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INVOLUTIVE SPACE CURVES

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Abstract: Let $C \subset \mathbf{CP}^3$ be a closed integral curve, which is involutive with respect to the symplectic form $dx_0 \wedge dx_1 + dx_2 \wedge dx_3$ and TC its tangent sheaf. Let s be the minimal degree of a surface of $\mathbf{CP}^3$ containing C. Here we give a proof that $h^0(C, TC(s-2)) \neq 0$, $\deg(TC) \geq d(2-s)$ and $\deg(TC) = d(2-s)$ if and only if $TC \cong \mathcal{O}_C(2-s)$. Assume that C is smooth. If $\omega_C \not\cong \mathcal{O}_C(s-2)$ and $(s-2)h^1(\mathbf{CP}^3, \mathcal{I}_C(s-1)) \geq (s-1)h^1(\mathbf{CP}^3, \mathcal{I}_C(s-2))$ (resp. either $h^1(\mathbf{CP}^3, \mathcal{I}_C(s-1)) = h^1(\mathbf{CP}^3, \mathcal{I}_C(s-1)) = 0$, or $(s-2)h^1(\mathbf{CP}^3, \mathcal{I}_C(s-1)) > s(h^1(\mathbf{CP}^3, \mathcal{I}_C(s-2))))$, then $s \leq 4$ (resp. $s \leq 3$).

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1. Introduction

Let $\pi : \mathbf{C}^4 \backslash \{0\} \rightarrow \mathbf{CP}^3$ be the quotient map and $C \subset \mathbf{CP}^3$ a closed integral subcurve. C is said to be *involutive* if $\pi^{-1}(C)$ is involutive with respect to the symplectic structure on $\mathbf{C}^4$ given by the form $dx_0 \wedge dx_1 + dx_2 \wedge dx_3$ (see [2]). By [2], Proposition 1, C is involutive if and only if for any homogeneous polynomial $F(x_0, x_1, x_2, x_3)$ vanishing on C. The Hamiltonian vector field $\mathbf{h}_F := (\partial F/\partial x_1)\partial_{x_0} - (\partial F/\partial x_0)\partial_{x_1} + (\partial F/\partial x_3)\partial_{x_2} - (\partial F/\partial x_2)\partial_{x_3}$ is tangent to C at each of its smooth points.

Let $s(C)$ be the minimal degree of an integral surface of $\mathbf{CP}^3$ containing C. Let $\mathcal{I}_C$ be the ideal sheaf of C in $\mathbf{CP}^3$. The coherent sheaf $\mathcal{I}_C/\mathcal{I}_C^2$ is an

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$\mathcal{O}_C$-sheaf and its dual $N_C := \text{Hom}\,(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ is called the normal sheaf of C in $\mathbf{CP}^3$. $N_C|C_{reg}$ is locally free of rank two. Since N_C is defined as a dual of an $\mathcal{O}_C$-sheaf, it is a torsion free rank two $\mathcal{O}_C$-sheaf. Let TC be the tangent sheaf of C, i.e. the dual of the cotangent sheaf Ω^1_C. Since TC is the dual of a sheaf, it is torsion free. The natural map $i : TC \to T\mathbf{CP}^3|C$ is injective, because it is injective on C_{reg} and TC is torsion free. We have an exact sequence of coherent sheaves on C:

$$0 \longrightarrow TC \xrightarrow{i} T\mathbf{CP}^3|C \xrightarrow{j} N_C \tag{1}$$

In Section 2 we will use the proofs and the ideas contained in [2] to prove the following result.

Theorem 1. *Let $C \subset \mathbf{CP}^3$ be an integral degree d involutive curve. Then $h^0(C, TC(s-2)) \neq 0$, $\deg(TC) \geq d(2-s)$ and $\deg(TC) = d(2-s)$ if and only if $TC \cong \mathcal{O}_C(2-s)$.*

The inequality $h^0(C, TC(s-2)) \neq 0$ is just Lemma 1, see [2].

Remark 1. Let C be a reduced curve (even not complete) and $P \in \text{Sing}(C)$. By [1], line 2 at p. 244, there is a map $c_C : \Omega^1_C \to \omega_C$ and hence a dual map $b_C : \omega^*_C \to TC$, which is an isomorphism at each smooth point of C. Since ω^*_C is torsion free, b_C is injective as a map of sheaves. Hence $\text{Coker}(b_C)$ is a skyscraper sheaf supported by $\text{Sing}(C)$. Let $\alpha(C,P)$ be the length of the part of $\text{Coker}(b_C)$ supported by P. If C has an ordinary node at P, then $\alpha(C,P) = 1$ (use that the part of $\text{Coker}(b_C)$ supported by P has length one by [1], Lemma 1.1.2). If C is complete we have $\deg(\omega_C) = 2g(C) - 2$.

Following [2] in Section 2 we will also prove the following result.

Theorem 2. *Let $C \subset \mathbf{CP}^3$ be a smooth and connected involutive curve. Set $s := s(C)$.*

(a) If $\omega_C \not\cong \mathcal{O}_C(s-2)$ and

$$(s-2)h^1(\mathbf{CP}^3, \mathcal{I}_C(s-1)) \geq (s-1)(h^1(\mathbf{CP}^3, \mathcal{I}_C(s-2))),$$

then $s \leq 4$.

