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GENERIC FAMILIES OF GENERALIZED LINEAR SYSTEMS

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Abstract: V.I. Arnold constructed smooth generic families of matrices with respect to similarity transformations depending smoothly on the entries of matrices with a small number of parameters. We solve the same problem for a family of triples of matrices (E, A, B), $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, representing generalized linear systems in the form $E\dot{x} = Ax + Bu$, with respect to feedback and derivative feedback equivalence.

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1. Introduction

We consider the space $\mathcal{M}$ of triples of matrices (E, A, B), where $E, A \in M_n(\mathbb{C})$, and $B \in M_{n\times m}(\mathbb{C})$ corresponding to a generalized time-invariant linear systems

$$E\dot{x} = Ax + Bu\,. \tag{1}$$

Many equivalence relations between triples of matrices are of interest in Con-

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trol Theory. Here we consider the equivalence relation, which generalizes the feedback equivalence between pairs of matrices.

Generalized systems of the form (1) arise naturally in a variety of circumstances, e.g. they are used in modelling of mechanical multibody systems (see Simeon et al [5]) and electrical circuits (see Günter et al [4]).

In this paper we consider families of triples of matrices smoothly depending on parameters. We can reduce each triple to canonical reduced form, but then we lose the smoothness relative to the parameters. It leads to the problem of reducing to normal form by a smoothly depending on parameters change of bases not only the matrices of the given triple, but of an arbitrary family of triples close to it.

2. State Feedback and Derivative Feedback Equivalence for Generalized Linear Systems

In $\mathcal{M}$ we consider the equivalence relation that generalizes in the natural way the feedback relation between pairs of matrices, defined in the following manner.

Definition 2.1. Two systems (E^i, A^i, B^i), $i = 1, 2$, are equivalent, if and only if, there exist matrices $P, Q \in Gl(n;\mathbb{C})$, $R \in Gl(m;\mathbb{C})$, and $U, V \in M_{m\times n}(\mathbb{C})$ such that

$$\begin{aligned} E_2 &= QE_1P + QB_1V, \\ A_2 &= QA_1P + QB_1U, \\ B_2 &= QB_1R, \end{aligned} \tag{2}$$

or written in a matrix form as follows:

$$(E_2, A_2, B_2) = Q(E_1, A_1, B_1)\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ V & U & R \end{pmatrix} \tag{3}$$

(here, we identify a triple (E, A, B) with the block matrix $\begin{pmatrix} E & A & B \end{pmatrix}$).

This relation is a generalization of the equivalence defined over standard systems

$$\begin{aligned} A_2 &= QA_1P + QB_1U, \\ B_2 &= QB_1 \end{aligned} \tag{4}$$

That is to say, the transformations permitted over generalized linear systems are basis change in the state space $x = Px_1$, in the input space $u = Ru_1$, feedback and derivative feedback $u = u_1 + Ux + V\dot{x}$, and pre-multiplication by a invertible matrix Q.

In the set of equivalent triple to a given a triple of matrices $(E, A, B) \in \mathcal{M}$ we can consider a triple in a simpler form in our case we can choose (see García-Planas et al ([3], Theorem 2.1)), the following reduced form

$$(E_c, A_c, B_c) = \left(\left(\begin{smallmatrix} 0 \\ M \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 \\ N \end{smallmatrix} \right), \left(\begin{smallmatrix} B_1 \\ 0 \end{smallmatrix} \right) \right),$$

where $B_1 = \begin{pmatrix} I_r & 0 \end{pmatrix}$, $r = \operatorname{rank} B$, and $M + \lambda N$ a $(n-r) \times n$-matrix pencil in its Kronecker reduced form.

In order to study families of triples can be useful to see the equivalence relation as induced by the action of a Lie group. Thus, let us consider in $\mathcal{G} = Gl(n; \mathbb{C}) \times Gl(n; \mathbb{C}) \times Gl(m; \mathbb{C}) \times M_{n\times m}(\mathbb{C}) \times M_{n\times m}(C)$ the group structure defined by

$$\begin{aligned} &(P_1, Q_1, R_1, U_1, V1) \circ (P_2, Q_2, R_2, U_2, V_2) \\ &\qquad = (P_2P_1, Q_1Q_2, R_2R_1, U_2P_1 + R_2U_1, V_2P_1 + R_2V_1). \end{aligned} \tag{5}$$

Obviously, it is a Lie group.

