YANG-MILLS CONNECTIONS IN
HOMOGENEOUS PRINCIPAL FIBRE BUNDLES

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\textbf{Abstract:} Let $K$ be a compact connected Lie-group of automorphisms of a principal fibre bundle $P(M,G)$ which acts fibre-transitively on $P$. We obtain a necessary and sufficient condition for a $K$-invariant connection in $P(M,G)$ to be a Yang-Mills connection, and give such examples.

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1. Introduction

Yang-Mills connections in a $G$-principal fibre bundle $P$ over a compact Riemannian manifold $M$ are the extrema of the Yang-Mills functional. In case of $\dim(M) = 4$, (anti-) self-dual connections are always Yang-Mills connections and the theory of (anti-) self-dual connections has been greatly developed. Nevertheless, to solve the Yang-Mills equation, itself, is still a difficult problem because of its non-linearity.

In this paper, for a compact connected Lie group $K$ of automorphisms of a
principal fibre bundle $P(M, G)$ which acts fibre-transitively on $P$, we obtain a necessary and sufficient condition for a $K$-invariant connection in $P(M, G)$ to be a Yang-Mills connection (cf. Theorem 4.1). Moreover as examples, we treat the case $K = SU(3)$ and $G = SU(2)$, and the base manifold $M = CP^2$ (cf. Proposition 5.2 and Theorem 5.4).

2. Yang-Mills Connections

For a compact Lie group $G$ with the Lie algebra $g$, let $P(M, G, \pi)$ be a principal $G$-bundle over a compact Riemannian manifold $(M, h)$. A $g$-valued 1-form $\omega$ on $P$ is called a connection (form) if it satisfies

$$\omega(A^*) = A, \quad A \in g,$$

where $A^*$ is a vector field on $P$ given by $A^*_u = d(u \cdot \exp(tA))/dt|_{t=0}$, $u \in P$, and the pull back $R_a^*\omega$ of $\omega$ by the action $R_a$, $a \in G$, of $G$ satisfies

$$R_a^*\omega = \text{Ad}(a^{-1})\omega, \quad a \in G.$$

We denote by $\mathcal{C}$ the set of all connections on $P$. Let $\{U_\alpha\}_{\alpha \in \mathcal{F}}$ be an open covering of $M$ with a family of isomorphisms $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times G$ and the corresponding family of transition functions $\psi_{\alpha\beta}: U_\alpha \cap U_\beta \to G$, $(\alpha, \beta \in \mathcal{F})$. For each $\alpha(\in \mathcal{F})$, let $\sigma_\alpha: U_\alpha \to P$ be the cross section on $U_\alpha$ defined by $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, 1_G)$, $x \in U_\alpha$, where $1_G$ is the identity of $G$. For each $\alpha(\in \mathcal{F})$, a $g$-valued 1-form $\omega_\alpha$ on $U_\alpha$ is defined by $\omega_\alpha := \sigma_\alpha^*\omega$. Then on each non-empty $U_\alpha \cap U_\beta$, $(\alpha, \beta \in \mathcal{F})$,

$$\omega = (L_{\psi_{\alpha\beta}})_*d\psi_{\alpha\beta} + \text{Ad}(\psi_{\alpha\beta}^*)\omega_\alpha. \quad (2.3)$$

Then, on $\pi^{-1}(U_\alpha)$, $(\alpha \in \mathcal{F})$,

$$\omega = \text{Ad}(s_{\alpha}^{-1})\pi^*\omega_\alpha + (L_{s_{\alpha}^{-1}})_*ds_{\alpha} \quad (\omega \in \mathcal{C}), \quad (2.4)$$

where $s_\alpha$ is the $G$-coordinate of the isomorphism $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times G$, $(\alpha \in \mathcal{F})$. The $g$-valued 2-form $\Omega^\omega$ on $P$, which is called the curvature form, is defined

$$\Omega^\omega = d\omega + (1/2)[\omega, \omega]. \quad (2.5)$$

On $\pi^{-1}(U_\alpha)$, $(\alpha \in \mathcal{F})$,

$$\Omega^\omega = \text{Ad}(s_{\alpha}^{-1})\pi^*\Omega_\alpha, \quad (2.6).$$
where $\Omega = d\omega + (1/2)[\omega, \omega]$. Fix an $\text{Ad}(G)$-invariant inner product $\langle , \rangle$ on $g$. Since the inner product $\langle \Omega^\alpha, \Omega^\alpha \rangle(u)$ depends only on the point $x = \pi(u)$ because of (2.2) and (2.6), the following functional called the Yang-Mills functional, is well defined:

