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MINIMUM PRINCIPLE-TYPE OPTIMALITY CONDITIONS FOR PARETO PROBLEMS

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Abstract: We study necessary optimality conditions for Pareto problems with three kinds of constraints: inequality constraints, equality constraints and a set constraint. We suppose that the objective function and the inequality constraints are Hadamard (directionally) differentiable at the optimal solution and the equality constraints are continuous around and Fréchet differentiable at the optimal solution. We provide minimum principle necessary optimality conditions for a problem with a convex set constraint whose interior may be empty. Some constraint qualifications are considered to get Kuhn-Tucker conditions. We also provide minimum principle necessary optimality conditions for a problem with an arbitrary set constraint under a generalized Bender constraint qualification (GBCQ), which is automatically satisfied by the interior tangent cone and the cone of quasi-interior directions to the constraint set.

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1. Introduction

The minimum principle optimality conditions play a crucial role in the theory of optimization. These conditions are usually connected with scalar optimization problems with three types of constraints: equality constraints, inequality constraints and an arbitrary set constraint Q. These conditions are also known as "generalized Lagrange multiplier rule" because they extend the usual Fritz John conditions.

The classical multiplier rule usually requires that the objective function and the inequality constraints be Fréchet differentiable, the equality constraints be continuously Fréchet differentiable, and the set constraint be convex with nonempty interior (see, for example, Mangasarian [18]).

During the last decades, this multiplier rule was extended in many directions. Some of these extensions were given in the direction of weakening the smooth differentiability of the equality constraints (keeping Fréchet differentiability) and considering directional differentiability (in several senses) or Lipschitz continuity for the inequality constraints and the objective function. Also, mixed assumptions have been considered (see Ye [24]).

Further, many constraint qualifications have been studied to obtain Kuhn-Tucker conditions from the Fritz John conditions. Thus, for example, Di [7] considers a scalar problem with equality, inequality and a closed convex set constraints, where all the functions are Fréchet differentiable at the optimal solution and continuous on a neighborhood, and obtains Kuhn-Tucker conditions under an extended Mangasarian-Fromovitz constraint qualification. These results have been extended by Jiménez and Novo [16] weakening the differentiability.

In this paper, we focus on (multiobjective optimization) Pareto problems, whose main feature is that the equality constraints are differentiable at the optimal solution. In Section 2, we state the notation and some preliminary results. In Section 3, we provide minimum principle necessary optimality conditions for a problem with a convex set constraint whose interior may be empty. Some constraint qualifications are considered to get Kuhn-Tucker conditions. In Section 4, we also provide minimum principle necessary optimality conditions for a problem with an arbitrary set constraint by means of a convex subset of the Bouligand tangent cone and a generalized Bender constraint qualification (GBCQ). In particular, if we take the interior tangent cone or the cone of quasi-interior directions (assumed convexity), the constraint qualification is not needed. With additional requirements, Kuhn-Tucker conditions are deduced.

2. Notations and Preliminaries

Let x and y be two points of $\mathbb{R}^p$. Throughout this paper, we shall use the following notations:

$x \leq y$ if $x_i \leq y_i$, $i = 1, \ldots, p$; $x < y$ if $x_i < y_i$, $i = 1, \ldots, p$.

Let M be a subset of $\mathbb{R}^n$. As usual, $\operatorname{cl} M$, $\operatorname{int} M$, $\operatorname{ri} M$, $\operatorname{co} M$, $\operatorname{cone} M$ and $\operatorname{lin} M$ will denote the closure, interior, relative interior, convex hull, generated cone and linear span by M, respectively. $B(x_0, \delta)$ is the open ball centered at x_0 and radius $\delta > 0$.

Given a point $x_0 \in M$ and a function $f : \mathbb{R}^n \to \mathbb{R}^p$, the following multiobjective optimization problem is considered

$$\operatorname{Min}\{f(x) : \ x \in M\}.$$

The point x_0 is a weak Pareto minimum point, denoted $x_0 \in \operatorname{WMin}(f, M)$, if there exists no $x \in M$ such that $f(x) < f(x_0)$. The point x_0 is a local weak Pareto minimum point, written $x_0 \in \operatorname{LWMin}(f, M)$, if the previous condition is verified on $M \cap B(x_0, \delta)$, for some $\delta > 0$.

The following cones (Definition 2.1) and directional derivatives (Definition 2.4) are considered.

Definition 2.1. Let $M \subset \mathbb{R}^n$ and $x_0 \in \operatorname{cl} M$.

