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# OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF SECOND ORDER NEUTRAL TYPE DIFFERENCE EQUATIONS

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**Abstract:** In this paper the authors establish some new criteria for the oscillation of all solutions of the equation

$$\Delta^2(x_n + a x_{n-k} - b x_{n+\ell}) = q_n x_{n-m} + p_n x_{n+r}\,,$$

and asymptotic behavior of nonoscillatory solutions of the equation

$$\Delta^2(x_n - p x_{n-k}) + Q_n x_{n+1+r} = 0.$$

Examples are inserted to illustrate the results.

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## 1. Introduction

Consider the second order neutral type difference equations of the forms

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$$\Delta^2(x_n + ax_{n-k} - bx_{n+\ell}) = q_n x_{n-m} + p_n x_{n+r}, \quad n \in \mathbb{N}, \tag{1}$$

and

$$\Delta^2(x_n - px_{n-k}) + Q_n x_{n+1+r} = 0, \quad n \in \mathbb{N}, \tag{2}$$

where $\mathbb{N} = \{1, 2, 3, \cdots\}$, $\Delta$ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, and the following assumptions hold:

($c_1$) $a$ and $b$ are positive constants, $0 < p < 1$, and $k$, $\ell$, $m$, and $r$ are nonnegative integers;

($c_2$) $\{p_n\}$, $\{q_n\}$, and $\{Q_n\}$ are positive real sequences.

By a *solution* of equation (1) or (2), we mean a real sequence $\{x_n\}$ that exists for all $n \geq n_0 - \theta$, $n_0 \in \mathbb{N}$, where $\theta = \max\{k, m\}$ for equation (1) and $\theta = k$ for equation (2), and that satisfies the equation for all $n \geq n_0$. A solution $\{x_n\}$ of equation (1) or (2) is said to be *oscillatory* if for every $N \in \mathbb{N}$, there exists an integer $n > N$ such that $x_n x_{n+1} \leq 0$, and it is said to be *nonoscillatory* otherwise. We should point out that equations (1) and (2) contain terms of both the delayed and advanced types. Such equations are sometimes referred to in the literature as being of the *mixed type*.

Recently Grace [3] and Agarwal and Grace [2] considered equations of type (1) with $q_n \equiv q$ and $p_n \equiv p$ and obtained conditions for the oscillation of all solutions with the restriction that, for second order equations, $k$, $\ell$, $m$, and $r$ must be even positive integers. For equation (2), there are only a few known results on the asymptotic properties of nonoscillatory solutions.

Our aim in this paper is to obtain criteria for the oscillation of all solutions of equation (1) without requiring $k$, $\ell$, $m$, and $r$ to be even. In addition, we wish to obtain sufficient conditions for the convergence to zero of all nonoscillatory solution of equation (2). For background results on the oscillation of neutral type difference equations, we refer the reader, for example, to the monographs of Agarwal [1] and Györi and Ladas [5].

## 2. Oscillation Results for Equation (1)

We begin with the following oscillation criteria, which is the main result of this section. Other oscillation criteria for equation (1) will be established on the basis of this theorem. The proof of our result is based on comparison theorems for difference equations.

**Theorem 1.** *Let $m > k$ and $r > \ell$. Let $\{p_n^*\}$ and $\{q_n^*\}$ be nonnegative real sequences such that*

$$q_{n+\ell} \le q_n^* \le \min\{q_n, q_{n-k}\} \quad \text{and} \quad p_{n+\ell} \le p_n^* \le \min\{p_n, p_{n-k}\}, \tag{3}$$

*for $n \ge n_0 \in \mathbb{N}$. Assume that:*

*(i) the difference inequality*

$$\Delta^2 y_n - \frac{p_n^*}{1+a} y_{n+r} \ge 0, \tag{4}$$

*has no eventually positive increasing solutions;*

*(ii) the difference inequality*

$$\Delta^2 y_n - \frac{q_n^*}{1+a} y_{n-m+k} \ge 0, \tag{5}$$

*has no eventually positive decreasing solutions;*

*(iii) the difference inequality*

$$\Delta^2 y_n + \frac{q_n^*}{b} y_{n-m-\ell} + \frac{p_n^*}{b} y_{n+r-\ell} \le 0 \tag{6}$$

