International Journal of Pure and Applied Mathematics
Volume 6 No. 2 2003, 159-171

ODD-SYMPLECTIC GROUP IN FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

L. Sbano
Department of Mathematics and Statistic
University of Surrey
Guildford GU2 7XH, UNITED KINGDOM
e-mails: L.Sbano@eim.surrey.ac.uk, Sbano@maths.warwick.ac.uk

Abstract: In this paper it is shown that the characteristic vector field associated to the first order PDE:

$$h(x, \nabla z(x)) = 0 \text{ with } x \in \mathbb{R}^n$$

has the same form of an infinitesimal generator of an odd-symplectic transformation with contact Hamiltonian $h(x,p)$ on the level set $h = 0$. We also study under which condition such PDE has a characteristic vector field commuting with a generator of an action of the odd-symplectic group.

AMS Subject Classification: 53D05, 53D10, 35F05
Key Words: symplectic geometry, contact geometry, first order PDE, theory of charcterics

1. Introduction

Recently R. Cushman [4] (see also [3]) presented the notion of *odd-symplectic group* in the context of geodesic flow. The odd symplectic group we use was defined in [6] but the same name was used for different groups in [7]. Loosely speaking, the odd-symplectic group lies between the standard symplectic group and the contact group. More precisely, it interpolates symplectic groups in different dimensions.

Received: January 10, 2003

Although the definition of the odd-symplectic group is mathematically natural, at the moment, there are few examples of where it arises. The aim of this paper is to show that odd-symplectic transformations rise naturally in the geometric theory of first order PDE. In particular, in the study of the characteristics of PDEs of the following form:

$$h(x, \nabla z(x)) = 0. \tag{1}$$

Here $x \in \mathbb{R}^n$ $h(.,.)$ is a smooth function on $\mathbb{R}^{2n}$.

The author is aware that the novelty of this paper relies in combining several well-known facts and looking at them from a different point of view.

Here is an outline of the paper. In Section 2 the odd-symplectic group is defined. In the following section we recall the theory of characteristics for first order PDEs and we show that the characteristic vector field associated to (1) coincides with an infinitesimal generator of an odd-symplectic transformation. Also we show that

$$h(x, \nabla z(x)) = \frac{1}{2}\langle g\nabla z, \nabla z\rangle + \langle v, \nabla z\rangle + k$$

with g constant $n \times n$ symmetric matrix, $k \in \mathbb{R}$ and $v \in \mathbb{R}^n$ has a symmetry, whose infinitesimal generator is given by an element in the Lie algebra of the odd-symplectic group. Finally, we exhibit that there is an isomorphism between certain class of first order PDE and the Lie algebra of the odd-symplectic group. Moreover, we show the connection of the odd-symplectic group with Hamilton-Jacobi theory and the Eikonal equation.

2. Odd-Symplectic Group and Contact Geometry

Let us consider a real symplectic vector space (V, ω), of dimension $2n + 2$. The even symplectic group is

$$Sp(V, \omega) \doteq \{A \in Gl(V) | A^*\omega = \omega\}. \tag{2}$$

The odd-symplectic group $Sp(V, \omega)_{v_0}$ can be defined as follows. Let $v_0 \in V$ be a non-zero vector, then:

$$Sp(V, \omega)_{v_0} \doteq \{A \in Sp(V, \omega) | Av_0 = v_0\}. \tag{3}$$

Sometimes $Sp(V, \omega)_{v_0}$ is denoted by $Sp(2n+1, \mathbb{R})$. This group has been studied in [3, 5]. It turns out that the odd symplectic group arises naturally in contact geometry. Let us follow [3]. Consider a contact manifold $(\mathcal{M}, \theta)$ ($\dim \mathcal{M} = 2n+1$), where locally, in Darboux coordinates, the contact form θ is:

$$\theta = dz - y_i dx^i \,. \tag{4}$$

In [1] it is shown that one can symplectify $(\mathcal{M}, \theta)$ to $(\mathcal{N}, \omega) = (\mathbb{R}_+ \times M, d(t\theta))$, where $t > 0$ is the coordinate on $\mathbb{R}_+$. Consider a one parameter group of symplectomorphisms $F_s : \mathcal{N} \to \mathcal{N}$ of the form:

$$F_s(t, m) = (t, G_s(t, m)). \tag{5}$$

As shown in [3] the infinitesimal generator of F_s at $t = 1, y = 0$ is an element of the Lie algebra $sp(2n+1, \mathbb{R})$ of $Sp(2n+1, \mathbb{R})$. If X is vector field on $\mathcal{N}$ such that

$$L_X(t) = 0 \text{ and } L_X(d(t\theta)) = 0\,, \tag{6}$$

then its flow is a map of the form (5). Since

$$L_X(d(t\theta)) = d(i_X(d(t\theta))) = 0\,.$$

Locally on $\mathcal{N}$ there exists a function $\widehat{h} : \mathcal{N} \to \mathbb{R}$ such that:

$$i_X d(t\theta) = d\widehat{h}\,. \tag{7}$$

To see what this means, let us use Darboux coordinates. If we denote

$$X_{\widehat{h}} = \frac{dt}{ds}\frac{\partial}{\partial t} + \frac{dz}{ds}\frac{\partial}{\partial z} + \frac{dx^i}{ds}\frac{\partial}{\partial x^i} + \frac{dy_i}{ds}\frac{\partial}{\partial y_i}\,,$$

then (6) and (7) give

$$\begin{cases} \dfrac{dt}{ds} = 0, \\ \dfrac{dx^i}{ds} = \dfrac{1}{t}\dfrac{\partial \widehat{h}}{\partial y_i}, \\ \dfrac{dy_i}{ds} = -\dfrac{1}{t}\dfrac{\partial \widehat{h}}{\partial x^i}, \\ \dfrac{dz}{ds} = -\dfrac{\partial \widehat{h}}{\partial t} + \dfrac{y_i}{t}\dfrac{\partial \widehat{h}}{\partial y_i}, \end{cases} \tag{8}$$

with the condition:

$$\frac{\partial \widehat{h}}{\partial z} = 0. \tag{9}$$

This has the following several consequences:

1. Since $dt/ds = 0$, the flow preserves $t =$const. Moreover,
$$L_{X_{\widehat{h}}}(\widehat{h}) = 0,$$
the flow of the vector field $X_{\widehat{h}}$ preserves the Hamiltonian function $\widehat{h}$.

2. For $t = 1$ and $\widehat{h} = t \cdot h(x, y)$ one recovers the contact vector field Y_h (defined in [3]), whose tangetial curves satisfy:
$$\begin{cases} \dfrac{dx^i}{ds} = \dfrac{\partial h}{\partial y_i}, \\ \dfrac{dy_i}{ds} = -\dfrac{\partial h}{\partial x^i}, \\ \dfrac{dz}{ds} = -h + y_i \dfrac{\partial h}{\partial y_i}. \end{cases} \tag{10}$$

3. The contactomorphisms generated by flows of vector fields Y_h form a subgroup of the group of contact transformation. In [2] V.Arnol'd defined a contact Hamiltonian vector field without requiring that the associated one parameter transformation has the form (5). Arnol'd's definition leads to a vector field that generates a contact transformation but does not preserve the contact Hamiltonian $\widehat{h}$.

4. Consider $\widehat{h}_u : \mathcal{N} \to \mathbb{R}$ such that:
$$\widehat{h}_u = \frac{1}{2}[-\langle \gamma x, x\rangle + 2t\langle \alpha x, y\rangle + \\ +t^2\langle \beta x, y\rangle] + \langle v_1, y\rangle - \langle v_2, x\rangle \frac{1}{t} - kt\,, \tag{11}$$
where α, β, γ are $n \times n$ real matrices with $\beta^T = \beta$ and $\gamma^T = \gamma$. One verifies that the vector field $X_{\widehat{h}_u}$ on $\{t = 1\}$ is the infinitesimal generator of the one parameter group $s \to \exp(s\ u)$ in $\mathbb{R} \to \mathbb{R}^{2n+2}$, where $u \in \mathrm{sp}(2n+1, \mathbb{R})$. In other coordinates,
$$X_{\widehat{h}_u}\Big|_{t=1} = \frac{d}{ds}\Big|_{s=0,t=1} \exp(su) \cdot (t, x, y, z). \tag{12}$$
Here
$$u = \begin{pmatrix} 0 & 0 & 0 \\ v & A & 0 \\ k & Jv & 0 \end{pmatrix}, \tag{13}$$

where J is the standard symplectic structure in $\mathbb{R}^{2n}$, $v \in \mathbb{R}^{2n}$, $k \in \mathbb{R}$ and $A \in \mathrm{sp}(2n, \mathbb{R})$ such that

$$JA = \begin{pmatrix} -\gamma & \alpha \\ \alpha^T & \beta \end{pmatrix}. \tag{14}$$