$$\text{YM}(\omega) = (1/2) \int_M \langle \Omega^\alpha, \Omega^\alpha \rangle dv_h,$$

$$\text{YM}(\omega_\alpha) = (1/2) \int_M \langle \Omega_\alpha, \Omega_\alpha \rangle dv_h,$$  \hspace{1cm} (2.7)

where $dv_h$ is the volume element of $(M,h)$. A connection $\omega$ in $C$ is a Yang-Mills connection if it is a critical point of the Yang-Mills functional $\text{YM}$. The following theorem is well known (cf. [2,4]):

**Theorem 2.1.** A connection $\omega$ in $C$ is a Yang-Mills connection if and only if

$$\sum_{j=1}^m \left\{ (\nabla_{e_j} \Omega)_{\alpha}(e_j, e_i) + [\omega(\alpha), e_j, e_i] \right\} = 0 \quad (i = 1, 2, \cdots, m),$$  \hspace{1cm} (2.8)

where $\nabla$ is the Levi-Civita connection of $(M,h)$ and $\{e_i | i = 1, 2, \cdots, m\}$ is a local orthonormal frame field on $(M,h)$.

### 3. Invariant Connections in $P = K \times_{(\lambda, H)} G$

The situation of this paper is the following. Let $K$ be a compact connected Lie group acting on a principal fibre bundle $P(M,G,\pi)$ as a group of automorphisms which acts fibre-transitively on $P$, i.e., (i) each $k(\in K)$ is a diffeomorphism such that $k(ua) = k(u)a$ $(u \in P, a \in G)$, and (ii) for any two fibres of $P$, there is an element of $K$ which maps one fibre into the other. Every element $k$ of $K$ induces a transformation of $M$ in a natural manner because of (i), which is denoted by $\tau_k$. For an arbitrary fixed point $u_0$ in $P$, with the projection $\pi(u_0) = x_0$, let $H$ be the isotropy subgroup of $K$ at $x_0$, i.e., $H := \{ k \in K | \tau_kx_0 = x_0 \}$. Then $K/H = M$ and $x_0 = \{H\}$. Moreover $P = K \times_{(\lambda, H)} G$. In fact, the identification $\Psi$ of $P$ with $K \times_{(\lambda, H)} G$ is given by

$$\Psi(u) := [(k,a)] \quad (u = k(u_0)a \in P, \ k \in K, \ a \in G).$$  \hspace{1cm} (3.1)

This correspondence (3.1) is well defined, then $u_0 = [(1_K, 1_G)]$ and $\pi(u) = \pi[(k,a)] = \pi_0(k)$, $(u = k(u_0)a)$. In this paper, $\pi_0$ is the natural projection
of $K$ onto $K/H$. Let $\lambda$ be the holonomy representation of $H$ into $G$, i.e., $h(u_0) = u_0\lambda(h)$ ($h \in H$). Now let us recall a work of H.C. Wang (cf. [7]) in which he considered the $K$-invariant connections on $P$. A connection $\omega \in \mathcal{C}$ is $K$-invariant if the full back $k^*\omega$, ($h \in H$), coincides with $\omega$. We denote by $\mathcal{C}_K$ the set of all $K$-invariant connections of the principal fibre bundle $P(M,G,\pi)$. For every $\omega \in \mathcal{C}_K$, a linear map $\wedge$ of $k$ into $g$ is defined by $\wedge(X) := \omega_{u_0}(\tilde{X})$, $X \in k$, where $k$ is the Lie algebra of $K$ and $\tilde{X}$ is a vector field on $P$ defined by $\mathcal{X}_u := d(\exp tX)(u)/dt|_{t=0}$, ($u \in P$). (3.2)

Then, (cf. [7]),

$$\begin{align*}
\wedge(X) &= \lambda(X), \quad (X \in h), \quad \text{and} \\
\wedge(\text{Ad}(h)X) &= \text{Ad}(\lambda(h))(\wedge(X)), \quad (h \in H, X \in k).
\end{align*}$$

(3.3)

Since $K$ is compact, the Lie algebra $k$ of $K$ can be decomposed into a direct sum of the Lie algebra $h$ of $H$ and an $\text{Ad}(H)$-invariant subspace $m$ as vector spaces, that is, $k = h \oplus m$ and $\text{Ad}(H)m \subset m$. Then we have the following theorem.