*has no eventually positive solutions.*

*Then every solution of equation (1) is oscillatory.*

*Proof.* Without loss of generality, we may assume that the equation (1) has an eventually positive solution $\{x_n\}$, say $x_n > 0$ for $n \ge n_0 \in \mathbb{N}$. Let $z_n = x_n + a x_{n-k} - b x_{n+\ell}$. Then there exists an integer $n_1$ such that

$$\Delta^2 z_n = q_n x_{n-m} + p_n x_{n+r} > 0, \quad n \ge n_1 \ge n_0,$$

which implies that there exists an integer $n_2 \ge n_1$ such that the sequences $\{z_n\}$ and $\{\Delta z_n\}$ are of one sign for $n \ge n_2$. We claim that $z_n > 0$ eventually. To see this, assume that $z_n < 0$ for all $n \ge n_2$. Then,

$$0 < u_n = -z_n = b x_{n+\ell} - a x_{n-k} - x_n \le b x_{n+\ell},$$

and so

$$x_n \ge \frac{u_{n-\ell}}{b}, \quad n \ge n_2 + \ell = n_3.$$

From equation (1), we obtain

$$\begin{aligned} 0 &= \Delta^2 u_n + q_n x_{n-m} + p_n x_{n+r} \\ &\geq \Delta^2 u_n + q_n \frac{u_{n-m-\ell}}{b} + p_n \frac{u_{n+r-\ell}}{b} \\ &\geq \Delta^2 u_n + \frac{q_n^*}{b} u_{n-m-\ell} + \frac{p_n^*}{b} u_{n+r-\ell}\,, \end{aligned}$$

for $n \geq n_4$ for some $n_4 \geq n_3$. Hence, $\{u_n\}$ is a positive solution of (6), which is a contradiction. Thus, $z_n > 0$ for $n \geq n_2$.

Now define

$$y_n = z_n + a z_{n-k} - b z_{n+\ell}. \tag{7}$$

Then,

$$\begin{aligned} \Delta^2 y_n = {} & q_n x_{n-m} + p_n x_{n+r} + a q_{n-k} x_{n-k-m} \\ & + a p_{n-k} x_{n+r-k} - b q_{n+\ell} x_{n+\ell-m} - b p_{n+\ell} x_{n+l+r}, \end{aligned}$$

and so we have

$$\Delta^2 y_n \geq q_n^* z_{n-m} + p_n^* z_{n+r} > 0. \tag{8}$$

Hence, $\{y_n\}$ and $\{\Delta y_n\}$ are eventually of one sign.

Next, we will prove that $y_n > 0$ eventually. If this is not the case, then we repeat the above procedure by letting

$$0 < v_n = -y_n = b z_{n+\ell} - a z_{n-k} - z_n \leq b z_{n+l}.$$

Thus,

$$z_n \geq \frac{v_{n-\ell}}{b},$$

and we eventually arrive at

$$\begin{aligned} 0 &\geq \Delta^2 v_n + q_n^* z_{n-m} + p_n^* z_{n+r} \\ &\geq \Delta^2 v_n + \frac{q_n^*}{b} v_{n-m-\ell} + \frac{p_n^*}{b} v_{n+r-\ell}. \end{aligned}$$

Hence, we have that $\{v_n\}$ is a positive solution of (6), which again is a contradiction.

Suppose $\Delta z_n < 0$ for $n \geq n_3$ for some $n_3 \geq n_2$. We claim that $\Delta y_n < 0$ for $n \geq n_3$. If not, we then have $y_n > 0$, $\Delta y_n > 0$, and $\Delta^2 y_n > 0$ which imply that $\lim_{n\to\infty} y_n = \infty$. On the other hand, $z_n > 0$ and $\Delta z_n < 0$ imply that

$\lim_{n\to\infty} z_n = c < \infty$. Taking limits on both sides of (7) we obtain a contradiction. Thus, $\Delta y_n < 0$ for $n \geq n_3$. Using the monotonicity of $\{z_n\}$, we have

$$\begin{aligned} y_{n-m} &= z_{n-m} + az_{n-m-k} - bz_{n+r-m} \\ &\leq z_{n-m} + az_{n-m-k} \\ &\leq (1+a)z_{n-k-m}, \end{aligned}$$

or

$$z_{n-m} \geq \frac{y_{n-m+k}}{(1+a)},$$

for $n \geq n_4$ for some sufficiently large $n_4 \geq n_3$. From (8), it follows that

$$\Delta^2 y_n \geq q_n^* z_{n-m} \geq \frac{q_n^*}{(1+a)} y_{n-m+k}.$$

Thus, $\{y_n\}$ is a positive decreasing solution of (5). This contradiction shows that $\Delta z_n > 0$ for $n \geq n_3$.