(a) The Bouligand tangent cone to M at x_0 is the set
$T(M, x_0) = \{v \in \mathbb{R}^n : \ \exists t_n \to 0^+, \ \exists x_n \in M \text{ such that } \dfrac{x_n - x_0}{t_n} \to v\}.$

(b) The cone of attainable directions is the set
$A(M, x_0) = \{v \in \mathbb{R}^n : \ \forall t_n \to 0^+, \ \exists x_n \in M \text{ such that } \dfrac{x_n - x_0}{t_n} \to v\}.$

(c) The interior tangent cone (or cone of interior directions) is the set

$$IT(M, x_0) = \{v \in \mathbb{R}^n : \ \exists \delta > 0 \text{ such that } x_0 + tu \in M \ \forall t \in (0, \delta] \ \forall u \in B(v, \delta)\}.$$

(d) The sequential interior tangent cone (or cone of quasi-interior directions) [9, Definition 6] is the set

$$IT_s(M,x_0) = \{v \in \mathbb{R}^n : \ \exists \delta > 0, \ \exists t_n \to 0^+ \text{ such that } \\ x_0 + t_n u \in M \ \forall n \in \mathbb{N} \ \forall u \in B(v,\delta)\}.$$

(e) The cone of linear directions (or radial tangent cone or cone of feasible directions) is the set
$Z(M,x_0) = \{v \in \mathbb{R}^n : \ \exists \delta > 0 \text{ such that } x_0 + tv \in M \ \forall t \in (0,\delta]\}$.

(f) The Clarke tangent cone is the set

$$T_{Cl}(M,x_0) = \{v \in \mathbb{R}^n : \ \forall t_n \to 0^+, \ \forall x_n \to x_0 \text{ with } x_n \in M, \\ \exists v_n \to v \text{ such that } y_n := x_n + t_n v_n \in M \ \forall n \in \mathbb{N}\}.$$

We have the following inclusion properties (see [2]).

Proposition 2.2. *Let M, M_1, M_2 be subsets of $\mathbb{R}^n$. Then:*

(i) $IT(M,x_0) \subset Z(M,x_0) \subset A(M,x_0) \subset T(M,x_0)$.

(ii) $IT(M,x_0) \subset IT_s(M,x_0) \subset T(M,x_0)$.

(iii) $T(M_1,x_0) \cap IT(M_2,x_0) \subset T(M_1 \cap M_2, x_0)$.

(iv) $A(M_1,x_0) \cap IT_s(M_2,x_0) \subset T(M_1 \cap M_2, x_0)$.

These cones have several other characterizations. We are interested in the following (see [9, Theorem 2]).

Proposition 2.3.

$$A(M,x_0) = \{v \in \mathbb{R}^n : \ \forall \delta > 0 \ \exists t_0 > 0 \ \forall t \in (0,t_0) \ \exists u \in B(v,\delta) \\ \text{such that } x_0 + tu \in M\}.$$

Let $D \subset \mathbb{R}^n$, the polar cone to D is $D^* = \{v \in \mathbb{R}^n : \ \langle v,x\rangle \leq 0 \ \forall x \in D\}$. The normal cone to M at x_0 is the polar of the Bouligand tangent cone: $N(M,x_0) = T(M,x_0)^*$. The Clarke normal cone to M at x_0 is the polar of the Clarke tangent cone: $N_{Cl}(M,x_0) = T_{Cl}(M,x_0)^*$. If M is a convex set, one has: $T_{Cl}(M,x_0) = T(M,x_0) = \operatorname{cl}\operatorname{cone}(M - x_0)$ and $N(M,x_0) = N_{Cl}(M,x_0) = (M - x_0)^*$.

Definition 2.4. Let $f : \mathbb{R}^n \to \mathbb{R}^p, \ x_0, v \in \mathbb{R}^n$.

(a) The Dini derivative (or directional derivative) of f at x_0 in the direction v is

$$Df(x_0, v) = \lim_{t \to 0^+} [f(x_0 + tv) - f(x_0)]/t.$$

(b) The Hadamard (Neustadt) derivative of f at x_0 in the direction v is

$$df(x_0, v) = \lim_{(t,u) \to (0^+, v)} [f(x_0 + tu) - f(x_0)]/t.$$

(c) f is Dini differentiable or directionally differentiable (resp. Hadamard differentiable) at x_0 if its Dini derivative (resp. Hadamard derivative) exists for all direction v. We say that f is Gâteaux differentiable at x_0 if it is Dini differentiable at x_0 with linear derivative.

We also will denote the derivatives by $Df(x_0)v = Df(x_0, v)$ and $df(x_0)v = df(x_0, v)$.

(d) If f is Fréchet differentiable at x_0, its Fréchet differential is denoted by $\nabla f(x_0)$.

(e) The Clarke derivative of a locally Lipschitz real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ at x_0 in the direction v is denoted by $f^0(x_0, v)$ and is defined as:

$$f^0(x_0, v) = \limsup_{(t,x) \to (0^+, x_0)} [f(x + tv) - f(x)]/t.$$

Remark 2.5. (1) The Hadamard derivative coincides with the Neustadt derivative, since the domain of f is finite-dimensional. This is not the case if the domain of f is not a finite-dimensional space.

(2) For locally Lipschitz mappings in normed spaces, Hadamard and Dini directional derivatives are equivalent (see [6, Proposition 1.3.2] and [23]).

(3) We say that f is Fréchet directionally differentiable at x_0 or that f is B-differentiable at x_0 (see Shapiro [23]) if f is Dini differentiable at x_0 and

$$f(x_0 + v) = f(x_0) + Df(x_0, v) + o(\|v\|),$$

i.e., if in the definition of $Df(x_0, v)$ the limit holds uniformly for all directions v.

