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AN IMPROVEMENT OF ARCHIMEDES METHOD OF APPROXIMATING π

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Abstract: About 255 B.C., Archimedes used the perimeters of inscribed and circumscribed polygons of 6, 12, 24, 38, and 96 sides to find upper and lower bounds for π. Using 96 sides, he showed that

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}.$$

In the following 1900 years, many improvements of Archimedes bounds were obtained using polygons with more sides, but little progress was made in improving his method. In Cyclometricus (1621), Willebrord Snell (Snellius) obtained a dramatic improvement by suggesting that the perimeter of the inscribed polygon of n sides converges to π twice as fast as the perimeter of the circumscribed polygon, though this was only first proved by Christian Huygens [2] in 1654. Using this, Snell obtained 7 digits of π using a 96 sided polygon, and obtained 34 digits of π with $n = 2^{30}$. In this paper, we improve the Snell-Huygens method by proving that areas of inscribed and circumscribed polygons of n sides, when averaged in much the same way that Huygens averaged perimeters, converge exactly $\frac{3}{8}$ as fast (in the limit). Using this, we obtain 10 digits of π when $n = 96$, and when $n = 2^{62}$ we obtain 110 digits of π. The table in Section 3 and analysis of the proof in Section 2 suggest for polygons of n sides, the Snell-Huygens method gives about twice as many digits of precision as Archimedes method, our method gives three times as many digits, and uses only regular polygons.

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1. Introduction and Summary

About 255 B.C., Archimedes (287-212 B.C.), in his famous treatise *On the Measurement of the Circle*, obtained a method of approximating π. Using a recursive algorithm he found the perimeters of inscribed and circumscribed polygons of 6, 12, 24, 48, and 96 sides. When $n = 96$, he obtained the lower and upper bounds

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}.$$

Since these upper and lower bounds have only two digits of precision in the approximation of π (n digits of precision are defined to be an error of less than 10^{-n}), Archimedes is often credited with only two digits of accuracy in his approximation of π, but simply averaging of his upper and lower bounds gives an error of about $.25 \times 10^{-4}$, so we credit him with 3 digits of precision in our table in Section 3.

In the succeeding years from 255 B.C. to 1654 A.D., there were many notable improvements in the bounds obtained by Archimedes, but almost all were obtained by using polygons with more and more sides. In 263 A.D., Liu Hui [2, p. 27] used a 3072-gon to obtain $\pi \approx 3.14159$. About 480 A.D., Tsu Chung-Chih and his son Tsu-Keng-Chih obtained the impressive bounds $3.1415926 < \pi < 3.1415927$, and the best approximation of π using fractions with 3 or fewer digits in the numerator and denominator, $\pi \approx \frac{355}{113}$. In 1424, Al Kashi obtained 14 decimal places using a polygon of 6×2^{27} sides, and in 1593, Romanus [2, p. 102] obtained 15 decimal places with a polygon of 2^{30} sides. In 1609, Ludolph van Ceulen, who was Snell teacher and who devoted his life to this problem, obtained 35 decimal places using a polygon of 2^{62} sides, though some authors say that Snell actually completed the computation of the last 3 decimal places. The 35 decimal places were published posthumously, and it is said that his widow, at his request, engraved the digits on his tombstone, though other historians say only the last 3 digits were engraved [2, p. 102]. The stone has long since been lost. In any case, π became known in Germany as the Ludolphine number (the symbol for π was not actually used until 1706).

The brilliant mathematician and physicist Willebrord Snell was a student of Ludolph van Ceulen and became interested in 1617 in finding a significant acceleration of Archimedes method which would make it possible to find as many digits of π as possible using a polygon of at most n sides. In Cyclometricus in 1621 he gave such a result. He indicated (without formal proof) that the perimeter of an inscribed polygon of n sides converges to π twice as fast (in the limit) as the perimiter of the circumscribed polygon. He and others also used, see, e.g. [2, p. 111-120], irregular polygons to obtain even sharper lower and upper bounds. In this paper we restrict ourselves to the regular polygons used by Archimedes. Christians Huygens [2, p. 114] gave the first formal proof of Snell result in 1654 in *De circuli magnitude inventa*. Using this result, which follows immediately from the first two terms of the MacClaurin expansions given in Section 2, he showed empirically that