Assume that $\Delta y_n < 0$. Proceeding similarly as above and using the monotonicity of $\{z_n\}$, we obtain

$$y_{n-m} \leq (1+a)z_{n-m}.$$

From (8) and the monotonicity of $\{y_n\}$, we have

$$\Delta^2 y_n \geq q_n^* z_{n-m} \geq \frac{q_n^*}{(1+a)} y_{n-m} \geq \frac{q_n^*}{(1+a)} y_{n-m+k}.$$

Again we have that $\{y_n\}$ is a positive decreasing solution of (5), which is a contradiction.

Finally, suppose that $\Delta y_n > 0$ for $n \geq n_3$. Then

$$y_{n+r} \leq (1+a)z_{n+r},$$

which, in view of (8), implies

$$\Delta^2 y_n \geq p_n^* z_{n+r} \geq \frac{p_n^*}{1+a} y_{n+r}.$$

That is, (4) possesses a positive increasing solution. This contradiction completes the proof of the theorem. □

**Remark 1.** Condition (3) may appear to be somewhat artificial, but it is satisfied, for example, if the sequences $\{q_n\}$ and $\{p_n\}$ are decreasing. In that case, we can set $p_n^* = p_n$ and $q_n^* = q_n$. Condition (3) could be eliminated if the sequences $\{q_n\}$ and $\{p_n\}$ are bounded from below, say $p_n \ge p > 0$ and $q_n \ge q > 0$. We would then need to replace the inequalities in (4)–(6) by

$$\Delta^2 y_n - \frac{p}{1+a} y_{n+r} \ge 0,$$

$$\Delta^2 y_n - \frac{q}{1+a} y_{n-m+k} \ge 0,$$

and

$$\Delta^2 y_n + \frac{q}{b} y_{n-m-\ell} + \frac{p}{b} y_{n+r-\ell} \le 0,$$

respectively.

By imposing conditions that will ensure that (4), (5), and (6) are satisfied, it is possible to construct various oscillation criteria for equation (1). As an example, we have the following new oscillation result.

**Theorem 2.** *Suppose that $m > k$, $r > \ell \ge 1$, and condition (3) holds. If*

$$\limsup_{n\to\infty} \sum_{s=n}^{n+r-1} (n+r-s-1) p_s^* > 1+a, \tag{9}$$

$$\limsup_{n\to\infty} \sum_{s=n-m+k}^{n} (s-n+m-k+1) q_s^* > 1+a, \tag{10}$$

*and the difference inequality (6) has no eventually positive solutions, then equation (1) is oscillatory.*

*Proof.* First we prove that condition (9) is sufficient to ensure that (4) has no eventually positive increasing solutions. Assume the contrary, that is, suppose there is a solution $\{y_n\}$ and an integer $N \in \mathbb{N}$ such that $y_n > 0$ and $\Delta y_n > 0$ for $n \ge N$. Summing (4) from $n$ to $s-1$, we have

$$\Delta y_s - \Delta y_n \ge \sum_{j=n}^{s-1} \frac{p_j^*}{1+a} y_{j+r}.$$

Summing the last inequality in $s$ from $n$ to $n+r-1$, we obtain

$$\begin{aligned} y_{n+r} - y_n - r\Delta y_n &\geq \sum_{s=n}^{n+r-1} \sum_{j=n}^{s-1} \frac{p_j^*}{1+a} y_{j+r} \\ &\geq y_{n+r} \sum_{s=n}^{n+r-1} (n+r-s-1) \frac{p_s^*}{1+a}\,, \end{aligned}$$

where we have used the fact that $\{y_n\}$ is increasing. From the last inequality, we have

$$0 > y_{n+r}\left[\sum_{s=n}^{n+r-1} (n+r-s-1)\frac{p_s^*}{1+a} - 1\right],$$

which is a contradiction.