If the domain of f is finite-dimensional, as in our case, we have:

(i) f Hadamard directionally differentiable implies f Fréchet directionally differentiable.

(ii) f Fréchet directionally differentiable and $Df(x,\cdot)$ continuous implies f Hadamard directionally differentiable.

Therefore, if the domain of f is finite-dimensional and f is locally Lipschitz the concepts of Hadamard, Dini and Fréchet directionally differentiability are equivalent.

For a Hadamard differentiable function $f:\mathbb{R}^n \to \mathbb{R}$ the Hadamard subdifferential can be defined (see [21]).

Definition 2.6. The Hadamard subdifferential of f at x_0 is the set

$$\partial_H f(x_0) = \{\xi \in \mathbb{R}^n : \ \langle \xi, v\rangle \le df(x_0,v) \ \forall v \in \mathbb{R}^n\}.$$

If $df(x_0,v)$ is convex in v, then there exists the classical subdifferential of the Convex Analysis of that function at $v=0$: $\partial df(x_0,\cdot)(0)$. This set is nonempty, compact and convex in $\mathbb{R}^n$ and $\partial_H f(x_0) = \partial df(x_0,\cdot)(0)$.

The generalized Jacobian of a locally Lipschitz function $f:\mathbb{R}^n \to \mathbb{R}^p$ at x_0 is given by

$$\partial_{Cl} f(x_0) = \mathrm{co}\{\lim_{n\to\infty} \nabla f(x_n) : \ x_n \to x_0, \ x_n \in \Omega_f\},$$

where Ω_f is the set of points for which f is Fréchet differentiable.

Consider now the following multiobjective optimization problem

$$\text{(MP)} \quad \mathrm{Min}\{f(x) : \ x \in S \cap Q\},$$

where

$$S = G \cap H, \ G = \{x \in \mathbb{R}^n : \ g(x) \le 0\}, \ H = \{x \in \mathbb{R}^n : \ h(x) = 0\},$$
$$f:\mathbb{R}^n \to \mathbb{R}^p, \quad g:\mathbb{R}^n \to \mathbb{R}^m, \quad h:\mathbb{R}^n \to \mathbb{R}^r,$$

and Q is an arbitrary set of $\mathbb{R}^n$.

Let f_i, $i \in I = \{1,\dots,p\}$, g_j, $j \in J = \{1,\dots,m\}$, h_k, $k \in K = \{1,\dots,r\}$ be the component functions of f, g and h, respectively. Given $x_0 \in S$, the set of active indexes at x_0 is $J_0 = \{j \in J : \ g_j(x_0) = 0\}$. The set of points "better" than x_0 is $F = \{x \in \mathbb{R}^n : \ f(x) \le f(x_0)\}$.

Hereafter we will suppose that the involved functions in (MP) are, at least, Dini differentiable at a point $x_0 \in S \cap Q$. We will consider the following cones:

$$C_0(G,x_0) = \{v \in \mathbb{R}^n : \ Dg_j(x_0,v) < 0 \ \forall j \in J_0\},$$
$$C(G,x_0) = \{v \in \mathbb{R}^n : \ Dg_j(x_0,v) \le 0 \ \forall j \in J_0\},$$
$$C_0(S,x_0) = C_0(G,x_0) \cap \operatorname{Ker} Dh(x_0,\cdot),$$

$C(S, x_0) = C(G, x_0) \cap \operatorname{Ker} Dh(x_0, \cdot)$,

$C_0(F, x_0) = \{v \in \mathbb{R}^n : \; Df_i(x_0, v) < 0 \; \forall i \in I\}$,

$C(F, x_0) = \{v \in \mathbb{R}^n : \; Df_i(x_0, v) \leq 0 \; \forall i \in I\}$,

($C(S, x_0)$ is called the linearized cone).

To obtain the Fritz John conditions for problem (MP), the next basic inclusion is necessary:

$$\operatorname{Ker} Dh(x_0) \subset T(H, x_0). \tag{2.1}$$

In the following result, equation (2.1) is proved under weak conditions of differentiability.

Theorem 2.7. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be continuous on a neighborhood of x_0 and Gâteaux differentiable at x_0 with nonnull derivative $Dh(x_0)$. Then*

$$\operatorname{Ker} Dh(x_0) \subset A(H, x_0) \subset T(H, x_0).$$

Proof. The second inclusion follows from Proposition 2.2(i). Let us prove the first one.

Let $v \in \operatorname{Ker} Dh(x_0)$ and $\delta > 0$ be arbitrary. Choose $w \in \mathbb{R}^n$ such that $Dh(x_0)w > 0$ and $\|w\| < \delta$. Then $Dh(x_0)(v + w) > 0$. Now, by the definition of Dini derivative, this last condition implies that $h(x_0 + t(v + w)) > 0$ for all $t \in (0, t_1)$ for some $t_1 > 0$.