**Theorem 3.1.** (cf. [7]) On the $G$-principal fibre bundle $P = K \times_{(\lambda,H)} G$ over $M$, the correspondence above $\omega \rightarrow \wedge$ gives a bijection between $\mathcal{C}_K$ and the set of all linear maps $\wedge$ satisfying

$$\wedge(m)(\text{Ad}(h)X) = \text{Ad}(\lambda(h))(\wedge(m)(X)) \quad (h \in H, X \in m),$$

(3.4)

and the curvature form $\Omega$ of the $K$-invariant connection defined by $\wedge(m)$ satisfies the following

$$2\Omega_{u_0}(\tilde{X},\tilde{Y}) = [\wedge(m)(X),\wedge(m)(Y)] - \wedge(m)([X,Y]_m) - \lambda([X,Y]_h),$$

(3.5)

where $\wedge(m)$ is the restriction of $\wedge$ to $m$, and $[X,Y]_m$ (resp. $[X,Y]_h$) denotes the $m$-component (resp. $h$-component) of $[X,Y] \in k$.

4. Main Results

We preserve the notations as in Section 2 and Section 3. For the $H$-principal fibre bundle $K(K/H,H,\pi_0)$, the following lemma (cf. [3, Lemma 4.1, p. 123]) is well known.

**Lemma 4.1.** There is a neighbourhood $V$ of 0 in the vector space $m$ which is mapped diffeomorphically under $\exp|_m$ and such that $\pi_0$ maps $N := \exp(V)$ diffeomorphically onto a neighbourhood $U$ of the point $\{H\}$ in $K/H$. 

Let $\sigma_0$ be a cross section of the neighbourhood $U$ of $\{H\}$ in Lemma 4.1 into $\pi^{-1}(U) (\subset K)$ which is defined by $\sigma_0(\pi_0(\exp X)) = \exp(X)$ $(X \in V)$. For each $u = [(k,a)] \in P, \pi(u) = \pi_0(k)$. For convenience in this paper, we denote by $U_\alpha$ the neighbourhood $U$ of $\{H\}$ $(\in M)$ in Lemma 4.1. Using the mapping $\sigma_0$, we can define a cross section $\sigma_\alpha$ of the neighborhood $U_\alpha$ into $\pi^{-1}(U_\alpha)(\subset P)$, which is defined by $\sigma_\alpha(\pi_0(\exp X)) := \exp X(u_0), (X \in V)$. Evidently, $\sigma_\alpha(x_0) = u_0$.

For the calculus, we define a vector field $X^*, X \in m = T_{\{H\}}M$, on the neighborhood $U_\alpha$ of $\{H\}$ in $K/H$ by

$$X^*_{xH} := (\tau_x)_\ast X \in T_{xH}(M), \quad x \in \exp(V) = N. \quad (4.1)$$

Let $\langle , \rangle$ be an inner product which is $\Ad(H)$-invariant on $m$. This inner product $\langle , \rangle$ determines a $K$-invariant Riemannian metric $h_{\langle , \rangle}$ on $K/H$. Let $\{X_i\}_{i=1}^m$ be an orthonormal basis on $(m, \langle , \rangle)$. Then $\{X_i^*\}_{i=1}^m$ is an orthonormal frame field on the neighborhood $U_\alpha$ of $\{H\}$ in $(K/H, h_{\langle , \rangle})$. Let $\{\theta_i^*\}_{j=1}^m$ be a system of 1-forms on $U_\alpha$ which is dual to $\{X_i^*\}_{i=1}^m$. Then, the Levi-Civita connection $\nabla$ of $(K/H,h_{\langle , \rangle})$ is given by

$$\nabla_X Y^* = (1/2)[X,Y]m + U(X,Y) \quad (X,Y \in m), \quad (4.2)$$

where $U(X,Y)$ is determined by

$$2 \langle U(X,Y), Z \rangle = \langle [Z,X]m, Y \rangle + \langle X, [Z,Y]m \rangle = \langle X, Y, Z \rangle. \quad (4.3)$$

