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SOME APPLICATIONS OF THE CRITERIA FOR GENERALIZED DIAGONALLY DOMINANT MATRICES

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Abstract: We have proposed a new efficient algorithm to determine a positive diagonal matrix D if matrix A is a generalized diagonally dominant matrix, where AD is a strictly diagonally dominant matrix [5]. In this paper, we show a Gauss form of this method, and by using this algorithm, we further propose some applications for distinguishing the symmetric positive definite matrices, M-matrices, the stability of nonlinear autonomous systems, computing the stable quantity of an real matrix, and estimating distribution of eigenvalues for generalized diagonally dominant matrices, etc.

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1. Introduction

Let A be an $n \times n$ complex matrix, $N = \{1, 2, ..., n\}$, and $N_1 = \{i \mid \mid a_{ii} \mid > \sum_{j \neq i} \mid a_{ij} \mid = S_i, i \in N\} \neq \Phi$. When $J = N$, A is said to be a strictly diagonally dominant matrix. If there exists a positive diagonal matrix D such that AD is a strictly diagonally dominant matrix, then we call that A is a generalized diagonally dominant matrix (simply, we call A is a GDDM or an H-matrix).

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The GDDM is an extensive class of special matrices, which is useful in many fields such as the convergence of iterative methods [10], and the stability of control systems [1], etc. GDDM has been discussed very much theoretically, and we have known a lot of conditions of equivalence. But we can not get any practical method by using these Theorems [2]. Some authors have shown some practical conditions [3, 4, 6], but the useful range of these conditions are pretty limited. Recently, we proposed a new efficient algorithm by an iterative process to determine a positive diagonal matrix D if matrix A is a GDDM, where AD is a strictly diagonally dominant matrix [5]. Many examples of numerical examination are expressing this method is pretty effective.

In this paper, we show a Gauss form of this method, and by using this algorithm, we note some applications for distinguishing the symmetric positive definite matrices, distinguishing M-matrices, distinguishing the stability of non-linear autonomous systems, computing the stable quantity of an real matrix, and estimating distribution of eigenvalues for the GDDM, etc.

2. An Iterative Criterion for GDDM

We first summarize the iterative method shown in [5] below.

Let $N_2 = N - N_1$, computing by the following procedure:

1) Compute $S_i = \sum_{j \neq i} \mid a_{ij} \mid, i = 1, 2, ..., n$.

2) Decide a positive diagonally matrix D by the following form :

$$D^{(1)} = \text{diag}\,\{d_i \mid d_i = \frac{S_i + \varepsilon}{\mid a_{ii} \mid + \varepsilon}, i \in N_1, \quad d_i = 1, i \in N_2\},$$

where ε is a small number with $0 < \varepsilon \ll 1$.

3) Compute $A^{(1)} = AD^{(1)}$.

4) If $A^{(1)}$ is not a strictly diagonally dominant matrix, then return to 1), until $A^{(k)}$ becomes a strictly diagonally dominant matrix.

Obviously, if $A^{(k)}$ is a strictly diagonally dominant matrix, then A is a GDDM, where the positive diagonal matrix D satisfies $D = D^{(1)} \cdot D^{(2)} \cdots D^{(k)}$. And we can also prove that if A is a GDDM, then this iterative method is certainly always valid [5].

Now, we shall show some numerical examples for verifying the efficiency of this method.

Example 1. Let

$$A = \begin{pmatrix} 1 & 0.1 & 0.1 & 0.1 & 0.8 \\ 0.35 & 1 & 0.1 & 0.7 & 0.2 \\ 0.1 & 0.2 & 1 & 0.1 & 0.02 \\ 0.1 & 0.06 & 0.03 & 1 & 0.02 \\ 0.1 & 0.2 & 0.2 & 0.2 & 1 \end{pmatrix},$$

then this method shows A is a GDDM by using only $k = 1$ iteration for any $0 < \varepsilon \leq 0.9$.

Example 2. Let

$$A = \begin{pmatrix} 1 & 0.0089 & 0.1305 & 0.0679 & 0.0252 \\ 0.2891 & 1 & 0.4724 & 0.2938 & 0.3628 \\ 0.1424 & 0.3383 & 1 & 0.0972 & 0.029 \\ 0.3454 & 0.3384 & 0.4843 & 1 & 0.2982 \\ 0.0363 & 0.1415 & 0.3680 & 0.1266 & 1 \end{pmatrix},$$

then this method shows A is a GDDM by using only $k = 1$ iteration for any $0 < \varepsilon \leq 0.1$.