Putting $(x, y) = w$, a simple computation shows that on level set $\{t = 1\}$:

$$X_{\widehat{h}_u}\Big|_{t=1} = (v + Aw)^T \frac{\partial}{\partial w} + (k + (Jv)^T w) \frac{\partial}{\partial z}\,. \tag{15}$$

3. Theory of Characteristics for First Order PDE and Odd-Symplectic Vector Fields

In this section we shall follow the presentation of first order PDE given in [1]. Let us consider the case, in which the manifold $\mathcal{M}$ is the first jet bundle associated to $\mathbb{R}^n$. Locally, such space can be modelled by a $x \in R^n$ and a germ of a function in x considered up to its gradient. To each $m \in J^1(\mathbb{R}^n)$ we associate $(x, z(x), \nabla z)$ with z a germ of a smooth function in $\mathbb{R}^n$.

The manifold $J^1(R^n)$ is a contact manifold with contact 1-form

$$\theta = dz - y_i dx^i,$$

which vanishes on any germ $z(.)$ such that $\partial z/\partial x^i = y_i$. The first jet space is the natural place to study the geometry of first order PDEs. In fact any PDE can be understood as a submanifold of $J^1(R^n)$ (see [1]). Let us suppose that we have a smooth function on $J^1(R^n)$:

$$h : J^1(R^n) \to \mathbb{R} \text{ with } \frac{\partial h}{\partial z} = 0.$$

$\{h = 0\}$ defines a manifold in $J^1(R^n)$, which corresponds to the following PDE

$$h\,(x, z(x), \nabla z(x)) = 0\,. \tag{16}$$

To equation (16) we can associate a contact vector field ζ_c, whose integral curves satisfy (10). We pose the natural question whether ζ_c is related to the *characteristic* vector field, which converts a first order PDE into a system of ordinary differential equations. In order to answer this question we need to recall the theory of characteristics (see [1] for more details).

3.1. Theory of Characteristics for First Order PDE

Let us assume that we have a first order PDE defined by the level set of $h : J^1(R^n) \to \mathbb{R}$. In the first jet space $J^1(\mathbb{R}^n)$ we can introduce *contact plane* Π_x at any $x \in \mathbb{R}^n$ spanned by the tangent vectors V

$$i_V\theta = 0\,. \tag{17}$$

Equation (17) determines a co-dimension 1 submanifold E^{2n} of $J^1(\mathbb{R}^n)$. Its tangent bundle TE^{2n} is the set of V such that:

$$i_V dh = 0\,. \tag{18}$$

Definition 3.1. The submanifold E^{2n} is called non-characteristic for the PDE defined by $h = 0$, if TE^{2n} is transversal to the contact plane Π at any every point in E^{2n}.

Definition 3.2. The intersection of vector spaces

$$P_x = T_x E^{2n} \cap \Pi_x$$

is called characteristic plane at x. Therefore P is defined by all vectors V such that

$$i_V\theta = 0 \quad \text{and} \quad i_V dh = 0. \tag{19}$$

We shall consider only the case, in which E^{2n} is not characteristic. Note that since TE^{2n} and Π lie in $J^1(R^n)$ we have

$$\dim P_x = \dim TE^{2n}_x + \dim \Pi_x - \dim TJ^1(R^n) = 2n - 1\,.$$

The 2-form $d\theta$, associated to the contact form θ, defines a non-degenerate skew product on $TJ^1(R^n)$. This allows us to select a vector in each characteristic plane P_x, which is skew-orthogonal to P_x. The characteristic vector field for (16) is given by ζ_c:

1. $i_{\zeta_c}\theta = 0$, ζ_c is a contact vector field.
2. $d\theta(\zeta_c, V) = 0$ for all $V \in P$.