For each $u \in \pi^{-1}(U_\alpha)$, there exist a unique pair $(X, a) \in (V \times G) \subset (m \times G)$ such that $u = [(\exp X, a)]$. A diffeomorphism $\psi_\alpha$ of $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times G$ is defined by

$$\psi_\alpha(u) = (\pi(u), a) = (\pi_0(\exp X), a), \quad (4.4)$$

for $u = \exp X(u_0) \in [(\exp X, a)], (X \in V$ and $a \in G)$. Then $\sigma_\alpha(\pi_0(\exp X)) = \psi_\alpha^{-1}(\pi_0(\exp X), 1_G) = (\exp X)(u_0), (X \in V)$. So, $s_\alpha(\sigma_\alpha(\tau_{\exp X}(\{H\}))) = 1_G, (X \in V)$. By virtue of (2.6), we have on $U_\alpha$

$$\Omega^\omega(\sigma_\alpha(X_i^*), \sigma_\alpha(X_j^*)) = \Omega(\alpha)(X_i^*, X_j^*). \quad (4.5)$$

Since $\sigma_\alpha(\pi_0(\exp tX_i)) = \exp(tX_i)(u_0)$ for sufficiently small $t$, $\widetilde{X}_i(u_0) = \sigma_\alpha(X_i^*_{\{H\}})$ for each $i$. Hence, by (2.6) we get on $U_\alpha$

$$(\sigma_\alpha^\ast \omega)(X_i^*_{\{H\}}) = \wedge m(X_i),$$

$$\Omega^\omega(\widetilde{X}_i, \widetilde{X}_j)(u_0) = \Omega(\alpha)(X_i^*, X_j^*)(x_0). \quad (4.6)$$
From now on, we use the following notations:

\{X_i\}_i: an orthonormal basis of \((\mathfrak{m}, \langle \cdot, \cdot \rangle)\), \{Y_a\}_a: a basis of \(\mathfrak{h}\)

\{E_\alpha\}_\alpha: an orthonormal basis of the Lie algebra \(\mathfrak{g}\) of the structure group \(G\) with respect to an \(\text{Ad}(G)\)-invariant inner product \(\langle \cdot, \cdot \rangle\),

\([X_i, X_j]_\mathfrak{m} =: \sum_k C_{ij}^k X_k, \quad [X_i, X_j]_\mathfrak{h} =: \sum_b C_{ij}^b Y_b,\)

\(d\lambda(Y_a) =: \lambda(Y_a) =: \sum_\beta \lambda_\alpha^\beta E_\beta, \quad [E_\alpha, E_\beta] =: \sum_\gamma G_{\alpha\beta}^\gamma E_\gamma,\)

\(U(X_i, X_j) =: \sum_k U_{ij}^k X_k, \quad \wedge \mathfrak{m} (X_j) =: \sum_\beta \wedge_j^\beta E_\beta,\)

\(\Omega_\alpha := \sum_{i,j,\beta} \Omega_{ij}^\beta (\theta^i \wedge \theta^j) \otimes E_\beta, \quad \Omega_\alpha(X_i^*, X_j^*) := \Omega_{ij}^\alpha \text{ on } U_\alpha,\)

\((\nabla X_i^*(H) \Omega_\alpha)(X_j^*, X_j^*) =: \nabla_k \Omega_{ji}^\alpha \text{ on } U_\alpha.\)

By (4.3), we get

\[ U_{ij}^k = (1/2)(C_{ki}^j + C_{kj}^i). \] 

(4.7)

We obtain by help of (4.2) and (4.7)

\[
\begin{align*}
(\nabla X_i^* X_j^*) &= (1/2) \sum_k (C_{ki}^j + C_{kj}^i + C_{ij}^k) X_k^*, \\
(\nabla X_i^* \theta^j) &= (-1/2) \sum_k (C_{ji}^k + C_{jk}^i + C_{ik}^j) \theta^k.
\end{align*}
\] 

(4.8)