Similarly, $Dh(x_0)(v - w) < 0$, so $h(x_0 + t(v - w)) < 0$ for all $t \in (0, t_2)$ for some $t_2 > 0$. As h is continuous on the segment $L_t = [x_0 + t(v - w), x_0 + t(v + w)]$ for all $t \in (0, t_0)$, with $t_0 = \operatorname{Min}\{t_1, t_2\}$, there exists $x_t \in L_t$ such that $h(x_t) = 0$, that is, $x_t \in H$. Now, x_t is described in the form $x_t = x_0 + t(v + w_t)$ with $w_t \in [-w, w]$, and so, $\|w_t\| < \delta$. Consequently, by Proposition 2.3, $v \in A(H, x_0)$. □

When in problem (MP) both equality and inequality constraints are present, $Q = \mathbb{R}^n$ (or $x_0 \in \operatorname{int} Q$) and the functions are continuously differentiable, we obtain the usual Fritz John conditions. These conditions are simply obtained from the inclusion $C_0(S, x_0) \subset T(S, x_0)$, which is deduced easily from (2.1).

If an arbitrary set constraint Q is incorporated, even if (2.1) holds, the condition

$$\operatorname{Ker} Dh(x_0) \cap T(Q, x_0) \subset T(H \cap Q, x_0) \tag{2.2}$$

may not hold. This last condition suffices to obtain the Fritz John conditions for (MP) if f, g and h are Hadamard differentiable, with continuous linear derivative at x_0 (Bender, [3, Theorem 5.1]). It must be noted that if a function $h : \mathbb{R}^n \to \mathbb{R}^r$ is Hadamard differentiable at x_0 with linear derivative, then h is Fréchet differentiable at x_0 (Flett [8, p. 266]).

Condition (2.2) is satisfied in the next case:

$h : \mathbb{R}^n \to \mathbb{R}^r$ is continuous on a neighborhood of x_0 and Fréchet differentiable at x_0, $Q \subset \mathbb{R}^n$ is a convex set and the following regularity condition holds at $x_0 \in H \cap Q$:

$$(R_1) \quad 0 \in \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + N(Q, x_0) \;\Rightarrow\; \nu = 0$$

(see Di, [7, Theorem 4.1] and [15, Theorem 3.2]; in both cases expression (2.2) is obtained with an improper inclusion, i.e., with an equality).

Condition (R_1) is equivalent (by Lemma 3.2(ii) in [14]) to:

$$(R_1') \quad 0 \in \operatorname{int} \nabla h(x_0)(Q - x_0).$$

We also obtain condition (2.2) for a nondifferentiable locally Lipschitz function.

Proposition 2.8. *If $h : \mathbb{R}^n \to \mathbb{R}^r$ is locally Lipschitz at x_0 and Hadamard differentiable at x_0, $Q \subset \mathbb{R}^n$ is an arbitrary set and the following regularity condition holds at $x_0 \in H \cap Q$:*

$$(R_2) \quad 0 \in \partial_{Cl}(\nu h)(x_0) + N_{Cl}(Q, x_0) \;\Rightarrow\; \nu = 0,$$

then

$$\operatorname{Ker} dh(x_0) \cap T(Q, x_0) = T(H \cap Q, x_0), \tag{2.3}$$

$$\operatorname{Ker} dh(x_0) \cap A(Q, x_0) = A(H \cap Q, x_0). \tag{2.4}$$

Proof. We only prove (2.3) because the proof of (2.4) is similar. We only have to prove the inclusion "$\subset$" because the reverse inclusion is always true since $T(H \cap Q, x_0) \subset T(H, x_0) \cap T(Q, x_0)$ and $T(H, x_0) \subset \operatorname{Ker} dh(x_0)$ is valid for Hadamard differentiable functions.

By Proposition 3.2 in Jourani [17], h is metrically regular at x_0 with respect to Q, i.e., there exist $\alpha > 0$ and $\delta > 0$ such that

$$d(x, h^{-1}(z) \cap Q) \leq \alpha \|h(x) - z\| \qquad \forall x \in Q \cap B(x_0, \delta),\; z \in B(h(x_0), \delta). \tag{2.5}$$

Take $v \in \operatorname{Ker} dh(x_0) \cap T(Q, x_0)$, then there exist sequences $x_n \in Q$ and $t_n \to 0^+$ such that $\lim_{n\to\infty}(x_n - x_0)/t_n = v$. Since h is Hadamard differentiable at x_0, we have $\lim_{n\to\infty}(h(x_n)-h(x_0)/t_n = dh(x_0,v) = 0$, that is, $\|h(x_n)-h(x_0)\| = o(t_n)$. From (2.5), it follows that

$$d(x_n, h^{-1}(0)\cap Q) \leq \alpha\|h(x_n) - 0\| = \alpha\|h(x_n) - h(x_0)\| = o(t_n).$$

Hence, there exists a sequence $y_n \in h^{-1}(0)\cap Q$ such that $\lim_{n\to\infty}(x_n - y_n)/t_n = 0$. Now,

$$\lim_{n\to\infty}(y_n - x_0)/t_n = \lim_{n\to\infty}((y_n - x_n)/t_n + (x_n - x_0)/t_n) = v,$$

therefore $v \in T(H\cap Q, x_0)$. □

Condition (R_2) is equivalent to

$$(R_2') \quad 0 \in \nu\partial_{Cl}h(x_0) + N_{Cl}(Q,x_0) \;\Rightarrow\; \nu = 0,$$

because $\nu\partial_{Cl}h(x_0) = \partial_{Cl}(\nu h)(x_0)$ (Clarke, [5, Theorem 2.6.6]). In (R_2') ν must be understand as a row vector.

