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CONTROLLABILITY OF ℓ-ORDER LINEAR SYSTEMS

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Abstract: Given a $(\ell+1)$-ple of matrices $(A_{\ell-1},\dots,A_0,B)$ representing ℓ-order time-invariant linear systems, $x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \dots + A_0x^{(0)} + Bu$, we analyze conditions in such a way that changing the control u by $u_1 = u - F_\ell x^{(\ell)} - \dots F_0x^{(0)}$ the system obtained has a stable solution.

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1. Introduction

We consider the space of $(\ell+1)$-ple of matrices $(A_{(\ell-1)},\dots,A_0,B)$, where $A_{(\ell-1)},\dots,A_0 \in M_n(\mathbb{C})$, and $B \in M_{n\times m}(\mathbb{C})$ corresponding to a ℓ-order time-invariant linear systems

$$x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \dots + A_0x^{(0)} + Bu \tag{1}$$

($x^{(i)}$ denotes the i-th-derivative).

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It is well known the great interest in the study of the linear systems in control theory. Recently, the interest is also focused in generalized linear systems in the form $Ex^1 = Ax + Bu$, and high order linear systems as (1), (see Antoniou [1], and García-Planas [2]). All of them are applied in engineering systems as well as economical and biological systems (see Marszalek et al [4], for example).

One of methods to study ℓ-order linear systems is linearizing the system (see Gohberg et al [3], for example), obtaining a linear system in the following manner. Given a ℓ-order linear system $x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)} + Bu$ or simply write $(A_{\ell-1}, \ldots, A_0, B)$, and calling

$$X = \begin{pmatrix} x^{(0)} \\ x^{(1)} \\ \vdots \\ x^{(\ell-1)} \end{pmatrix},$$

we have

$$X^{(1)} = \mathbf{A}X + \mathbf{B}u\,, \tag{2}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & I_n & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I_n \\ A_0 & A_1 & \ldots & A_{\ell-1} \end{pmatrix} \in M_{\ell n}(\mathbb{C}),\ \mathbf{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ B \end{pmatrix} \in M_{\ell n \times m}(\mathbb{C})\,. \tag{3}$$

In this paper using linearization method and state space approach, we obtain necessary and sufficient conditions to ensure the existence of stable solutions for the ℓ-order system.

2. State Feedback Equivalence in ℓ-order Linear Systems

Given a linearized ℓ-order linear system it can be interesting to consider a feedback equivalent linear system in the form (3), for that we need to restrict the feedback group to the subgroup formed by $(\ell + 2)$-ple of matrices $(P, Q, F_0, \ldots, F_{\ell-1})$ $P \in Gl(n; \mathbb{C})$, $Q \in Gl(m; \mathbb{C})$, and $F_i \in M_{m\times n}(\mathbb{C})$ acting over the space of this kind of systems in the following manner

Definition 2.1. Two systems $(A^i_{\ell-1}, \ldots, A^i_0, B^i)$, $i = 1, 2$, are equivalent, if and only if, the exist matrices $P \in Gl(n; \mathbb{C})$, $Q \in Gl(m; \mathbb{C})$, and

$F_i \in M_{m\times n}(\mathbb{C})$ such that

$$\begin{pmatrix} 0 & I_n & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & & & & \dots \\ 0 & 0 & \dots & I_n & 0 \\ A_0^2 & A_1^2 & \dots & A_{\ell-1}^2 & B^2 \end{pmatrix} = \begin{pmatrix} P^{-1} & & 0 \\ & \ddots & \\ 0 & & P^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & I_n & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & & & \\ \dots & & & & \dots \\ 0 & 0 & \dots & I_n & 0 \\ A_0^1 & A_1^1 & \dots & A_{\ell-1}^1 & B^1 \end{pmatrix} \begin{pmatrix} P & & 0 & 0 \\ & \ddots & & \\ 0 & & P & 0 \\ F_0 & \dots & F_{\ell-1} & Q \end{pmatrix}. \quad (4)$$

That is to say, the transformations permitted over ℓ-order standard systems are basis change in the state space $x = Px_1$, in the input space $u = Qu_1$ and i-order derivative feedback $(i = 0, \dots, \ell-1)$ $u = u_1 + F_0x^{(0)} + \dots + F_{\ell-1}x^{(\ell-1)}$.

From the above definition, we have the following proposition.