Using (3.5) and (4.6), we obtain

\[ \Omega_{ij}^\alpha = (1/2)(\sum_\beta \wedge_i^\beta \wedge_j^\gamma G_{\beta \gamma}^\alpha - \sum_k C_{ij}^k \wedge_k^\alpha - \sum_\alpha C_{ij}^\alpha \lambda_\alpha^\alpha). \] 

(4.9)

By virtue of (4.6) and (4.8), we get

\[ \sum_j \nabla_j \Omega_{ji} = \sum_{j,k,\alpha} \{\Omega_{ik}^\alpha C_{kj}^j + (1/2)\Omega_{kj}^\alpha (C_{ki}^j + C_{kj}^i + C_{ij}^k)\} E_\alpha, \] 

(4.10)

\[ \sum_j [\wedge \mathfrak{m}(X_j), \Omega_{ji}] = (1/2) \sum_{j,\alpha,\beta,\delta} \wedge_j^\beta (\sum_{\gamma,\mu} \wedge_j^\gamma \wedge_i^\mu G_{\gamma \mu}^\delta - \sum_k C_{ji}^k \wedge_k^\delta - \sum_\alpha C_{ji}^\alpha \lambda_\alpha^\delta) G_{\beta \delta}^\alpha E_\alpha. \] 

(4.11)
Thus, by (4.10) and (4.11) we obtain the following theorem.

**Theorem 4.1.** Let $K$ be a compact connected Liegroup of automorphisms of $P(M,G)$ which acts fibre-transitively on $P$. Then a $K$-invariant connection in the principal fibre bundle $P(M,G)$ is a Yang-Mills connection if and only if

$$
\sum_{k,j} \{2\Omega_{ik}^\alpha C_{kj}^\beta + \Omega_{kj}^\alpha (C_{kj}^i + C_{ki}^j + C_{ji}^k)\} + \sum_{j,\beta,\delta} \wedge_j^\beta G_{\beta\delta}^\alpha \\
\times \left( \sum_{\gamma,\mu} \wedge_j^\gamma \wedge_i^\mu G_{\gamma\mu}^\delta - \sum_k C_{ji}^k \wedge_k^\delta - \sum_a C_{ji}^a \lambda_a^\delta \right) = 0. \quad (4.12)
$$

**Corollary 4.2.** Assume the base manifold $(M,h<>,>)$ in the principal fibre bundle $P(M,G)$ is symmetric. Then a $K$-invariant connection in the bundle $P(M,G)$ is a Yang-Mills connection if and only if

$$
\sum_{j,\beta,\delta} \wedge_j^\beta G_{\beta\delta}^\alpha \left( \sum_{\gamma,\mu} \wedge_j^\gamma \wedge_i^\mu G_{\gamma\mu}^\delta - \sum_a C_{ji}^a \lambda_a^\delta \right) = 0. \quad (4.13)
$$

## 5. Examples

We consider the case when $K = SU(3), H = S(U(1) \times U(2))$ and $G = SU(2)$. Note that $U(1) \times SU(2)$ is a double covering of $U(2)$ and $U(2)$ is isomorphic with $S(U(1) \times SU(2))$ by group homomorphisms

$$
U(1) \times SU(2) \longrightarrow U(2) \longrightarrow S(U(1) \times U(2)),
$$

which are given by

$$(e^{i\theta}, A) \text{ or } (e^{i(\theta+\pi)}, -A) \longmapsto e^{i\theta} A =: B \longmapsto \begin{pmatrix} \det(B^{-1}) & 0 \\ 0 & B \end{pmatrix}.\$$

If $l$ is an even integer, a group homomorphism $\lambda$ of $S(U(1) \times U(2))$ into $SU(2)$ via $e^{i\theta} A \longmapsto \text{diag}(e^{il\theta}, e^{-il\theta})$, $(e^{i\theta} \in U(1), \ A \in SU(2))$, is well defined.

