posner's result for derivations satisfying this identity. We work here with similar situations for a pair of derivations. For instance, we show (Theorem 2.2) that if $R$ is a semiprime ring, $I$ a nonzero two-sided ideal of $R$ and $f, g$ is a pair of derivations of $R$ such that $f(x)y + yg(x) = 0$ for all $x, y \in I$, then $f(u)[x, y] = [x, y]g(u) = 0$ for all $x, y \in I$. Further, $f$ and $g$ map $I$ into $Z(R)$. In particular, if $R$ is a noncommutative prime ring, then $f = g = 0$ (Corollary 2.3). These results also improve earlier work of Thaheem [7, 8]. Further, some known facts follow as applications of the results here.

## 2. The Results

We shall need the following easy lemma in the proof of Theorem 2.2. We include it here for completeness.

**Lemma 2.1**. *Let $R$ be a semiprime ring, $I$ a nonzero two-sided ideal of $R$ and $a \in R$ such that $axa = 0$ for all $x \in I$, then $a = 0$.*

*Proof.* Let $v \in R$. Then, by assumption, $avxa = 0$; that is, $xavxa = 0$ for all $v \in R$. By the semiprimeness of $R$, $xa = 0$ for all $x \in I$. Similarly, $ax = 0$ for all $x \in I$. Therefore, $ax = xa$ for all $x \in I$ and hence $a \in Z(I) \subseteq Z(R)$. So, by hypothesis, $a^3 = 0$. Since the center of $R$ is free from nonzero nilpotents, therefore $a = 0$. □

We now prove our main result for semiprime rings. In case of prime rings, this is closely related to a result of Brešar [3].

**Theorem 2.2**. *Let $R$ be a semiprime ring, $I$ a nonzero two-sided ideal of $R$ and $f, g$ be derivations of $R$ such that*

$$f(x)y + yg(x) = 0 \quad \text{for all} \ \ x, y \in I. \tag{1}$$

*Then $f(u)[x, y] = [x, y]g(u) = 0$ for all $x, y, u \in I$; in particular, $f$ and $g$ map $I$ into $Z(R)$.*

*Proof.* Replacing $x$ by $yx$ in (1), we get

$$\begin{aligned} f(yx)y + yg(yx) &= f(y)xy + yf(x)y + yg(y)x + y^2g(x) \\ &= (f(y)xy + yg(y)x) + y(f(x)y + yg(x)) \\ &= f(y)xy + yg(y)x = 0. \end{aligned}$$

That is,

$$f(y)xy + yg(y)x = 0 \quad \text{for all} \ \ x, y \in I. \tag{2}$$

By (1), $yg(y) = -f(y)y$. So from (2), we get

$$f(y)xy - f(y)yx = f(y)(xy - yx) = f(y)[x, y] = -f(y)[y, x] = 0.$$

That is,

$$f(y)[y, x] = 0 \quad \text{for all} \ \ x, y \in I. \tag{3}$$

Let $w \in I$. Replacing $x$ by $wx$ in (3), we get $f(y)[y, wx] = f(y)w[y, x] + f(y)[y, w]x = 0$ and by (3), we get

$$f(y)w[y, x] = 0 \quad \text{for all} \ \ x, y, w \in I. \tag{4}$$

Linearizing (3) (in $y$) and using (3), we get

$$\begin{aligned} f(y + u)[y + u, x] &= (f(y) + f(u))([y, x] + [u, x]) \\ &= (f(y)[y, x] + f(u)[u, x]) + f(y)[u, x] + f(u)[y, x] \\ &= f(y)[u, x] + f(u)[y, x] = 0. \end{aligned}$$

That is, $f(y)[u, x] = -f(u)[y, x]$. So,

$$f(y)[u, x] = f(u)[x, y] \quad \text{for all} \ \ x, y, u \in I. \tag{5}$$

We want to prove that $f(u)[x, y] = 0$. For this purpose, let $v \in R$ and consider $f(u)[x, y]vf(u)[x, y]$. Then by (5), we have

$$f(u)[x, y]vf(u)[x, y] = f(u)[x, y]vf(y)[u, x] \quad \text{for all} \ \ x, y, u \in I \text{ and all } v \in R. \tag{6}$$

Put $w = [x, y]vf(y)$. As $[x, y] \in I$ and $I$ is an ideal, so $w \in I$. Therefore, by (4) and (6), we get $f(u)[x, y]vf(u)[x, y] = f(u)w[u, x] = 0$ for all $v \in R$ and hence by the semiprimeness of $R$, $f(u)[x, y] = 0$ for all $x, y, u \in I$. That is

$$f(u)[x, y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{7}$$

This proves the first identity.