(b) If either

$$h^1(\mathbf{CP}^3, \mathcal{I}_C(s-1)) = h^1(\mathbf{CP}^3, \mathcal{I}_C(s-1)) = 0$$

or

$$(s-2)h^1(\mathbf{P}^3, \mathcal{I}_C(s-1)) > s(h^1(\mathbf{CP}^3, \mathcal{I}_C(s-2))),$$

then $s \leq 3$.

Every involutive curve contained in a plane is a line (see [2], Proposition 2). Every smooth involutive curve contained in a quadric surface is rational (see [2], Proposition 3).

2. The Proofs

Proof of Theorem 1. By the very definition of the integer $s = s(C)$ there is a degree s non-zero homogeneous polynomial F vanishing on C. Since the curve C is involutive, the image in $N_C(s-2)$ of the restriction to C of the Hamiltonian vector field $\mathbf{h}_F$ vanishes. Thus, $\mathbf{h}_F$ induces $j(F) \in H^0(C, TC(s-2))$. If $j(F) \neq 0$, then $\deg(TC(s-2)) \geq 0$ and $\deg(TC) = d(2-s)$ if and only if $TC \cong \mathcal{O}_C(2-s)$, i.e. all the assertions of Theorem 1 are true in this case. Hence to prove Theorem 1 it is sufficient to assume $j(F) = 0$ and obtain a contradiction. We have $j(F) = 0$ if and only if $\mathbf{h}_F$ vanishes on C, i.e. if and only if it induces a non-zero section $i(F)$ of $H^0(\mathbf{CP}^3, \mathcal{I}_C \otimes T\mathbf{CP}^3(s-2))$. Set $A(F) = x_0\partial(F)/\partial x_3 - x_2\partial(F)\partial x_1$. By [2], Corollary at p. 155, the zero-locus of $\mathbf{h}_F$ is contained in the zero-locus of $A(F)$. First assume $A(F) \equiv 0$, i.e.

$$x_0\partial(F)/\partial x_3 = x_2\partial(F)\partial x_1\,. \tag{2}$$

If F depends from x_3 or from x_1, then the right hand side and the left hand side of (2) have different degrees with respect to x_3 or x_1, contradiction. Thus, F depends only from the variables x_0 and x_2. Thus $A(F)$ is the product of s linear forms in the variables x_0 and x_2. Since C is irreducible, we obtain $s = 1$. By [2], Proposition 2, C is a line and Theorem 1 is true in this case. Now assume that $A(F)$ is a non-zero degree s polynomial vanishing on C. We may assume that F contains at least one of the variables x_3 or x_1, say x_1. Let $k > 0$ be the maximal exponent of x_1 in a monomial occuring in F and call $D(x_0, x_2, x_3)x_1^k$ be the sum of the monomials of F containing x_1^k. Choosing a different F we may also assume that every non-zero polynomial of degree $s(C)$ vanishing on C has degree at least k with respect to x_1. Let $m \geq 0$ be the degree of $D(x_0, x_2, x_3)$ with respect to x_3. If $m = 0$, then $A(F)$ contains no monomial of degree at least k in x_1, contradiction. If $m > 0$, then every monomial containing x_1^k contains x_3 at most with power $m-1$. Iterating the action of the differential operator A at most $m-1$ times we obtain a contradiction. □

Proof of Theorem 2. Set $d = p_a(C)$ and $g = g(C)$. By the smoothness of C and Theorem 1 we have $2g-2 \leq d(s-2)$. Hence $h^1(C, \mathcal{O}_C(s-2)) \leq 1$ and $h^1(C, \mathcal{O}_C(s-2)) = 1$ if and only if $\omega_C \cong \mathcal{O}_C(s-2)$ (Serre duality). First assume

$h^1(C, \mathcal{O}_C(s-2)) = 0$. Set $\alpha = h^1(\mathbf{CP}^3, \mathcal{I}_C(s-1))$ and $\beta = h^1(\mathbf{CP}^3, \mathcal{I}_C(s-2))$. By Riemann-Roch we obtain:

$$(s-2)d + 1 - g = \binom{s+1}{3} + \beta\,, \tag{3}$$

$$(s-1)d + 1 - g = \binom{s+2}{3} + \alpha\,. \tag{4}$$

Subtracting (3) from (4) we obtain:

$$d = (s+1)s/2 + \alpha - \beta\,, \tag{5}$$

and hence by (4)

$$2(g-1) = (s+1)s(2s-5)/3 + 2((s-2)\alpha - (s-1)\beta)\,, \tag{6}$$

contradicting (5) and the inequality $2g-2 \le d(s-2)$ if either $s \ge 4$, $(s-2)\alpha > s\beta$ or $s \ge 4$ and $\alpha = \beta = 0$ or $s > 4$ and $(s-2)\alpha \ge s\beta$. Now assume $\omega_C \cong \mathcal{O}_C(s-2)$. Hence $h^1(C, \mathcal{O}_C(s-1)) = 0$. The only difference is that in the right hand side of (3) we must add -1, while (4) is still true. Since $d(s-2) = 2g-2$ in this case, we conclude by our assumptions on α and β. □

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