Remark 2.9. As consequence of Proposition 2.8, taking $Q = \mathbb{R}^n$, if there is no $\nu \in \mathbb{R}^r \setminus \{0\}$ such that $0 \in \nu\partial_{Cl}h(x_0)$, then

$$\operatorname{Ker} dh(x_0) = A(H,x_0) = T(H,x_0).$$

Proposition 2.8 does not guarantee that $A(H\cap Q, x_0) = T(H\cap Q, x_0)$ as the following example shows.

Example 2.10. Let $M \subset \mathbb{R}^2$ be the set given by

$$M = \{(x,y):\; x\geq 0,\; y \geq 0,\; x+y = 2^{-n} \text{ for some } n = 0,1,2,\dots\} \cup \{(0,0)\}.$$

Here it is $T(M,(0,0)) = \mathbb{R}^2_+$, $A(M,(0,0)) = T_{Cl}(M,(0,0)) = \{(0,0)\}$.

Now let $Q \subset \mathbb{R}^3$ with $Q = M\times\mathbb{R}$ and $x_0 = (0,0,0)$. One has $T(Q,x_0) = \mathbb{R}^2_+\times\mathbb{R}$, $A(Q,x_0) = T_{Cl}(Q,x_0) = \{(0,0)\}\times\mathbb{R}$ and consequently $N_{Cl}(Q,x_0) = \mathbb{R}^2\times\{0\}$.

With $h:\mathbb{R}^3\to\mathbb{R}$ according to $h(x,y,z) = z$ (which is continuously differentiable such that $\partial_{Cl}h(x_0) = \{\nabla h(x_0)\} = \{(0,0,1)\}$, the assumption ($\mathrm{R}_2$) of Proposition 2.8 is fulfilled. Nevertheless, with $H = h^{-1}(0) = \mathbb{R}^2\times\{0\}$, it is

$$A(H\cap Q, x_0) = \{(0,0,0)\} \neq T(H\cap Q,x_0) = \mathbb{R}^2_+\times\{0\}.$$

3. Fritz John and Kuhn-Tucker Conditions with a Convex Set Constraint

Robinson [22, Theorem 3] obtains the Fritz John conditions for (MP) with a convex set constraint Q and in infinite spaces for Fréchet differentiable functions without using (apparently) any regularity condition. We say apparently because he requires one condition: the separation property [22, Definition 5]. If in problem (MP) the involved sets are finite-dimensional, we can deduce the "minimum principle type" necessary conditions of Robinson in the following way.

The next lemma will be used in the proof.

Lemma 3.1. *Suppose that g_j, $j \in J_0$ are Hadamard differentiable at x_0 and g_j, $j \in J \setminus J_0$ continuous at x_0 and Q an arbitrary subset of $\mathbb{R}^n$, then:*

(i) $C_0(G, x_0) \subset IT(G, x_0)$.

(ii) $C_0(G, x_0) \cap T(Q, x_0) \subset T(G \cap Q, x_0)$.

Proof. Part (i) is easily proved. Part (ii) follows from part (i) and Proposition 2.2(iii). □

Theorem 3.2. *Let $Q \subset \mathbb{R}^n$ be a convex set, $x_0 \in S \cap Q$, and assume the following:*

(a) h is continuous on a neighborhood of x_0 and Fréchet differentiable at x_0.

(b) f and g_j, $j \in J_0$, are Hadamard differentiable at x_0.

If $x_0 \in \mathrm{LWMin}(f, S \cap Q)$, we have:

(i) If (R_1) holds at x_0, then $C_0(S, x_0) \cap T(Q, x_0) \cap C_0(F, x_0) = \emptyset$.

(ii) If the Hadamard derivatives of f and g_j, $j \in J_0$, at x_0 are convex, then there exists $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^{J_0} \times \mathbb{R}^r$ such that $(\lambda, \mu) \geq 0$, $(\lambda, \mu, \nu) \neq 0$ and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_H f_i(x_0) + \sum_{j \in J_0} \mu_j \partial_H g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + T(Q, x_0)^*.$$

Proof. We only prove part (ii), as the proof of part (i) is contained in the proof of part (ii).

If $\nabla h_1(x_0), \ldots, \nabla h_r(x_0)$ are linearly dependent, the conclusion is evident. So, suppose that they are linearly independent. Then, we have $\mathrm{Ker}\, \nabla h(x_0) = T(H, x_0)$ by Theorem 3.3 of Di [7].