Now, we define an action α of $\mathcal{G}$ over $\mathcal{M}$ in the following manner.

Definition 2.2.

$$\begin{aligned} \alpha : \mathcal{G} \times \mathcal{M} &\longrightarrow \mathcal{M}, \\ ((P, Q, R, U, V), (E, A, B)) &\longrightarrow Q(E, A, B) \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ V & U & R \end{pmatrix}. \end{aligned} \tag{6}$$

Hence, we have the following proposition.

Proposition 2.1. *Two triples $(E_1, A_1, B_1), (E_2, A_2, B_2) \in \mathcal{M}$ are equivalent if and only if*

$$\alpha(P, Q, R, U, V), (E_1, A_1, B_1) = (E_2, A_2, B_2), \tag{7}$$

for some $(P, Q, R, U, V) \in \mathcal{G}$.

As a consequence, the set of equivalent triples to (E, A, B) is a orbit with regard to action α, which we denote this manifold by $\mathcal{O}(E, A, B)$.

3. Tangent and Normal Space to the Orbit

In order to study the local structure of the orbit of a given triple (E, A, B) we determine the tangent space to the orbit.

Proposition 3.1. *For any triple of matrices $(E, A, B) \in \mathcal{M}$ the tangent space $T_{(E,A,B)}\mathcal{O}(E, A, B)$ to its orbit $\mathcal{O}(E, A, B)$ at (E, A, B) is given by*

$$T_{(E,A,B)}\mathcal{O}(E, A, B) = \big\{(X, Y, Z) : \\ X = EP + QE + BV, Y = AP + QA + BU, Z = BR + QB\big\} \quad (8)$$

$\forall P, Q \in M_n(\mathbb{C}), R \in M_m(\mathbb{C}),\ U, V \in M_{m\times n}(\mathbb{C})$.

Proof. Let us consider the parametrization of $\mathcal{O}(E, A, B)$ induced by α

$$\begin{aligned} \alpha_{(E,A,B)} : \mathcal{G} &\longrightarrow \mathcal{M}\,, \\ (P, Q, R, U, V) &\longrightarrow Q(E, A, B)\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ V & U & R \end{pmatrix}. \end{aligned} \quad (9)$$

Hence, the tangent space to $\mathcal{O}(E, A, B)$ at (E, A, B) is the image of the differential of $\alpha_{(E,A,B)}$ at the unit element $I = (I_n, I_n, I_m, 0, 0)$ of $\mathcal{G}$, that is to say

$$T_{(E,A,B)}\mathcal{O}(E, A, B) = d\alpha_{(E,A,B)_I}(T_I\mathcal{G})\,, \quad (10)$$

where $T_I\mathcal{G}$ is the tangent space to $\mathcal{G}$ at I. Obviously, $T_I\mathcal{G} = M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times M_m(\mathbb{C}) \times M_{n\times m}(\mathbb{C}) \times M_{n\times m}(\mathbb{C})$.

Then, it is sufficient to compute $d\alpha_{(E,A,B)_I}(P, Q, U, V, R)$, where $(P, Q, R, U, V) \in T_I\mathcal{G}$. To do that, we compute

$$\alpha_{(E,A,B)}(I_n + \varepsilon P, I_n + \varepsilon Q, I_m + \varepsilon R, \varepsilon U, \varepsilon V)\,,$$

and to consider the linear approximation. □

In order to know the local structure of a triple (that is to say, to know the different structures that can be in a neighborhood of a triple), we compute the orthogonal space to the tangent space $T_{(E,A,B)}(E, A, B)$ with respect some Hermitian product previously defined in $\mathcal{M}$, (orthogonality in $\mathcal{M}$).