Next, we show that under condition (10), the inequality (5) has no eventually positive decreasing solutions. Assume to the contrary that there is a solution $\{y_n\}$ and an integer $N \in \mathbb{N}$ such that $y_n > 0$ and $\Delta y_n \leq 0$ for $n \geq N$. Summing (5) from $s$ to $n$, we have

$$\Delta y_{n+1} - \Delta y_s \geq \sum_{j=s}^{n} \frac{q_j^*}{1+a} y_{j-m+k},$$

and summing this inequality in $s$ from $n-m+k$ to $n$, we obtain

$$\begin{aligned} &(m-k+1)\Delta y_{n+1} - y_{n+1} + y_{n-m+k} \\ &\qquad \geq \sum_{s=n-m+k}^{n} (s-n+m-k+1)\frac{q_s^*}{1+a} y_{s-m+k} \\ &\qquad\qquad \geq y_{n-m+k} \sum_{s=n-m+k}^{n} (s-n+m-k+1)\frac{q_s^*}{1+a}. \end{aligned}$$

Hence, for $n \geq N$, we obtain

$$y_{n+1} + y_{n-m+k}\left(\sum_{s=n-m+k}^{n} (s-n+m-k+1)\frac{q_s^*}{1+a} - 1\right) \leq 0,$$

which is a contradiction. The result now follows from Theorem 1. $\square$

## 3. Asymptotic Behavior of Nonoscillatory Solutions of Equation (2)

In this section, we study the asymptotic behavior of the nonoscillatory solutions of equation (2).

**Theorem 3.** *If*

$$\sum_{n=n_0}^{\infty} Q_n = \infty, \quad n_0 \in \mathbb{N}, \tag{11}$$

*then every nonoscillatory solution of equation (2) converges to zero as* $n \to \infty$.

*Proof.* Without loss of generality, assume that $\{x_n\}$ is a positive solution of the equation (2) and define

$$z_n = x_n - p x_{n-k}. \tag{12}$$

From equation (2), we have $\Delta^2 z_n \le 0$ for all $n \ge n_0$. If $\Delta z_n < 0$ eventually, then $\lim_{n\to\infty} z_n = -\infty$. But if $z_n < 0$, then

$$x_n < p x_{n-k} < p^2 x_{n-2k} < \cdots < p^j x_{n-jk},$$

which, in view of $(c_1)$, implies that $\lim_{n\to\infty} x_n = 0$. This contradicts the fact that $\lim_{n\to\infty} z_n = -\infty$, and so $\Delta z_n > 0$ for all large $n$, say $n \ge n_1$. There are two possible cases for $\{z_n\}$.

*Case 1.* Let $z_n > 0$ for $n \ge n_2$ for some $n_2 \ge n_1$. From (2) and (12), we have

$$\Delta^2 z_n + Q_n z_{n+1+r} \le 0. \tag{13}$$

The monotonicity of $\{z_n\}$ implies there exists a constant $c > 0$ and an integer $n_3 \ge n_2$ such that $z_{n+1+r} \ge c$ for $n \ge n_3$. Then, from (13), we have

$$\Delta^2 z_n + c Q_n \le 0,$$

and summing, we obtain

$$\Delta z_n - \Delta z_{n_3} + c \sum_{s=n_3}^{n-1} Q_s \le 0,$$

or

$$c \sum_{s=n_3}^{n-1} Q_s \le \Delta z_{n_3},$$

for $n \geq n_3$. This contradicts (11) and completes the proof in this case.

*Case 2.* Let $z_n < 0$ for $n \geq n_2$. In this case, as observed above, we are led to conclude that $\lim_{n\to\infty} x_n = 0$. This competes the proof of the theorem. □

**Example 1.** Consider the difference equation

$$\Delta^2(x_n - px_{n-k}) + (p2^k - 1)2^{m-1}x_{n+m+1} = 0\,, \tag{14}$$

where $0 < p < 1$, $p > \frac{1}{2^k}$, and $k$ and $m$ are positive integers. By Theorem 3, every nonoscillatory solution of the equation (14) tends to zero as $n \to \infty$. One such solution is $\{x_n\} = \left\{\frac{1}{2^n}\right\}$.

In case the series $\sum\limits_{n=n_0}^{\infty} Q_n$ converges, we have the following result.