If $\operatorname{Ker}\nabla h(x_0) \cap \operatorname{ri}(Q - x_0) = \emptyset$, then by the Separation Theorem, there exists $b \in \mathbb{R}^n \setminus \{0\}$ such that

$$\langle b, x - x_0 \rangle \leq 0 \leq \langle b, y \rangle \quad \forall x \in Q,\ \forall y \in \operatorname{Ker}\nabla h(x_0). \tag{3.1}$$

It follows that $-b \in (\operatorname{Ker}\nabla h(x_0))^* = \operatorname{lin}\{\nabla h_1(x_0), \ldots, \nabla h_r(x_0)\}$. Hence, $-b = \sum_{k=1}^r \nu_k \nabla h_k(x_0)$ for some $\nu \in \mathbb{R}^r$, $\nu \neq 0$ since $b \neq 0$. From (3.1) we deduce that $b \in T(Q, x_0)^*$. Therefore, we have the conclusion taking $\lambda = 0$ and $\mu = 0$.

So, we can suppose that $\operatorname{Ker}\nabla h(x_0) \cap \operatorname{ri}(Q - x_0) \neq \emptyset$. This is equivalent to $0 \in \operatorname{ri}\nabla h(x_0)(Q - x_0)$ since $\operatorname{ri}\nabla h(x_0)(Q - x_0) = \nabla h(x_0)(\operatorname{ri}(Q - x_0))$. Assume that $\operatorname{int}\nabla h(x_0)(Q - x_0) = \emptyset$. As $\nabla h(x_0)(Q - x_0)$ is a convex set, there exists an hyperplane which contains it, that is, there exists $\nu \in \mathbb{R}^r$, $\nu \neq 0$, such that

$$\langle \nu, \nabla h(x_0)(x - x_0) \rangle = \sum_{k=1}^r \nu_k \nabla h_k(x_0)(x - x_0) = 0 \quad \forall x \in Q.$$

Taking $-b = \sum_{k=1}^r \nu_k \nabla h_k(x_0) \in T(Q, x_0)^*$ we conclude as before.

Thus, we can suppose that $0 \in \operatorname{int}\nabla h(x_0)(Q - x_0)$ (i.e., condition (R_1') holds). But this condition is equivalent to (R_1), which is condition (RC) of Theorem 3.2 in [15]. Using this theorem we obtain that

$$\operatorname{Ker}\nabla h(x_0) \cap T(Q, x_0) = T(H \cap Q, x_0). \tag{3.2}$$

Now, by applying successively Lemma 3.1(i) and Proposition 2.2, we have

$$\begin{aligned} C_0(G, x_0) \cap \operatorname{Ker}\nabla h(x_0) \cap T(Q, x_0) &\subset IT(G, x_0) \cap T(H \cap Q, x_0) \\ &\subset T(G \cap H \cap Q, x_0), \end{aligned}$$

that is, $C_0(S, x_0) \cap T(Q, x_0) \subset T(S \cap Q, x_0)$. Since x_0 is a local weak minimum and f is Hadamard differentiable at x_0, it is known that $T(S \cap Q, x_0) \cap C_0(F, x_0) = \emptyset$, and therefore

$$C_0(S, x_0) \cap T(Q, x_0) \cap C_0(F, x_0) = \emptyset.$$

From here, using Theorem 3.9 in [14] we obtain the conclusion. □

Remark 3.3. We want to stress that in Mangasarian-Fromovitz [19], Mangasarian [18], and Jahn [12, 13] it is explicitly required that $\operatorname{int} Q \neq \emptyset$ (besides the usual assumptions on Fréchet differentiability of f and g and the continuous differentiability of h).

The proof illustrates clearly the possible regularity conditions to impose for getting $(\lambda, \mu) \neq 0$ or $\lambda \neq 0$ (Kuhn-Tucker conditions).

Some Constraint Qualifications to Obtain Kuhn-Tucker Conditions

Consider the following conditions:

(1) (CQ_1) $C_0(S, x_0) \cap \operatorname{ri}(Q - x_0) \neq \emptyset$.

(2) (CQ_1') $C_0(S, x_0) \cap (Q - x_0) \neq \emptyset$ and $0 \in \operatorname{ri} \nabla h(x_0)(Q - x_0)$.

(3) (CQ_2) $C_0(S, x_0) \cap (Q - x_0) \neq \emptyset$ and $0 \in \operatorname{int} \nabla h(x_0)(Q - x_0)$.

(4) (CQ_2') $0 \in \sum_{j\in J_0} \mu_j \partial_H g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + T(Q, x_0)^*$, $\mu \geq 0 \Rightarrow (\mu, \nu) = (0, 0)$.

(5) (BCQ) $\operatorname{Ker} \nabla h(x_0) \cap T(Q, x_0) \subset T(H \cap Q, x_0)$. (6) (CC) The cone $D := \operatorname{cone} \operatorname{co}(\cup_{j\in J_0} \partial_H g_j(x_0)) + \operatorname{lin}\{\nabla h_k(x_0) : k = 1, \ldots, r\} + N(Q, x_0)$ is closed.

(7) (ACQ) $C(S, x_0) \cap T(Q, x_0) \subset T(S \cap Q, x_0)$.

(CQ_2') is named by some authors the no nonzero abnormal multiplier constraint qualification [24, Definition 4.2], (CQ_1) can be named generalized Mangasarian-Fromovitz constraint qualification, (BCQ) is the Bender constraint qualification [3, Condition (3)], (CC) is the closedness condition used in [14], and (ACQ) is the Abadie constraint qualification. (CQ_2') has been used by Jiménez and Novo to obtain equality in (ACQ) [16, Theorem 3.3]. Notice that when g is differentiable at x_0, (CQ_2') becomes the basic constraint qualification (B_2) of Di [7].