Example 3. Let

$$A = \begin{pmatrix} 0.9 & -0.1 & -0.2 & -0.1 \\ -0.9 & 0.9 & -0.7 & -0.8 \\ -0.1 & -0.1 & 0.9 & -0.1 \\ -0.3 & -0.1 & -0.2 & 0.9 \end{pmatrix},$$

then this method shows A is a GDDM by using only $k = 2$ iterations for any $0 < \varepsilon \leq 0.1$.

Example 4. Let

$$A = \begin{pmatrix} 0.9 & 0.1 & 0.05 & 0.05 & 0.1 & 0.1 \\ 0.1 & 1.05 & 0.05 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.9 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.2 & 0.1 & 0.7 & 0 & 0 \\ 0.5 & 0.4 & 0.02 & 0.3 & 0.98 & 0.01 \\ 0.5 & 0.5 & 0.01 & 0.3 & 0 & 0.92 \end{pmatrix},$$

then this method shows A is a GDDM by using only $k = 1$ iteration for any $0 < \varepsilon \leq 0.5$.

Example 5. Let

$$A = \begin{pmatrix} 1 & -0.2 & -0.1 & -0.2 & -0.1 \\ -0.4 & 1 & -0.2 & -0.1 & -0.1 \\ -0.9 & -0.2 & 1 & -0.1 & -0.1 \\ -0.3 & -0.7 & -0.3 & 1 & -0.1 \\ -1 & -0.3 & -0.2 & -0.4 & 1 \end{pmatrix},$$

then this method shows A is a GDDM by using $k = 12$ iterations for any $0 < \varepsilon \leq 0.02$.

3. Gauss Form of the Algorithm

Obviously, this algorithm is an iterative method of the Jacobi type. If we use the Gauss type iteration, then the number of iteration will be further decreased.

Gauss-Seidel version of the new method:

A) For $i = 1, 2, ..., n$, do
  A1) Compute $S_i = \sum_{j \neq i} \mid a_{ij} \mid$.
  A2) If $i \in N_1$, then

$$d_i = \frac{S_i + \varepsilon}{\mid a_{ii} \mid + \varepsilon},$$

$$a_{ji} = a_{ji} * d_i, \quad j = 1, 2, ..., n\,,$$

    else $d_i = 1$.
B) Return to A) until all $d_i < 1$, $i = 1, 2, ..., n$.

Obviously, the positive diagonal matrix D satisfies $D = D^{(1)} \cdot D^{(2)} \cdots$, where

$$D^{(m)} = \text{diag}\,(d_1^{(m)}, ..., d_n^{(m)}), \qquad m = 1, 2, \ldots.$$

For the Example 5 shown in the above section, if we use this Gauss-Seidel type method, then we have only $k = 6$ iterations for any $0 < \varepsilon \leq 0.02$.

4. How to Decide the ε

From many computing examples, we find that when $\varepsilon \leq \varepsilon_0$, the number k of iteration attains minimum value, and be a constant after that, where ε_0 is a small real number. In the above examples 1 to 5, ε_0 will be 0.9, 0.1, 0.1, 0.5, 0.02 separately. For optional matrix, it is a difficult problem to find ε_0. But actually, because if $0 < \varepsilon \leq \varepsilon_0$, then $k = k_{min}$ becomes a constant, so we only need to find the smallest ε_{min} which can be discriminated by the using computer.

We assume that the smallest digit in the computer screen is 10^{-n} (generally, in the inside of computer, the number of digits is more than n). Let

$$\frac{S_i + \varepsilon_{min}}{| a_{ii} | + \varepsilon_{min}} \cdot | a_{ii} | - S_i > 10^{-n}, \qquad i \in N_1,$$

then

$$\varepsilon_{min} > \frac{10^{-n} | a_{ii} |}{| a_{ii} | - S_i - 10^{-n}}, \qquad i \in N_1.$$

So, we can take

$$\varepsilon_{min} = \min \left\{ \frac{10^{-n}(| a_{ii} | + 1)}{| a_{ii} | - S_i}, \quad i \in N_1 \right\}.$$

Generally, this ε_{min} can be used until stop the iteration, but as theoretical way, it is ideal if we decide $\varepsilon_{min}^{(k)}$, $k = 1, 2, ...$, based on above formula in every iteration step.

Note. From the results shown in [12], it is obvious that we can let $\varepsilon = 0$ when A is an irreducible metrix.

5. Distinguishing the Symmetric Positive Definite Matrices

Definition 1. Let A be $n \times n$ real symmetric matrix. If for any vector $x = (x_1, x_2, \ldots, x_n)^T \neq 0$, it always satisfies $x^T A x > 0$, then we call that A is a symmetric positive definite matrix.