Let $E_{ij}$ denote a square matrix of order 3 with the $(i,j)$-entry being 1, and all the other entries being 0. Then we put:

$$
X_1 := (1/\sqrt{12})(E_{12} - E_{21}), \quad X_2 := (\sqrt{-1}/\sqrt{12})(E_{12} + E_{21}),
$$
$$
X_3 := (1/\sqrt{12})(E_{13} - E_{31}), \quad X_4 := (\sqrt{-1}/\sqrt{12})(E_{13} + E_{31}),
$$
$$
Y_5 := (1/\sqrt{12})(E_{23} - E_{32}), \quad Y_6 := (\sqrt{-1}/\sqrt{12})(E_{23} + E_{32}),
$$
Y_7 := (\sqrt{-1}/\sqrt{12}) \operatorname{diag}(0,1,-1), \quad Y_8 := (\sqrt{-1}/6) \operatorname{diag}(-2,1,1).

Then \{Y_5, Y_6, Y_7, Y_8\}_R = \mathfrak{h}. Let B be the Killing form of \mathfrak{su}(n), i.e., 
B(X,Y) = \operatorname{Trace}(\text{ad}(X)\text{ad}(Y)) = 2n\operatorname{Trace}(XY) \ (X,Y \in \mathfrak{su}(n)).
We define an inner product \langle , \rangle on \mathfrak{su}(n) by
\[ \langle X,Y \rangle = -B(X,Y) = -2n \ \operatorname{Trace}(XY) \ (X,Y \in \mathfrak{su}(n)). \quad (5.1) \]

We put \{X_1, X_2, X_3, X_4\}_R =: \mathfrak{m}, and then \{\mathfrak{h}, \mathfrak{m}\} \subset \mathfrak{m}. Moreover, \{X_i\}_{i=1}^4 is an orthonormal basis of \langle \mathfrak{m}, \langle , \rangle \rangle.

Similarly, we put \begin{align*}
E_1 &:= (1/\sqrt{8})(E_{12} - E_{21}), \quad E_2 := (\sqrt{-1}/\sqrt{8})(E_{12} + E_{21}), \\
E_3 &:= (\sqrt{-1}/\sqrt{8})\operatorname{diag}(1,-1) \text{ in } \mathfrak{su}(2). \quad \text{Then, } \{E_1, E_2, E_3\} \text{ is an orthonormal basis of } \mathfrak{su}(2) \text{ with respect to the } \operatorname{Ad}(\mathfrak{su}(2))-\text{invariant inner product } \langle , \rangle \text{ which is induced by the Killing form } B \text{ on } \mathfrak{su}(2).
\end{align*}

By straightforward computations, we have:
\begin{align*}
C_{12}^7 &= (-\sqrt{12})^{-1}, \quad C_{12}^8 = (-1/2), \quad C_{13}^5 = (-\sqrt{12})^{-1}, \\
C_{14}^6 &= (-\sqrt{12})^{-1}, C_{23}^6 = (\sqrt{12})^{-1}, \quad C_{24}^5 = (-\sqrt{12})^{-1}, \\
C_{34}^7 &= (\sqrt{12})^{-1}, \quad C_{34}^8 = (-1/2), \quad \text{and the others are zero;} \\
G_{12}^3 &= G_{23}^1 = G_{31}^2 = (1/\sqrt{2}), \quad \text{and the others are zero;} \\
\lambda^3 &= (\sqrt{2}/3) \quad \text{and the others are zero.} \quad (5.2)
\end{align*}

Using Theorem 4.1 and Corollary 4.2, we have the following result.

**Proposition 5.1.** Let \( P \) be the principal fibre bundle 
\[ SU(3) \times_{(\lambda, \operatorname{SU}(U(1) \times U(2)))} SU(2) =: P_\lambda \]
over the Riemannian manifold \((CP^2, g_{< , >})\). Then, a \( SU(3) \)-invariant connection \( w \) in \( P_\lambda \) is a Yang-Mills connection if and only if
\begin{align*}
3 \sum_{j \not= k} & \{ \wedge k^2(\wedge j^1 \wedge j^2) + \wedge k^3(\wedge j^1 \wedge j^3) - \wedge k^1(\wedge j^2 + \wedge j^3)^2 \} \\
& = \begin{cases} \
\ell \wedge 2^2 & \text{if } k = 1, \\
-\ell \wedge 1^2 & \text{if } k = 2,
\end{cases}
\end{align*}
\begin{align*}
3 \sum_{j \not= k} & \{ \wedge k^1(\wedge j^1 \wedge j^2) + \wedge k^3(\wedge j^2 \wedge j^3) - \wedge k^2(\wedge j^1 + \wedge j^3)^2 \} \\
& = \begin{cases} \
-\ell \wedge 2^1 & \text{if } k = 1, \\
\ell \wedge 1^1 & \text{if } k = 2,
\end{cases}
\end{align*}
and