**Theorem 4.** *If*

$$\limsup_{n\to\infty}\left\{\sum_{s=n}^{n+r-1}\sum_{t=s}^{\infty} Q_t + \sum_{s=n-r}^{n-1}\sum_{t=s+r}^{\infty} Q_t\right\} > 1, \tag{15}$$

*then every nonoscillatory solution of equation (2) converges to zero as $n \to \infty$.*

*Proof.* Let $\{x_n\}$ be an eventually positive solution of the equation (2) and set $z_n = x_n - px_{n-k}$. Then $\Delta^2 z_n \leq 0$ eventually. Similar to the proof of Theorem 3, we are led to conclude that $\Delta z_n > 0$ eventually. Again, there are two possibilities for $\{z_n\}$. For Case 2, i.e., $z_n$ is eventually negative, the proof is similar to that of Case 2 in the proof of Theorem 3 and is omitted. In Case 1, we proceed exactly as in the proof of Theorem 3 and obtain inequality (13). Summing (13) from $n$ to $\infty$, we have

$$\Delta z_n \geq \sum_{s=n}^{\infty} Q_s z_{s+1+r}, \tag{16}$$

and summing again from $n$ to $n + r$, we obtain

$$\begin{aligned} z_{n+r+1} - z_n &\geq \sum_{s=n}^{n+r}\sum_{t=s}^{\infty} Q_t z_{t+r+1} \\ &\geq z_{n+r+1}\sum_{s=n}^{n+r}\sum_{t=s}^{\infty} Q_t\,, \end{aligned} \tag{17}$$

where we have used the fact that $\{z_n\}$ is increasing. If we now sum (16) from $n_2$ to $n-1$, we obtain

$$\begin{aligned} z_n &\geq \sum_{s=n_2}^{n-1} \sum_{t=s}^{\infty} Q_t z_{t+r+1} \\ &\geq \sum_{s=n-r}^{n-1} \sum_{t=s+r}^{\infty} Q_t z_{t+r+1} \\ &\geq z_{n+r+1} \sum_{s=n-r}^{n-1} \sum_{t=s+r}^{\infty} Q_t \,, \end{aligned} \tag{18}$$

where we again used the fact that $\{z_n\}$ is increasing. From (17) and (18), we have

$$z_{n+r+1} \geq z_{n+r+1} \left\{ \sum_{s=n}^{n+r} \sum_{t=s}^{\infty} Q_t + \sum_{s=n-r}^{n-1} \sum_{t=s+r}^{\infty} Q_t \right\},$$

or

$$\left\{ \sum_{s=n}^{n+r} \sum_{t=s}^{\infty} Q_t + \sum_{s=n-r}^{n-1} \sum_{t=s+r}^{\infty} Q_t \right\} \leq 1,$$

which contradicts (15). The proof is now complete. □

**Example 2.** Consider the neutral difference equation

$$\Delta^2(x_n - p x_{n-k}) + \frac{a}{n(n+1)} x_{n+r+1} = 0, \quad n \geq 1, \tag{19}$$

where $a \geq 1$ and $0 < p < 1$. All conditions of Theorem 4 are satisfied and hence every nonoscillatory solution of equation (19) tends to zero as $n \to \infty$.

The next criteria that we obtain for the nonoscillatory solutions of equation (2) to converge to zero as $n \to \infty$ depend explicity on the parameter $p$.

**Theorem 5.** *Assume that there exists an integer $\alpha$ such that*

$$r - \alpha k \leq 1. \tag{20}$$

*Furthermore, assume that there exists an integer $\beta \geq \alpha$ such that*

$$\sum_{n=n_0}^{\infty} \left( p^{\alpha} \frac{1-p^{\beta-\alpha+2}}{1-p} (s+r) Q_s - \frac{1}{4(s+r)} \right) = \infty. \tag{21}$$

*Then every nonoscillatory solution of equation (2) converges to zero as $n \to \infty$.*

*Proof.* Without loss of generality, let $\{x_n\}$ be an eventually positive solution of the equation (2). As in the proof of Theorem 3, we again obtain Case 1 and Case 2. If $z_n > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$, then using (12), we have

$$\Delta^2 z_n + Q_n z_{n+r+1} + pQ_n x_{n+r+1-k} \leq 0. \tag{22}$$

Iterating this substitution, we obtain

$$\Delta^2 z_n + Q_n \sum_{i=0}^{j} p^j z_{n+r+1-ik} + p^{j+1} Q_n x_{n+r+1-(j+1)k} \leq 0,$$