Proposition 3.4. *Let Q be a convex set, suppose that conditions (a), (b) of Theorem 3.2 hold and that f and g_j, $j \in J_0$, have convex Hadamard derivatives. We have the following relations:*

(i) (CQ_1) $\Leftrightarrow$ (CQ_1').

(ii) (CQ_2) $\Leftrightarrow$ (CQ_2').

(iii) (CC)+(ACQ) $\Rightarrow$ Kuhn-Tucker conditions.

(iv) (CQ_1) $\Rightarrow$ (CC). (v) (CQ_1)+(BCQ) $\Rightarrow$ (ACQ). (vi) (CQ_2) $\Rightarrow$ (CQ_1) and (BCQ).

Proof. (i) See Lemma 3.10 in [14].

(ii) See Theorem 3.9 and Remark 3.11(1) in [14].

(iii) It follows from Theorem 4.1 in [14]. (iv) It follows from Proposition 3.6 (taking into account part (i)) and Theorem 3.5, both of them in [14].

(v) From (BCQ) and Lemma 3.1+Proposition 2.2 it follows $C_0(S, x_0) \cap T(Q, x_0) \subset T(S \cap Q, x_0)$. But, ($CQ_1$) using [11, Proposition III.2.1.10] implies that $\operatorname{cl}[C_0(S, x_0) \cap$

$T(Q,x_0)] = C(S,x_0) \cap T(Q,x_0)$. Therefore, because $T(S \cap Q, x_0)$ is closed, we have (ACQ).

(vi) Taking the second part of (CQ_2) into account, we obtain (3.2), which is (BCQ), exactly as in the proof of Theorem 3.2. The implication (CQ_2) $\Rightarrow$ (CQ_1) is obvious, taking part (i) into account. □

Therefore, each of the conditions (CQ_2), (CQ_1)+(BCQ), (CC)+ (ACQ) is sufficient to establish the Kuhn-Tucker conditions, being (CC) + (ACQ) the weakest.

4. Fritz John and Kuhn-Tucker Conditions with an Arbitrary Set Constraint

Theorem 4.1. *Let $Q \subset \mathbb{R}^n$ be an arbitrary set, $x_0 \in S \cap Q$, $P \subset \mathbb{R}^n$ a convex set with $0 \in P$ and assume that conditions (a), (b) of Theorem 3.2 hold, that the Hadamard derivatives of f and g_j, $j \in J_0$ are convex and that the following regularity condition holds*

$$(GBCQ) \quad \operatorname{Ker} \nabla h(x_0) \cap P \subset T(H \cap Q, x_0).$$

If $x_0 \in \operatorname{LWMin}(f, S \cap Q)$, then there exists $(\lambda,\mu,\nu) \in \mathbb{R}^p \times \mathbb{R}^{J_0} \times \mathbb{R}^r$ such that $(\lambda,\mu) \geq 0$, $(\lambda,\mu,\nu) \neq 0$ and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_H f_i(x_0) + \sum_{j \in J_0} \mu_j \partial_H g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + P^*. \tag{4.1}$$

Proof. Using (GBCQ) and applying Lemma 3.1(ii), we have

$$\begin{aligned} C_0(G,x_0) \cap \operatorname{Ker} \nabla h(x_0) \cap P \subset C_0(G,x_0) \cap T(H \cap Q, x_0) \\ \subset T(G \cap H \cap Q, x_0), \end{aligned}$$

that is, $C_0(S,x_0) \cap P \subset T(S \cap Q, x_0)$. Since x_0 is a local weak minimum and f is Hadamard differentiable at x_0, it is known that $T(S \cap Q, x_0) \cap C_0(F,x_0) = \emptyset$, and therefore $C_0(S,x_0) \cap P \cap C_0(F,x_0) = \emptyset$, i.e., the system

$$df(x_0,v) < 0, \ dg_j(x_0,v) < 0 \ (j \in J_0), \ \nabla h(x_0)v = 0, \ v \in P$$

is incompatible in $v \in \mathbb{R}^n$. From here, using Theorem 3.9 in [14] we obtain the conclusion. □

Remark 4.2. We obtain the same conclusion (Fritz John conditions) for $P = IT(Q, x_0) \cup \{0\}$ and $P = IT_s(Q, x_0) \cup \{0\}$ without (GBCQ) (i.e., $IT(Q, x_0)$ and $IT_s(Q, x_0)$ are assumed to be convex).
In fact, if the gradients $\nabla h_1(x_0), \ldots, \nabla h_r(x_0)$ are linearly dependent, the conclusion is evident. If they are linearly independent, then, we have $\operatorname{Ker} \nabla h(x_0) = A(H, x_0) = T(H, x_0)$. Hence,

$$\operatorname{Ker} \nabla h(x_0) \cap IT(Q, x_0) = T(H, x_0) \cap IT(Q, x_0) \subset T(H \cap Q, x_0)\,,$$

by Proposition 2.2(iii). Also

$$\operatorname{Ker} \nabla h(x_0) \cap IT_s(Q, x_0) = A(H, x_0) \cap IT_s(Q, x_0) \subset T(H \cap Q, x_0)\,,$$

by Proposition 2.2(iv). In both cases, (GBCQ) holds and we can apply Theorem 4.1.