In symplectic geometry one can show that the skew-orthogonal complement to a k dimensional vector space is a $2n-k$ vector space. The characteristic plane P has dimension $2n - 1$. Therefore the characteristic direction ζ_c is uniquely determined.

For a given $h : J^1(R^n) \to \mathbb{R}$, in local coordinates the vector field ζ_c has components:

$$\begin{cases} \dfrac{dx^i}{ds} = \dfrac{\partial h}{\partial y_i}, \\ \dfrac{dy_i}{ds} = -\dfrac{\partial h}{\partial x^i} - y_i \dfrac{\partial h}{\partial z}, \\ \dfrac{dz}{ds} = y_i \dfrac{\partial h}{\partial y_i}. \end{cases} \tag{20}$$

3.2. Cauchy Problem

The theory of characteristics is used to solve the Cauchy problem. For an equation

$$h(x, z(x), \nabla z(x)) = 0\,,$$

Cauchy data are: a manifold $\gamma \subset \mathbb{R}^n$ and the values $z_{|\gamma} = \phi(x)$. We have seen that the equation $h = 0$ corresponds to submanifold E^{2n} in $J^1(\mathbb{R}^n)$. Cauchy data define a submanifold $N \subset E^n$ called the *manifold of initial data*. It turns out that the Cauchy problem is solvable (locally) if N is not characteristic, that is, if the projection of the characteristic direction to $\mathbb{R}^n$ is transversal to the tangent plane to N (see [1]). Given a parametrization $x_\gamma(\lambda)$ of γ, it is possible to construct the family of initial data

$$x_0 = x_\gamma(\lambda), \quad z_0 = \phi(x_\gamma(\lambda)), \quad p_0 = p_0(\lambda)$$

where $p_0(\lambda)$ is such that:

$$h(x_\gamma(\lambda), \phi(x_\gamma(\lambda)), p_0(\lambda)) = 0.$$

This forms the family of initial data for the characteristic vector field (20).

Remark 3.1. The characteristic vector field (20) does not depend on whether the PDE is given by $h = 0$ or $h = c$ with $c \neq 0$, but the Cauchy problem does. In fact the value of the level set of h is contained in p_0.

4. The Main Result

We can state the following theorem.

Theorem 4.1. *For all the PDEs represented in $J^1(R^n)$ by a smooth function $h : J^1(R^n) \to \mathbb{R}$ of the form:*

$$h(x, \nabla z(x)) = 0,$$

the odd-symplectic Hamiltonian vector field (10) commutes with the characteristic vector field (20). Moreover, on $h = 0$, the two vector fields coincide.

Proof. Consider the function $h : J^1(\mathbb{R}^n) \to \mathbb{R}$ as a contact Hamiltonian. Then by inspection one finds that the infinitesimal generator Y_h can be written as follows:

$$Y_h = \zeta_c - h\frac{\partial}{\partial z}. \tag{21}$$

Since h does not depend on z, therefore $\zeta_c(h) = 0$. Also, the Lie bracket:

$$[\zeta_c, Y_h] = -\left[\zeta_c, h\frac{\partial}{\partial z}\right] = 0$$

vanishes.

Now we want to compare Y_h (10) and ζ_c (20). We notice that if $h : J^1(R^n) \to \mathbb{R}$ does not depend on z, the first two components are equal. The component dz/ds differs because of the term $-h$ in (10). We have

$$L_{Y_h}(h) = 0\,,$$

that is h is preserved by the flow of Y_h. Therefore in the manifold $\{h = 0\}$ the vector field Y_h is equal to the characteristic vector field ζ_c. □

4.1. First Order PDE and $sp(2n + 1, \mathbb{R})$

Let us consider a particular class of first order PDEs in $\mathbb{R}^n$, namely,

$$c_{ij}\frac{\partial z(x)}{\partial x^i}\frac{\partial z(x)}{\partial x^j} + b_{ij}\frac{\partial z(x)}{\partial x^i}x^j + a_{ij}x^i x^j + e_i\frac{\partial z(x)}{\partial x^i} + f_i x^i = h_0\,, \tag{22}$$

or more compactly:

$$\langle B\nabla z(x), \nabla z(x)\rangle + \langle e, \nabla z(x)\rangle + \langle f, x\rangle = h_0 \tag{23}$$

Then we have the following proposition.