or

$$\Delta^2 z_n + Q_n \sum_{i=0}^{j+1} p^j z_{n+r+1-ik} \leq 0,$$

and so,

$$\Delta^2 z_n + Q_n \sum_{i=\alpha}^{j+1} p^j z_{n+r+1-ik} \leq 0.$$

Letting

$$y_j(n) = \sum_{i=\alpha}^{j+1} p^i z_{n+r-ik},$$

we have

$$\Delta^2 z_n + Q_n y_j(n+1) \leq 0. \tag{23}$$

Define

$$v_n = \frac{(n+r)\sum\limits_{i=\alpha}^{j+1} p^i}{y_j(n)} \Delta z_n\,,$$

for $n \geq n_1$. Then $v_n > 0$ and

$$\begin{aligned}\Delta v_n &= \frac{(n+r)\sum\limits_{i=\alpha}^{j+1} p^i}{y_j(n+1)} \Delta^2 z_n + \frac{\sum\limits_{i=\alpha}^{j+1} p^i \Delta z_{n+1}}{y_j(n+1)} - \frac{(n+r)\sum\limits_{i=\alpha}^{j+1} p^i \Delta z_n}{y_j(n) y_j(n+1)} \Delta y_j(n) \\ &\leq -(n+r) Q_n \sum_{i=\alpha}^{j+1} p^i + \frac{v_{n+1}}{(n+1+r)} - \frac{(n+r)}{(n+1+r)} v_{n+1} \frac{\Delta y_j(n)}{y_j(n)}.\end{aligned}$$

Since $\{y_j(n)\}$ is increasing and $\{\Delta z_n\}$ is decreasing, we have

$$\Delta z_{n+r-ik} \geq \Delta z_{n+r-\alpha k}\,,$$

for $i \geq \alpha$, and by (20) we obtain $\Delta z_{n+r-\alpha k} \geq \Delta z_{n+1}$. Thus,

$$\Delta v_n \leq \frac{v_{n+1}}{n+r+1} - \frac{(n+r)}{(n+r+1)^2} v_{n+1}^2 - (n+r)Q_n \sum_{i=\alpha}^{j+1} p^i$$

$$\leq \frac{1}{4(n+r)} - (n+r)Q_n \sum_{i=\alpha}^{j+1} p^i.$$

Summing the last inequality from $n_1$ to $n-1$, we have

$$v_n \leq v_{n_1} - \sum_{s=n_1}^{n-1} \left( p^\alpha \frac{(1-p^{j-\alpha+2})}{(1-p)} (s+r)Q_s - \frac{1}{4(s+r)} \right).$$

Letting $n \to \infty$, (21) yields a contradiction to the fact that $v_n > 0$.

Case 2 again leads to $\lim_{n\to\infty} x_n = 0$. This competes the proof of the theorem. □

**Corollary 6.** *Assume that there exists an integer $\alpha$ such that (20) holds. If*

$$\liminf_{n\to\infty} (n+r)^2 Q_n > \frac{1}{4p^\alpha(1+p)}, \tag{24}$$

*then every nonoscillatory solution of equation (2) converges to zero as $n \to \infty$.*

*Proof.* Let $c = \liminf_{n\to\infty} (n+r)^2 Q_n$ and choose $\beta = \alpha$. Condition (24) implies that for

$$\varepsilon = \frac{1}{2}\left[ c - \frac{1}{4p^\alpha(1+p)} \right]$$

there is a positive $N_\varepsilon$ such that $n \geq N_\varepsilon$ implies

$$(n+r)^2 Q_n > c - \varepsilon = \frac{1}{2}\left[ c + \frac{1}{4p^\alpha(1+p)} \right].$$

Hence, for $n \geq N_\varepsilon$, we have

$$\frac{p^\alpha(1-p^{\beta-\alpha+2})}{1-p}(n+r)Q_n - \frac{1}{4(n+r)}$$

$$= \frac{p^\alpha(1-p^2)}{(1-p)(n+r)} \left[ (n+r)^2 Q_n - \frac{1}{4p^\alpha(1+p)} \right]$$

$$\geq \frac{p^\alpha(1+p)}{(n+r)} \left[ c - \varepsilon - \frac{1}{4p^\alpha(1+p)} \right]$$

$$\geq \frac{p^\alpha(1+p)}{(n+r)} \varepsilon.$$

This implies that (21) holds and completes the proof of the theorem. □

We conclude this paper with the following example.

**Example 3.** Consider the neutral difference equation

$$\Delta^2(x_n - px_{n-k}) + \frac{(p2^k - 1)}{2^{1+r}} y_{n+1-r} = 0. \tag{25}$$

For $0 < p < 1$, $k > \frac{1}{p \log 2}$, and $\alpha \geq 1$, all conditions of Corollary 6 are satisfied, and hence all nonoscillatory solutions of (25) tend to zero as $n \to \infty$. One such solution of this equation is $\{x_n\} = \left\{\frac{1}{2^n}\right\}$.

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