We now show that $f(u) \in Z(I)$, where $Z(I)$ denotes the center of $I$. Replacing $x$ by $xf(u)$ in (7) (and using (7)), we get

$$\begin{aligned} f(u)[xf(u), y] &= f(u)x[f(u), y] + f(u)[x, y]f(u) \\ &= f(u)x[f(u), y] = 0. \end{aligned}$$

So,

$$f(u)x[f(u), y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{8}$$

Replacing $x$ by $yx$ in (8), we get

$$f(u)yx[f(u), y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{9}$$

Multiplying (8) by $-y$ on the left, we get

$$-yf(u)x[f(u), y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{10}$$

Adding (9) and (10), we get

$$(f(u)y - yf(u))x[f(u), y] = [f(u), y]x[f(u), y] = 0.$$

That is,

$$[f(u), y]x[f(u), y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{11}$$

By Lemma 2.1, $[f(u), y] = 0$ for all $u, y \in I$.

This shows that $f(u) \in Z(I)$. Since $u$ is an arbitrary element of $I$, we get $f(I) \subseteq Z(I) \subseteq Z(R)$. This proves another assertion about $f$.

We now show that $[x, y]g(u) = 0$ for all $x, y, u \in I$. Since $f(u) \in Z(R)$, therefore by (7), we have $f(u)[x, y] = [x, y]f(u) = 0$. So,

$$\begin{aligned} [x, y]f(u) &= xyf(u) - yxf(u) = xf(u)y - yf(u)x \\ &= -xyg(u) + yxg(u) = (yx - xy)g(u) \\ &= [y, x]g(u) = 0. \end{aligned}$$

That is,

$$[x, y]g(u) = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{12}$$

This gives the second identity.

We now show that $g(I) \subseteq Z(I)$. Let $u \in I$. Replace $x$ by $xg(u)$ in (12) (and using (12)), we get

$$\begin{aligned}[xg(u), y]g(u) &= [x, y]g(u)g(u) + x[g(u), y]g(u) \\ &= x[g(u), y]g(u) = 0.\end{aligned}$$

So,

$$[g(u), y]g(u)x[g(u), y]g(u) = 0 \quad \text{for all } u, y, x \in I. \tag{13}$$

From (13) and Lemma 2.1, we have $[g(u), y]g(u) = 0$. That is,

$$[g(u), y]g(u) = 0 \quad \text{for all } y, u \in I. \tag{14}$$

Replacing $y$ by $yv$ in (14) (and using (14)), we get

$$\begin{aligned}[g(u), yv]g(u) &= [g(u), y]vg(u) + y[g(u), v]g(u) \\ &= [g(u), y]vg(u) = 0.\end{aligned}$$

That is,

$$[g(u), y]vg(u) = 0 \quad \text{for all } u, y, v \in I. \tag{15}$$

Multiplying (15) by $y$ on the right, we get

$$[g(u), y]vg(u)y = 0 \quad \text{for all } u, y, v \in I. \tag{16}$$

Also, replacing $v$ by $vy$ in (15), we get

$$[g(u), y]vyg(u) = 0 \quad \text{for all } u, y, v \in I. \tag{17}$$

Subtracting (17) from (16), we get

$$[g(u), y]v[g(u), y] = 0 \quad \text{for all } u, y, v \in I. \tag{18}$$

From (18) and Lemma 2.1, we get $[g(u), y] = 0$ for all $u, y \in I$. This implies that $g(u) \in Z(I)$; that is, $g(I) \subseteq Z(I) \subseteq Z(R)$. □

By Theorem 2.2, $f(x), g(x) \in Z(R)$ for all $x \in I$. Therefore, $f$ and $g$ are trivially centralizing on $I$ and hence by Bell and Martindale [1, Theorem 4], $f = g = 0$. This establishes the following corollary.

**Corollary 2.3**. *Let $R$ be a noncommutative prime ring, $I$ a nonzero two-sided ideal of $R$ and $f, g$ be derivations of $R$ such that*

$$f(x)y + yg(x) = 0 \quad \textit{for all } x, y \in I. \tag{19}$$

*Then* $f = g = 0$ *on* $R$.

**Remark 2.4**. (a) We shall make use of the following well-known results:

(i) Let $R$ be a prime ring and $I$ a nonzero two-sided ideal of $R$. Then $I$ is a prime subring;

(ii) A noncommutative prime ring does not contain nonzero commutative left ideals.

(b) If $R$ is a noncommutative prime ring, $I$ a nonzero two-sided ideal of $R$ such that $f(x)x = 0$ for all $x \in I$, then $f = 0$ on $R$. This follows from Corollary 2.3. Indeed, put $g = 0$ and $y = x$ in (19), we get $f(x)x = 0$ for all $x \in I$ and hence $f = 0$ on $R$.

**Theorem 2.5**. *Let* $R$ *be a noncommutative prime ring,* $I$ *a nonzero two-sided ideal of* $R$ *and* $f, g$ *be derivations of* $R$ *such that*

$$f(x)xy + yg(x)x = 0 \quad \text{for all} \;\; x, y \in I. \tag{20}$$

*Then* $f = g = 0$.