Proposition 4.1. *For a given first order PDE of the form (23), if*

$$B \doteq \frac{1}{2}\begin{pmatrix} 2a & b \\ b^T & 2c \end{pmatrix}$$

(where $.^T$ is the transpose) is symmetric, then we can associate $u \in \mathrm{sp}(2n+1,\mathbb{R})$

$$u = \begin{pmatrix} 0 & 0 & 0 \\ v & A & 0 \\ h_0 & Jv & 0 \end{pmatrix}, \tag{24}$$

where $k = h_0$, $Jv = (e, f)$ and $-JB = A$, to the characteristic vector field of (22).

If $a = b = 0$ and $v = (v_1, 0)$ in B then the characteristic vector field X_c associated to (22) commutes u.

Proof. The proof is essentially based on Theorem 4.1 and on Remark 4. According to Theorem 4.1, equation (23) can be interpreted as a contact Hamiltonian $h(x, y)$, where $y = \nabla z(x)$, and the contact vector field on $h = 0$ is equal to the characteristic vector field. If the conditions on the matrices a, b, c are satisfied the PDE (23), on $J^1(\mathbb{R}^n)$ gives rise to a contact Hamiltonian of the form:

$$\widehat{h}(x, y, 1) = \langle w, JAw\rangle + \langle Jv, w\rangle + h_0, \text{ where } w = (x, y)$$

According to 4, we can associate to such Hamiltonian an element of $\mathrm{sp}(2n+1,\mathbb{R})$ of the form (24). Let us consider the Hamiltonian

$$\widehat{h}_u = \langle ax, x\rangle + 2t\langle bx, y\rangle + t^2\langle cx, y\rangle + \langle v_1, y\rangle - \langle v_2, x\rangle\frac{1}{t} - h_0 t \tag{25}$$

The corresponding infinitesimal generator $Y_{\widehat{h}_u}$ is:

$$Y_{\widehat{h}_u} = \xi + Z_{\widehat{h}}\frac{\partial}{\partial z}, \tag{26}$$

where

$$Z_{\widehat{h}} = \langle v_1, y\rangle - \langle v_2, x\rangle + h_0\,. \tag{27}$$

The characteristic vector field ζ_c of (23) is

$$\zeta_c = \xi + Z_c\frac{\partial}{\partial z}, \tag{28}$$

where

$$\xi = 2(bx + cy)\cdot\frac{\partial}{\partial x} - 2(ax + by)\cdot\frac{\partial}{\partial y} \tag{29}$$

and

$$Z_c = 2\langle bx, y\rangle + 2\langle cy, y\rangle + \langle v_1, y\rangle\,. \tag{30}$$

Since h does not depend on z, the commutator is given by:

$$[Y_{\widehat{h}}, \zeta_c] = (\xi(Z_c) - \xi(Z_{\widehat{h}}))\frac{\partial}{\partial z}\,.$$

After some simplification one finds

$$\begin{aligned}\xi(Z_c) - \xi(Z_{\widehat{h}})& \\ &= 4\langle bx, b^T y\rangle + 2\langle cy, b^T y\rangle + \langle bx, v_2\rangle + \langle cy, v_2\rangle \\ &\quad - 2\langle bx, ax\rangle - 2\langle bx, by\rangle - 4\langle ax, cy\rangle - 4\langle by, cy\rangle\,. \end{aligned} \tag{31}$$

Last expression is identically zero for every (x, y) only for $a = b = 0$ and $v_2 = 0$. □

4.2. Hamilton-Jacobi Theory

In certain cases in Hamiltonian mechanics the construction of coordinate transformations that allows to integrate explicitly the equations of motion is based on the solution of Hamilton-Jacobi equation. We refer the reader to [1]. Suppose that we have a Hamiltonian system on $(\mathbb{R}^{2n}, \omega)$. The Cartan one-form, written in local coordinates, is:

$$\Theta = p_i dq^i - H(p, q, t)dt\,, \tag{32}$$

where $H : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ in the Hamiltonian. Θ is an integral invariant for the Hamiltonian flow. Consider the extended phase space $\mathcal{M} = \mathbb{R}^{2n} \times \mathbb{R}$ and two sets of coordinates

$$(q, p, t) \quad \text{and} \quad (x, y, \tau).$$

There are two functions $K(x, y, \tau)$ and S such that

$$p_i dq^i - H(p, q, t)dt = y_i dx^i - K(x, y, \tau)d\tau + dS\,. \tag{33}$$

The function S is called a "generating function", because it allows one to define a canonical transformation.