$$\frac{2}{3}n\sin\left(\frac{\pi}{n}\right)+\frac{1}{3}n\tan\left(\frac{\pi}{n}\right)\to\pi \tag{1.1}$$

faster than the simple average of $n\sin\left(\frac{\pi}{n}\right)$ and $n\tan\left(\frac{\pi}{n}\right)$, (the perimeters of the inscribed and circumscribed regular polygons of n sides, respectively). Indeed, using a 96-gon, Snell obtained 7 digits of precision, whereas Archimedes method yields only 3 digits. Moreover, Snell showed that using (1.1) with $n=2^{30}$ yields 34 decimal places of π, greatly reducing the 2^{62} sides required by Ludolph van Ceulen to obtain 35 places.

In this paper we improve the Snell-Huygens method. Of course, in the last 400 years, many powerful methods have been developed to compute π so that our improvement is, as Snell was, only an improvement in the sense that one obtains as many digits of π as possible using information from polygons of n sides. Recently, Kanada formulas have been used to compute 1.2411×10^{12} digits of π using more powerful methods.

The key to our improvement is not to ignore the information obtained from looking at the areas of these polygons. It has long been known that areas are less useful than perimeters in approximating π as they converge less rapidly. It seem not to have been observed, however, the convergence is exactly $\frac{3}{8}$ as rapid when areas are treated in the same way that Snell treated perimeters. Using this result, we prove in Section 2 that

$$\frac{n}{30}\sin\left(\frac{\pi}{n}\right)+4\tan\left(\frac{\pi}{n}\right)-3\sin\left(\frac{2\pi}{n}\right)\to\pi \tag{1.2}$$

much faster than $\frac{2n}{3}\sin\left(\frac{\pi}{n}\right)+\frac{n}{3}\tan\left(\frac{\pi}{n}\right)$. Indeed, when $n=6$ (hexagon), we obtain 3 digits of precision with an error of only .00067..., and when $n=96$, we obtain 10 digits of precision in the approximation of π in comparison to 3 for Archimedes and 7 for Snell-Huygens. When $n=2^{62}$, we obtain 110 digits of π. See the table in Section 3 for further comparisons of Archimedes method, Snell-Huygens method, and our method. In the next section we prove that our method has an error of $O\left(\frac{1}{n^6}\right)$; see (2.3)-(2.5). In light of this, and the corresponding $O\left(\frac{1}{n^4}\right)$ for the Snell-Huygens method and $O\left(\frac{1}{n^2}\right)$ for Archimedes method, it is not surpirising that the values in the table in Section 3 indicate that for a given value of n, the Snell-Huygens methods yields about twice as many digits of precision as Archimedes method, and our method yields three times as many.

2. Proof of the Theorem

Let I_P and O_P denote the perimeters of inscribed and circumscribed polygons of n sides (resp.) on a circle of circumference π. It is easy to show that

$$I_P = n\sin\left(\frac{\pi}{n}\right) \quad \text{and} \quad O_P = n\tan\left(\frac{\pi}{n}\right). \tag{2.1}$$

Similarly, let I_A and O_A denote the areas of these polygons. Then, it is any easy exercise in trigonometry to show that

$$I_A = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right) \quad \text{and} \quad O_A = n\tan\left(\frac{\pi}{n}\right). \tag{2.2}$$

Let $U = \frac{2}{3}I_P + \frac{1}{3}O_P$ and $V = \frac{2}{3}O_A + \frac{1}{3}I_A$. We prove the folowing theorem.

Theorem 2.1. $\lim\limits_{n\to\infty} \dfrac{V-\pi}{U-\pi} = \dfrac{8}{3}$.