Remark 4.3. The Fritz John necessary optimality conditions in terms of the polar of the cone $IT(Q, x_0)$ have been obtained by Bazaraa and Goode [1] and in terms of the polar of the cone of quasi-interior directions by Giorgi and Guerraggio [10] and under milder differentiability assumptions by Jiménez and Novo [15].

If equality constraints are absent in problem (MP), we can obtain the necessary Fritz John-type conditions without imposing (GBCQ), being P any convex subcone of $T(Q, x_0)$ (see [14, Theorem 4.8]).

If Q is a convex set we have $IT(Q, x_0) = IT_s(Q, x_0) = \operatorname{cone}(\operatorname{int} Q - x_0)$. If, moreover, $\operatorname{int} Q \neq \emptyset$, then

$$IT(Q, x_0) = \operatorname{int} T(Q, x_0),$$
$$T(Q, x_0) = \operatorname{cl} IT(Q, x_0),$$

and so, if $\operatorname{int} Q \neq \emptyset$ (Q convex), we obtain the usual minimum principle-type optimality conditions of Mangasarian-Fromovitz stated in Section 3 under milder differentiability assumptions. However, if $\operatorname{int} Q = \emptyset$, we have $IT(Q, x_0) = \emptyset$, and so $IT(Q, x_0)^* = \mathbb{R}^n$ and condition (4.1) becomes trivially satisfied and not useful. This shows the opportunity for the analysis made in Section 3.

Another approach to problem (MP) is followed by Canon, Cullum and Polak [4], who obtain a Fritz John-type necessary condition, under the usual differentiability assumptions on f, g and h and by means of the following conical approximation. A convex cone $C(W, x_0)$ is called a *linearization of the first kind* of the set W at $x_0 \in W$ if for any finite collection $\{v_1, \ldots, v_k\}$ of linearly independent vectors in $C(W, x_0)$ there exists an $\varepsilon > 0$, possibly depending on x_0, $v_1, \ldots, v_k$, such that

$$\mathrm{co}\{x_0,\ x_0 + \varepsilon v_1, \ldots, x_0 + \varepsilon v_k\} \subset W.$$

If W is convex it can be easily proved that $C(W, x_0)$ becomes the cone of linear directions $Z(W, x_0)$. But the convexity of W entails the equality $Z(W, x_0) = \mathrm{cone}(W - x_0)$, so the minimum principle-type necessary conditions of Section 3 can be obtained. It must however be noted (see Palata [20]) that the linearization of the first kind cannot in general be compared with the cones $IT(W, x_0)$ and $IT_s(W, x_0)$, more used in optimization theory.

Next we proceed to get the Kuhn-Tucker conditions under the assumptions of the present section.

Theorem 4.4. *Let the assumptions of Theorem 4.1 be verified. If, moreover, h is locally Lipschitz, (R_2) holds, and the following constraint qualification holds: $C_0(S, x_0) \cap T_{Cl}(Q, x_0) \neq \emptyset$, then the Kuhn-Tucker conditions hold in* (4.1) *with $P^* = N_{Cl}(Q, x_0)$.*

Proof. By Proposition 2.8, we have

$$\mathrm{Ker}\,\nabla h(x_0) \cap T(Q, x_0) = T(H \cap Q, x_0).$$

Taking $P = T_{Cl}(Q, x_0) \subset T(Q, x_0)$, (GBCQ) in Theorem 4.1 is satisfied. Then (4.1) is verified with $P^* = N_{Cl}(Q, x_0)$, $(\lambda, \mu, \nu) \neq 0$, $(\lambda, \mu) \geq 0$.

On the other hand, (R_2) implies (R_1) because $\nabla h(x_0) \in \partial_{Cl} h(x_0)$, but (R_1) and $C_0(S, x_0) \cap T_{Cl}(Q, x_0) \neq \emptyset$ are equivalent to

$$(CQ_3)\ 0 \in \sum_{j \in J_0} \mu_j \partial_H g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + N_{Cl}(Q, x_0),\ \ \mu \geq 0 \quad \Rightarrow\ \ (\mu, \nu) = (0, 0)$$

(as in Proposition 3.4(ii)). Now, it is clear that $\lambda \neq 0$ in (4.1). □

Remark 4.5. (CQ_3) is implied by

$$(CQ_4)\ 0 \in \sum_{j \in J_0} \mu_j \partial_H g_j(x_0) + \nu \partial_{Cl} h(x_0) + N_{Cl}(Q, x_0),\ \ \mu \geq 0\ \ \Rightarrow \quad (\mu, \nu) = (0, 0).$$

Moreover, (CQ_4) $\Rightarrow$ (R_2) and $C_0(S, x_0) \cap T_{Cl}(Q, x_0) \neq \emptyset$.

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