Equation (33) can be used to find the canonical transformation which realizes $(q, p, t) \to (x, y, \tau)$. We consider this transformation when $t = \tau$. In this case, by means of Legendre transformation one can write (33) as

$$p_i dq^i - H(p, q, t)dt = x^i dy_i - K(x, y, t)dt + dS(q, y, t). \tag{34}$$

Choosing $K = 0$ one obtains

$$\begin{cases} \dfrac{\partial S}{\partial y_i} = x_i, \\ \dfrac{\partial S}{\partial q^i} = p_i, \\ H\left(q, \dfrac{\partial S}{\partial q}\right) + \dfrac{\partial S}{\partial t} = 0. \end{cases} \tag{35}$$

The third equation of (35) is the Hamilton-Jacobi equation and determines the function S, which completely describes the canonical transformation $(q, p, t) \to (x, y, t)$. The Hamilton-Jacobi equation is defined in $J^1(\mathbb{R}^{n+1})$ and fulfills the conditions of Theorem 4.1.

4.3. Eikonal Equation

In geometric optics and WKB approximation of quantum mechanics WKB approximation the Eikonal equation is relevant. This is a first order PDE given by

$$\frac{1}{2}\delta^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} - N(x) = 0. \tag{36}$$

Here S represents the wave front and N is a function:

$$N(x) = (n(x)/c)^2,$$

which in optics is the square ratio of the refraction index $n(x)$ to the speed of light c; whereas in WKB theory

$$N(x) = V(x) - E,$$

which is the potential minus the total energy.

When in optics we consider a medium with a piece-wise constant refraction index (multilayers), $N(x)$ can be easily approximated by piece-wise constant function. In quantum mechanics when we can approximate the potential by piece-wise constant function. In these cases Proposition 4.1 can be applied in such layer. For instance if

$$N(x) = n_k^2/c^2,$$

for x in the k-layer, then in k-th layer we have symmetry generated by

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -JA & 0 \\ n_k^2/c^2 & 0 & 0 \end{pmatrix}, \tag{37}$$

with

$$G = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

A similar analysis can be carried out in the case of semi-classical description of system which have piece-wise constant potential e.g. multi-well potential.

5. Conclusion

We showed that the odd-symplectic group arises naturally in the geometrical study of first order PDEs. This amounted to study the properties of the theory of characteristics for equations of the form $h(\nabla z(x), x) = 0$.

Acknowledgments

I really wish to thank Richard Cushman for his useful suggestions and long discussions we had at the Warwick University Mathematics Institute and his careful reading my manuscript. I want also to thank J.Sniatycki and L.Bates for the interesting and stimulating discusions.

This paper has been supported by Marie Curie Fellowship (Contract n. HPMFCT-2000-00541) and by M.A.S.I.E. European network.

References

[1] V.I. Arnol'd, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer Verlag (1988).

[2] V.I. Arnol'd, *Mathematical Methods in Classical Mechanics*, Springer Verlag (1989).

[3] L. Bates, R. Cushman The odd symplectic group in geometry, To appear in: *Internat. J. of Pure and Applied Math.* (2002)

[4] *Workshop on Geometry of Momentum Map and Hamiltonian Mechanics*, UMIST - Manchester (July 2002).

[5] R. Cushman, B. Westbury, *Conjugacy Classes in the Odd Symplectic Group*, Preprint, Warwick (2002).

[6] M.I. Gelfand, A.V. Zelevinsky, Models of representations and their hidden symmetries, *Akad. Nauk SSSR, Insti. Prikl. Mat.*, **71** (1984), 26.

[7] A.R. Proctor, Odd symplectic groups, *Inventiones Math.*, **92** (1988), 307-332.

[8] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Oxford Uni. Press (1998).