The inner product chosen for that, is the following:

Definition 3.1. Let $(E_1, A_1, B_1), (E_2, A_2, B_2) \in \mathcal{M}$, we define

$$< (E_1, A_1, B_1), (E_2, A_2, B_2) >= \mathrm{tr} E_1\overline{E}_2^t + \mathrm{tr} A_1\overline{A}_2^t + \mathrm{tr} B_1\overline{B}_2^t\,. \quad (11)$$

We have the following description of the orthogonal space to the tangent space $T_{(E,A,B)}\mathcal{O}(E,A,B)^{\perp}$.

Proposition 3.2. *We assume in $\mathcal{M}$ the inner product defined before. Then*

$$T_{(E,A,B)}\mathcal{O}(E,A,B)^{\perp} = \left\{ \begin{array}{r} (X,Y,Z) : \overline{X}^t E + \overline{Y}^t A = 0, E\overline{X}^t + A\overline{Y}^t + B\overline{Z}^t = 0 \\ \overline{X}^t B = 0, \overline{Y}^t B = 0, \overline{Z}^t B = 0 \end{array} \right\}.$$

4. Generic Families of Matrix Pencils

We study families of triples of matrices $(E(\lambda), A(\lambda), B(\lambda))$, $\lambda = (\lambda_1, \dots, \lambda_k)$, holomorphic at $\lambda = 0$. These are matrices, whose entries are convergent in the power series expansion of complex parameters $\lambda_1, \dots, \lambda_k$ in a neighborhood of 0 (the germ of a family $(E(\lambda), A(\lambda), B(\lambda)$ at 0 is called a *deformation* of the triple $(E,A,B) = (E(0), A(0), B(0))$, see Arnold [1]).

Two families $(E_1(\lambda), A_1(\lambda), B_1(\lambda))$ and $(E_2(\lambda), A_2(\lambda), B_2(\lambda))$ are called *equivalent* if there exist matrices $P(\lambda)$, $Q(\lambda)$, $R(\lambda)$, $U(\lambda)$ and $V(\lambda)$ holomorphic at 0 such that $P(0) = I_n$, $Q(0) = I_n$, $R(0) = I_m$, $U(0) = 0$, $V(0) = 0$ and

$$(E_2(\lambda), A(\lambda_2), B_2(\lambda)) = Q(\lambda)(E_1(\lambda), A_1(\lambda), B_1(\lambda)) \begin{pmatrix} P(\lambda) & 0 & 0 \\ 0 & P(\lambda) & 0 \\ V(\lambda) & U(\lambda) & R(\lambda) \end{pmatrix},$$

in a neighborhood of 0.

A family $(E(\lambda), A(\lambda), B(\lambda))$ is called *versal* if every family $E(\mu), A(\mu), B(\mu))$ with $(E(0), A(0), B(0)) = (E,A,B)$ is equivalent to a family $(E(\varphi(\lambda), A(\varphi(\lambda),\ B(\varphi(\lambda)))$, where $\varphi(\lambda) = \mu$, $\varphi(0) = 0$, are power series convergent in a neighborhood of 0. A versal family with the minimum possible number k of parameters is said to be *miniversal*.

For every triple of matrices (E,A,B), a miniversal family $(E(\lambda), A(\lambda), B(\lambda))$ with $(E(0), A(0), B(0)) = (E,A,B)$ can be obtained using the following result proved by Arnold for the square matrices under similarity.

Proposition 4.1. *A holomorphic family* $(E(\lambda), A(\lambda), B)\lambda)$ *with* $(E(0), A(0), B(0)) = (E, A, B)$ *is versal if and only if it is transversal to the orbit* $\mathcal{O}(E, A, B)$ *at* (E, A, B).

In particular, we have the following corollary.

Corollary 4.1. *Let us fixed any inner product in* $\mathcal{M}$*, (for example the inner product given in (3.1)). Then, for any triple of matrices* $(E, A, B) \in \mathcal{M}$ *the linear variety*

$$(E, A, B) + T_{(E,A,B)}(\mathcal{O}(E, A, B))^{\perp} \tag{12}$$

is a miniversal family for (E, A, B).

Remark. It suffices to construct a miniversal family for a canonical reduced triple.

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