\[
\sum_{i(j \neq k)} \left[ \wedge_k^1(\wedge_j^1 \wedge_j^3) + \wedge_k^2(\wedge_j^2 \wedge_j^3) - \wedge_k^3\{(\wedge_j^1)^2 + (\wedge_j^2)^2\}\right] = 0,
\]

for each \( k (k = 1, 2, 3, 4) \). (5.3)

The Hodge star operator \( * \) satisfies \( *^2 = id \). In case of \( \text{dim}(M) = 4 \), if \( *\Omega^\nu = \Omega^\nu \) (resp. \( *\Omega^\nu = -\Omega^\nu \)), \( w \) is self-dual (resp. anti-self-dual), which is always a Yang-Mills connection.

By help of (4.9) and (5.2), we have:

\[
\begin{align*}
\Omega_{12}^1 &= c(\wedge_1^1 \wedge_2^3 - \wedge_1^3 \wedge_2^1), \quad \Omega_{12}^2 = c(\wedge_1^3 \wedge_2^1 - \wedge_1^1 \wedge_2^3), \\
\Omega_{13}^1 &= c(\wedge_1^2 \wedge_3^3 - \wedge_1^3 \wedge_3^2), \quad \Omega_{13}^2 = c(\wedge_1^3 \wedge_3^1 - \wedge_1^1 \wedge_3^3), \\
\Omega_{14}^1 &= c(\wedge_1^2 \wedge_4^3 - \wedge_1^3 \wedge_4^2), \quad \Omega_{14}^2 = c(\wedge_1^3 \wedge_4^1 - \wedge_1^1 \wedge_4^3), \\
\Omega_{23}^1 &= c(\wedge_2^1 \wedge_3^3 - \wedge_2^3 \wedge_3^1), \quad \Omega_{23}^2 = c(\wedge_2^3 \wedge_3^1 - \wedge_2^1 \wedge_3^3), \\
\Omega_{24}^1 &= c(\wedge_2^1 \wedge_4^3 - \wedge_2^3 \wedge_4^1), \quad \Omega_{24}^2 = c(\wedge_2^3 \wedge_4^1 - \wedge_2^1 \wedge_4^3), \\
\Omega_{34}^1 &= c(\wedge_3^1 \wedge_4^2 - \wedge_3^2 \wedge_4^1), \quad \Omega_{34}^2 = c(\wedge_3^2 \wedge_4^1 - \wedge_3^1 \wedge_4^3), \\
\Omega_{34}^3 &= c\{\wedge_3^1 \wedge_4^2 - \wedge_3^2 \wedge_4^1 + (\ell/3)\},
\end{align*}
\]

where \( c := (2\sqrt{2})^{-1} \). We obtain from (5.4) the following proposition.

**Proposition 5.2.** In the principal fibre bundle

\[
SU(3) \times_{\langle\lambda, S(U(1)) \times U(2)\rangle} SU(2),
\]

over \( CP^2 \), \( w \) is self-dual (resp. anti-self-dual) if and only if

\[
\begin{align*}
\wedge_1^2 \wedge_2^3 - \wedge_1^3 \wedge_2^2 &= \wedge_3^2 \wedge_4^3 - \wedge_3^3 \wedge_4^2 \quad (\text{resp. } \wedge_3^3 \wedge_4^2 - \wedge_3^2 \wedge_4^3), \\
\wedge_1^3 \wedge_2^1 - \wedge_1^2 \wedge_3^2 &= \wedge_3^3 \wedge_4^1 - \wedge_3^1 \wedge_4^3 \quad (\text{resp. } \wedge_3^1 \wedge_4^3 - \wedge_3^3 \wedge_4^1), \\
\wedge_1^1 \wedge_2^2 - \wedge_1^2 \wedge_2^1 &= \wedge_3^1 \wedge_4^2 - \wedge_3^2 \wedge_4^1 \quad (\text{resp. } \wedge_3^2 \wedge_4^1 - \wedge_3^1 \wedge_4^2 - (2\ell)/3), \\
\wedge_1^1 \wedge_3^\alpha - \wedge_1^\beta \wedge_3^\alpha &= \wedge_4^\alpha \wedge_2^\beta - \wedge_4^\beta \wedge_2^\alpha \quad (\text{resp. } \wedge_4^\beta \wedge_2^\alpha - \wedge_4^\alpha \wedge_2^\beta), \\
\wedge_1^\alpha \wedge_4^\beta - \wedge_1^\beta \wedge_4^\alpha &= \wedge_2^\alpha \wedge_3^\beta - \wedge_2^\beta \wedge_3^\alpha \quad (\text{resp. } \wedge_2^\beta \wedge_3^\alpha - \wedge_2^\alpha \wedge_3^\beta),
\end{align*}
\]