*Proof*. Replacing $x$ by $x + y$ in (20), we get

$$\begin{aligned} &f(x+y)(x+y)y + yg(x+y)(x+y) \\ &\quad = (f(x) + f(y))(x+y)y + y(g(x) + g(y))(x+y) \\ &\quad = (f(x) + f(y))(xy + y^2) + (yg(x) + yg(y))(x+y) \\ &\quad = (f(x)xy + yg(x)x) + (f(y)y^2 + yg(y)y) + f(x)y^2 \\ &\qquad + f(y)xy + yg(x)y + yg(y)x = 0. \end{aligned}$$

By (20), we get

$$(f(x)y + f(y)x)y + y(g(x)y + g(y)x) = 0 \quad \text{for all} \;\; x, y \in I. \tag{21}$$

Replacing $x$ by $xy$ in (21), we get

$$\begin{aligned} &(f(xy)y + f(y)(xy))y + y(g(xy)y + g(y)xy) \\ &\quad = (f(x)y^2 + xf(y)y + f(y)xy)y + y(g(x)y^2 + xg(y)y + g(y)xy) \\ &\quad = f(x)y^3 + xf(y)y^2 + f(y)xy^2 + yg(x)y^2 + yxg(y)y + yg(y)xy \\ &\quad = (f(x)y^3 + f(y)xy^2 + yg(x)y^2 + yg(y)xy) + (xf(y)y^2 + yxg(y)y) \\ &\quad = (f(x)y^2 + f(y)xy + yg(x)y + yg(y)x)y + (xf(y)y^2 + yxg(y)y) \\ &\quad = ((f(x)y + f(y)x)y + y(g(x)y + g(y)x))y \\ &\qquad + xf(y)y^2 + yxg(y)y = 0. \end{aligned} \tag{22}$$

By (21) and (22), we get

$$xf(y)y^2 + yxg(y)y = 0 \qquad \text{for all } \ x, y \in I. \tag{23}$$

Putting $x = y$ in (20), we get

$$f(y)y^2 + yg(y)y = 0 \qquad \text{for all } \ y \in I. \tag{24}$$

Multiplying (24) by $x$ on the left, we get

$$xf(y)y^2 + xyg(y)y = 0 \quad \text{for all } \ x, y \in I. \tag{25}$$

Subtracting (25) from (23), we get

$$yxg(y)y - xyg(y)y = (yx - xy)g(y)y = [y, x]g(y)y = 0.$$

That is,

$$[x, y]g(y)y = 0 \quad \text{for all } \ x, y \in I. \tag{26}$$

Replacing $x$ by $xv$ in (26) (and using (26)), we get

$$[xv, y]g(y)y = x[v, y]g(y)y + [x, y]vg(y)y = [x, y]vg(y)y = 0.$$

That is,

$$[x, y]vg(y)y = 0 \ \text{ for all } \ x, y, v \in I. \tag{27}$$

Since $I$ is noncommutative (Remark 2.4 (a)), therefore identity (27) together with the primeness of $I$ (Remark 2.4(a)) implies that $g(y)y = 0$ for all $y \in I$. Then $g = 0$ on $R$ (Remark 2.4(b)).

We now prove that $f = 0$. Since $g = 0$, therefore by (20), $f(x)xy = 0$ for all $x, y \in I$. Replacing $y$ by $vy$, we get $f(x)xvy = 0$ for all $v \in I$. Since $I \neq (0)$ and $R$ is prime, we get $f(x)x = 0$ for all $x \in I$ and hence $f = 0$ on $R$. □

We conclude the paper with following remark.

**Remark 2.6**. A mapping $f$ of a ring $R$ is called *skew-commuting* on a subset $S$ of $R$ if for any $x \in S$, $f(x)x + xf(x) = 0$; it is called *semi-commuting* on $S$ if for any $x \in S$, either $f(x)x + xf(x) = 0$ or $f(x)x - xf(x) = 0$. If we substitute $y = x$ and $g = f$ in Corollary 2.3, then we have:

(a) Let $R$ be a noncommutative prime ring, $I$ a nonzero ideal of $R$ and $f$ a derivation of $R$ which is skew-commuting on $I$, then $f = 0$ on $R$.

If we substitute $y = x$ and $g = -f$ in Corollary 2.3, then by (a) above, we have:

(b) Let $R$ be a noncommutative prime ring, $I$ a nonzero ideal of $R$ and $f$ a derivation of $R$ which is semi-commuting on $I$, then $f = 0$ on $R$.

(c) We observe that (a) and (b) are analogs of Posner Theorem [6, Lemma 3] for skew-commuting and semi-commuting derivations, respectively.

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