Proposition 2.1. *Let $x^{(\ell)} = A^i_{\ell-1}x^{(\ell-1)} + \dots + A^i_0x^{(0)} + B^iu$ $i = 1, 2$ two equivalent ℓ-order standard linear systems. Then the linearized systems are feedback equivalent.*

Notice that if $(A^i_{\ell-1}, \dots, A^i_0, B^i)$ $i = 1, 2$ are two equivalent systems, then each one of the pairs of matrices $(A^1_{\ell-1}, B^1)$, …, (A^1_0, B^1) is feedback equivalent to the pair $(A^2_{\ell-1}, B^2)$, …, (A^2_0, B^2) respectively. Then, and if it is necessary we can take systems $(A_{\ell-1}, \dots, A_0, B)$, where one of the pairs $(A_{\ell-1}, B)$, … or (A_0, B) is in a canonical reduced form (Kronecker reduced form, for example).

3. Controllability

We can apply controllability results of linear systems to analyze the existence of stable solutions for ℓ-order linear systems. For that we introduce the concept of controllability of ℓ-order linear systems.

Definition 3.1. The ℓ-order linear system (1.1) is controllable if and only if there exists a control $u_1 = u - F_{\ell-1}x^{(\ell-1)} - \dots - F_0x^{(0)}$, with $F_i \in M_{m\times n}(\mathbb{C})$ such that the equation

$$x^\ell = (A_{\ell-1} + BF_{\ell-1})x^{\ell-1} - \dots - (A_0 + BF_0)x^{(0)} \quad (5)$$

has a stable solution.

Taking into account that $X_1 = \begin{pmatrix} x_1^{(0)} \\ \vdots \\ x_1^{(\ell-1)} \end{pmatrix}$ is a solution of the linear system associated $X^{(1)} = \mathbf{A}X + \mathbf{B}u$, if and only if $x_1^{(0)}$ is a solution of the equation (1.1), it is not difficult to prove the following proposition.

Proposition 3.1. *The ℓ-order linear system (1.1) is controllable if and only if the linear system associated (1.2) is controllable.*

Proof. The controllability of $X^{(1)} = \mathbf{A}X + \mathbf{B}u$ ensures the existence of $\mathbf{F} \in M_{m\times \ell n}(\mathbb{C})$ such that $X^{(1)} = \mathbf{A}X + \mathbf{B}u_1$ with $u_1 = u - \mathbf{F}X$ has a stable solution. Partitioning the matrix $\mathbf{F}$ into ℓ-blocks $\mathbf{F} = \begin{pmatrix} F_0 & \dots & F_{\ell-1} \end{pmatrix}$ we have that the equation $x^\ell = A_{\ell-1}x^{(\ell-1)} + \dots + A_0x^{(0)} + Bu_1$ with $u_1 = u - F_0x^{(0)} - \dots - F_{\ell-1}x^{\ell-1}$ has a stable solution. Converse is analogous. □

Now we present the main result that permit us to analyze the controllability character directly from the initial equation (1.1).

Theorem 3.1. *The ℓ-order linear (1.1) is controllable if and only if,*

$$\operatorname{rank} \begin{pmatrix} s^\ell I - s^{\ell-1}A_{\ell-1} - \dots - sA_1 - A_0 & B \end{pmatrix} = n. \tag{6}$$

for all $s \in \mathbb{C}$.

Proof. It is well known that the linear system (2.4) is controllable if and only if

$$\operatorname{rank} \begin{pmatrix} sI_{\ell n} - \mathbf{A} & \mathbf{B} \end{pmatrix} = \ell \cdot n.$$

So, making row and column elementary transformations we obtain

$$\ell \cdot n = \operatorname{rank} \begin{pmatrix} 0 & I_n & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & & & & \dots \\ 0 & 0 & \dots & I_n & 0 \\ A_0 & A_1 & \dots & A_{\ell-1} & B \end{pmatrix}$$

$$= \operatorname{rank} \begin{pmatrix} 0 & I_n & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & & & & \dots \\ 0 & 0 & \dots & I_n & 0 \\ A_0 + sA_1 + \dots + s^{\ell-1}A_{\ell-1} - s^\ell I & 0 & 0 & 0 & B \end{pmatrix}$$

$$= (\ell - 1)n + \operatorname{rank} \begin{pmatrix} s^\ell I - s^{\ell-1}A_{\ell-1} - \dots - sA_1 - A_0 & B \end{pmatrix}. \qquad \square$$

For some particular cases we have the following results for second-order linear systems

Example 3.1. (1) Case $A_1 = 0$. The linearized system is controllable if and only if the pair of matrices (A_0, B) is controllable.

(2.) Case $A_0 = 0$. It is not difficult to prove that the linearized system is controllable if and only if $n \geq m$ and the matrix B has full rank.

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