Proof. Using well-known MacClaurin series expansions for $\sin x$ and $\tan x$ we have

$$n\sin\left(\frac{\pi}{n}\right) = I_P = \pi - \frac{\pi^3}{6}n^2 - \frac{\pi^5}{120}n^4 + O\left(\frac{1}{n^6}\right), \tag{2.3}$$

$$n\tan\left(\frac{\pi}{n}\right) = O_P = \pi + \frac{\pi^3}{3}n^2 + \frac{2\pi^5}{15}n^4 + O\left(\frac{1}{n^6}\right), \tag{2.4}$$

$$\frac{n}{2}\sin\left(\frac{2\pi}{n}\right) = I_A = \pi - \frac{2\pi^3}{3}n^2 + \frac{2\pi^5}{15}n^4 + O\left(\frac{1}{n^6}\right). \tag{2.5}$$

From the first two terms we see that $I_P \to \pi$ twice as fast as O_P as was observed by Snell and proved by Huygens, and $O_A \to \pi$ twice as fast as I_A (with error $O\left(\frac{1}{n^4}\right)$). For large n we have

$$O_P - \pi \to 2\pi - 2I_P \quad \text{or} \quad \frac{2}{3}O_P + \frac{1}{3}I_P \to \pi.$$

Similarly,

$$\pi - O_A \to 2O_A - 2\pi \quad \text{or} \quad \frac{2}{3}O_A + \frac{1}{3}I_A \to \pi.$$

Now using (2.3), (2.4), and (2.5), we have after cancellation and letting terms which tend to zero vanish, that

$$\lim_{n\to\infty} \frac{V-\pi}{U-\pi} = \frac{\left(\frac{4\pi^5}{45} + \frac{2\pi^5}{45}\right)\left(\frac{1}{n^4}\right)}{\left(\frac{\pi^5}{180} + \frac{2\pi^5}{45}\right)\left(\frac{1}{n^4}\right)} = \frac{8}{3}.$$

We note that $V - \pi \to 8U - 3V$ is equivalent to $\frac{8U-3V}{5} \to \pi$. Moreover, $\frac{8U-3V}{5}$ simplifies easily to

$$\frac{n}{30}\left(32\sin\left(\frac{\pi}{n}\right) + 4\tan\left(\frac{\pi}{n}\right) - 3\sin\left(\frac{2\pi}{n}\right)\right). \tag{2.6}$$

3. Numerical Results

The following table gives the number of digits of precision obtained in the evaluation of π using Archimedes method, Snell-Huygens method and our method.

n	Archimedes	Snell-Huygens	Chakrabarti-Hudson
6	1	2	3
96	3	7	10
2^{10}	5	10	16
2^{11}	6	12	18
2^{30}	17	34	52
2^{62}	36	73	110

The numbers in the last three columns give the digits of precision. A number has precision n if $|n - \pi| < 10^{-n}$. The values for Archimedes are calculated by taking the simple average of his lower and upper bounds. Ludolph van Ceulen obtained 20 digits of π using a polygon of 60×2^{33} sides. He used several methods to accelerate convergence to π, but simple averaging, $\frac{I_P+O_P}{2}$ gives the 35 digits of π which were reputedly engraved on his tombstone (in fact the simple average gives 36 digits). Snell-Huygens obtain 37 more digits and we obtain 70 more digits of precision using the same inscribed and circumscribed polygons of 2^{62}

sides used by van Ceulen. When $n = 6$ (hexagons), the error using Archimedes method is .090458..., using Snell-Huygens method, is .013078..., and using our method, is .000672..., and the digits of precision are 1, 2, and 3 respectively.

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References

[1] Le Petit Archimede, No. Hors Serie, Le Nombre π (1980).

[2] P. Beckmann, *A History of Pi*, St. Martin's Press, New York (1971).

[3] J. Arndt, C. Haenel, π-Unleashed, Springer (2001).

[4] J.P. Delahaye, *Le fascinant nomber* π, Bibliotheque pour la Science, Berlin (1997).

[5] H. Engels, Quadrature of the circle in Ancient Egypt, *Historia Mathematica*, **4** (1977), 137-140.

[6] T.L. Heath, *The Works of Archimedes*, Cambridge University Press (1897).

[7] L.Y. Lam, T.S. Ang, Circle measurements in Ancient China, *Historia Mathematica*, **13** (1986), 325-340.

[8] G.M. Phillips, Archimedes the numerical analyst, *The American Mathematical Monthly*, **88** (1981), 165-169.

[9] H.C. Schelper, The chronology of Pi, *Mathematics Magazine* (1950).

[10] F. Viete, *Opera Mathematica*, Reprinted, Georg Olms Verlag, Hildesheim, New York (1970).

[11] E. Zebrowski, *A History of the Circle: Mathematical Reasoning and the Physical Universe*, Rutgers University Press, New Brunswick, New Jersey (1999).