In [7], we have shown that if $A = (a_{ij})$ is a symmetric matrix, $a_{ii} > 0$, $i = 1, 2, \ldots, n$, and A is a GDDM, then A surely is the symmetric positive definite. Therefore, we can distinguish A is a symmetric positive definite matrix by using the algorithm proposed in [5] or the Gauss form of this method above-mentioned.

Example 6. Let

$$A = \begin{pmatrix} 0.9 & 0.4 & -0.1 & -0.2 \\ 0.4 & 0.9 & -0.4 & 0.45 \\ -0.1 & -0.4 & 0.9 & 0.15 \\ -0.2 & 0.45 & 0.15 & 0.9 \end{pmatrix},$$

then by using the Gauss type algorithm, we know A is a symmetric positive definite matrix, where the number of iterations is $k = 3$ and for any $0 < \varepsilon \leq 0.1$ (when using the Jacobi type algorithm, it will be $k = 4$).

6. Distinguishing the M-Matrices

Definition 2. Let A be $n \times n$ real matrix, if A is satisfied with $a_{ii} > 0$, $i = 1, 2, \ldots, n$, $a_{ij} \leq 0$, $i \neq j$, $i, j = 1, 2, \ldots, n$, and $A^{-1} \geq 0$, then we call that A is an M-matrix.

It is well known that if A is satisfied with $a_{ii} > 0$, $i = 1, 2, \ldots, n$, $a_{ij} \leq 0$, $i \neq j$, $i, j = 1, 2, \ldots, n$, then A is an M-matrix if and only if A is a GDDM. Therefore, we can distinguish A is or not is an M-matrix by the algorithm proposed in [5] or the Gauss form of this method above-mentioned.

Example 7. Let

$$A = \begin{pmatrix} 0.9 & -0.1 & -0.05 & -0.05 & -0.4 & -0.1 \\ -0.1 & 1.05 & -0.05 & -0.2 & -0.1 & -0.1 \\ -0.1 & -0.2 & 0.9 & -0.2 & -0.2 & -0.2 \\ -0.1 & -0.2 & -0.4 & 0.7 & 0 & 0 \\ -0.5 & -0.4 & -0.02 & -0.3 & 0.98 & -0.01 \\ -0.5 & -0.5 & -0.01 & -0.3 & 0 & 0.92 \end{pmatrix},$$

then by using the Gauss type algorithm, we know A is an M-matrix, where the number of iterations is $k = 1$ and for any $0 < \varepsilon \leq 0.1$.

7. Distinguishing the Stability of Systems

Consider the following nonlinear autonomous system with separated variable type :

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} f_{ij}(x_j), \qquad f_{ij}(0) = 0, \qquad i, j = 1, 2, \ldots, n. \tag{1}$$

Theorem 1. (see [8]) *If the system (1) is satisfied with the following conditions, then its trivial solutions are global stable.*

1) $f_{ii}(x_i) \cdot x_i < 0$, *for any* $x_i \neq 0$, $i = 1, 2, \ldots, n$.

2) $\left| \dfrac{f_{ij}(x_j)}{f_{ii}(x_i)} \right| \leq a_{ij}, \qquad i \neq j, \quad i, j = 1, 2, \ldots, n.$

3) Matrix $A = (a_{ij})$ is a GDDM.

Theorem 2. (see [8]) *If the system (1) is satisfied with the following conditions, then its trivial solutions are global unstable.*

1) $f_{ii}(x_i) \cdot x_i < 0, \qquad i = 1, 2, \ldots, m-1,$
$f_{ii}(x_i) \cdot x_i > 0, \qquad i = m, m+1, \ldots, n, \qquad 1 \le m \le n, \qquad x_i \ne 0.$

2) $\mid \dfrac{f_{ij}(x_j)}{f_{ii}(x_i)} \mid \le a_{ij}, \qquad i \ne j, \quad i, j = 1, 2, \ldots, n.$

3) Matrix $A = (a_{ij})$ is a GDDM.

Next, we consider the following system which can be changed to a system with separated variable type:

$$\frac{dx_m}{dt} = f_m(\sum_{j=1}^{n} a_{mj} x_j), \qquad m = 1, 2, \ldots, n,$$

$$f_m(0) = 0, \qquad y_m \cdot f_m(y_m) > 0, \qquad \text{if} \quad y_m \ne 0\,. \tag{2}$$

Theorem 3. (see [8]) *If the system (2) is satisfied with following conditions, then its trivial solutions are global stable.*

1) $\displaystyle\int_0^{\pm\infty} f_j(y_j) \cdot dy_j = +\infty, \qquad j = 1, 2, \ldots, n.$

2) $\mid \dfrac{a_{ij}}{a_{jj}} \mid \le b_{ij}, \qquad i \ne j, \qquad i, j = 1, 2, \ldots, n.$

3) B is a GDDM.