(5.5)
where \( \alpha, \beta = 1, 2, 3, \) and \( \alpha \neq \beta. \)

By Proposition 5.2, it can be shown that if \( (\wedge_{i}^{\alpha}) \) satisfies \( \wedge_{i}^{1} = \wedge_{i}^{2} = 0 \) for each \( i \) \((i = 1, 2, 3, 4), \) i.e., \( < \wedge m(X_{i}), E_{1} > = < \wedge m(X_{i}), E_{2} > = 0 \) for each \( i, \) the \( SU(3) \)-invariant connection \( w \) in \( P_{\lambda} \) corresponding to \( (\wedge_{i}^{\alpha}) \) is self-dual. Since a self-dual (or anti-self-dual) connection \( \omega \) is a Yang-Mills connection, combining Proposition 5.1 and Proposition 5.2 we get the following result.

**Proposition 5.3.** For a \( SU(3) \)-invariant connection in the principal fibre bundle \( P_{\lambda} \) over \( CP^{2} \), the sufficient and necessary conditions to be a Yang-Mills connection are

\[
< \wedge m(X_{i}), E_{1} >= < \wedge m(X_{i}), E_{2} >= 0 \quad \text{for each} \quad i \ (i = 1, 2, 3, 4). \quad (5.6)
\]

By virtue of Proposition 5.1 and Proposition 5.3, we obtain the following theorem.

**Theorem 5.4.** Let \( P_{\lambda} \) be the principal fibre bundle \( K \times_{(\lambda,H)} G, \) \((K := SU(3), \ H := S(U(1) \times U(2)), \ G := SU(2)), \) over \( (CP^{2}, g_{\langle , \rangle}) \). Then the sufficient and necessary conditions for a \( K \)-invariant connection \( \omega \) in \( P_{\lambda} \) to be a Yang-Mills connection are

\[
\sum_{j(j \neq k)} [\wedge_{k}^{2}(\wedge_{j}^{1}\wedge_{j}^{2}) + \wedge_{k}^{3}(\wedge_{j}^{1}\wedge_{j}^{3}) - \wedge_{k}^{1}\{(\wedge_{j}^{2})^{2} + (\wedge_{j}^{3})^{2}\}] \\
= \sum_{j(j \neq k)} [\wedge_{k}^{1}(\wedge_{j}^{1}\wedge_{j}^{2}) + \wedge_{k}^{3}(\wedge_{j}^{2}\wedge_{j}^{3}) - \wedge_{k}^{2}\{(\wedge_{j}^{1})^{2} + (\wedge_{j}^{3})^{2}\}] \\
= \sum_{j(j \neq k)} [\wedge_{k}^{1}(\wedge_{j}^{1}\wedge_{j}^{3}) + \wedge_{k}^{2}(\wedge_{j}^{2}\wedge_{j}^{3}) - \wedge_{k}^{3}\{(\wedge_{j}^{1})^{2} + (\wedge_{j}^{2})^{2}\}] \\
= 0, \quad (5.7)
\]

for each \( k \) \((k = 1, 2, 3, 4). \)

By Proposition 5.3, we derive the following corollary.

**Corollary 5.5.** Let \( \ell \) be a non-zero even integer. Then for each \( SU(3) \)-invariant connection \( \omega \) in \( P_{\lambda}, \) it is a Yang-Mills connection if and only if \( \omega \) is self-dual.

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References