Therefore, from Theorems 1, 2 and 3, we can distinguish the global stability for the nonlinear autonomous systems (1) and (2) by the algorithms proposed in above-mentioned.

8. Computing the Stable Quantity of GDDM

Definition 3. If an real matrix $A = (a_{ij})$ has the eigenvalues with only negative real part, then we say that the matrix A is stable. Further, if there exists a constant $h > 0$, which is satisfied with $\mathrm{Re}(\lambda_i(A)) < -h$, $i = 1, 2, \ldots, n$, then we say that A has the stable quantity h at least.

Theorem 4. (see [9]) *If A is satisfied with the following conditions, then A has stable quantity h at least.*

a) There exists a constant $h \geq 0$, *and* $a_{ii} + h < 0$, $i = 1, 2, \ldots, n$,

b) $A + hI$ *is a GDDM,*

So, we can computing the stable quantity for a GDDM by using the above-mentioned algorithms. In fact, when A *satisfies* $a_{ii} < 0$, $i = 1, 2, \ldots, n$, *the stable quantity* h *of* A *will be located in* $0 \leq h < \min\{-a_{ii}\}$. *So, we can computing* h *by the dichotomy on interval* $[0, \min\{-a_{ii}\}]$. *And in the every iteration, we call the algorithm to check that* $A + hI$ *is or is not a GDDM.*

Example 8. Let

$$A = \begin{pmatrix} -1 & 0.1 & -0.1 & 0.1 & 0.8 \\ 0.35 & -1 & 0.1 & -0.7 & 0.2 \\ 0.1 & -0.2 & -1 & 0.1 & 0.02 \\ 0.1 & 0.06 & 0.03 & -1 & 0.02 \\ 0.1 & -0.2 & 0.2 & -0.2 & -1 \end{pmatrix},$$

then we shall get the following interval sequences:

$$\begin{aligned} h \in [0,1) &\supseteq [0, 0.5] \supseteq [0.25, 0.5] \supseteq [0.25, 0.375] \supseteq [0.25, 0.3125] \\ &\supseteq [0.2812, 0.3125] \supseteq [0.2969, 0.3125] \supseteq [0.2969, 0.3047] \\ &\supseteq [0.2969, 0.3008] \supseteq \cdots \to 0.2988. \end{aligned}$$

So, we know the stable quantity of A is about $h = 0.2988$.

9. Estimating Distribution of Eigenvalues

Let A be a GDDM, from the above-mentioned algorithm, we can get a positive diagonal matrix D, where AD is the strictly diagonally dominant. Consider matrix

$$B = D^{-1}AD = (a_{ij} \cdot \frac{d_j}{d_i}).$$

Obviously, the eigenvalues of matrix A and B are same. Therefore, we can estimate distribution of eigenvalues of B by using some properties of eigenvalues for the strictly diagonally dominant matrix. So, we shall know the range of eigenvalues for the matrix A.

Theorem 5. (see [11]) *Let* $A = (a_{ij})$ *be a GDDM,* a_{ii} $(a_{ii} \neq 0)$, $i = 1, 2, \ldots, n$, *are all real numbers, and the numbers of positive and negative in* a_{ii}, $i = 1, 2, \ldots, n$, *are* r *and* s *separetely. Then the numbers of eigenvalues* $\lambda_i(A)$ *with positive real part and negative real part are also* r *and* s *separetely.*

So, from the above Theorem, we can know the numbers of eigenvalues $\lambda_i(A)$ with positive real part and negative real part separetely.

Example 9. Let

$$A = \begin{pmatrix} 1 & -0.8 & -0.1 \\ -0.5 & -1 & 0.3 \\ -0.8 & -0.6 & 1 \end{pmatrix},$$

then we know A is a GDDM by using the Gauss type method with only $k = 2$ iterations (let $\varepsilon = 0$). From $a_{11} > 0$, $a_{22} < 0$, $a_{33} > 0$, we have $\mathrm{Re}\{\lambda_1(A)\} > 0$, $\mathrm{Re}\{\lambda_2(A)\} < 0$, $\mathrm{Re}\{\lambda_3(A)\} > 0$.

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