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AN APPROXIMATE ANALYTICAL SOLUTION TO A FAMILIAR CONJUGATE HEAT TRANSFER PROBLEM

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Abstract: An approximate analytical solution technique for partial differential equations is described and illustrated for a two-dimensional conjugate heat transfer problem. The steady-state heat transfer solution developed is based upon an enhanced version of the conservative averaging method [3] that ensures continuity of temperature and flux at all points in the primary conducting medium and thus circumvents apparent failings of the original method.

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1. Introduction

For heat transfer problems where exact analytical solutions are either impossible to determine or are impractical to develop, approximate analytical techniques

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can be useful for identifying key solution features. For example, Heggs and Stones [6] gave analytical solutions for 1D models of longitudinal and annular fin assemblies, and Manzoor [8] constructed a solution for a two-dimensional assembly by using a series truncation method. Buikis et al [5] and Malik [7] looked at two-dimensional models using finite-differences.

The aim of this work is to describe a modification to one particular approximate solution technique known as the conservative averaging method (CAM), [3]. For illustrative purposes we consider a two-dimensional steady-state model of a longitudinal rectangular fin attached to a plane wall (Figure 1(a)).

An assumption is made regarding the form of the temperature distribution in a direction perpendicular to (x-axis) or parallel to (y-axis) the primary surface (wall). An approximate solution is then constructed in such a way that the two-dimensional problem is transformed to a one-dimensional problem by averaging along the chosen axis. Buikis [3] presented a means of obtaining a one-dimensional model in y by equating the temperature distributions (averaged with respect to x) in the upper and lower parts of the wall (Figure 1(a)). However, temperature continuity did not appear to hold along the interface between the upper and lower parts of the wall. Buike et al [2] presented an improved method to solve the problem shown in Figure 1(a), while satisfying temperature and flux continuity conditions along the wall-fin interface. Buikis et al [4] solved the same conjugate problem by using a general exponential function to construct temperature profiles, but again temperature continuity within the primary surface was not satisfied.

Here we show how to construct an approximating function that differs from the previous proposals [3, 2, 4], and which ensures bi-directional temperature and flux continuity in the primary surface.

2. Model Problem

A two-dimensional steady-state model of a fin assembly is considered, comprising a rectangular fin attached to a plane wall (Figure 1(a)). By introducing the dimensionless variables

$$\theta_w = \frac{U_w - U_a}{U_b - U_a}, \quad \theta_f = \frac{U_f - U_a}{U_b - U_a}, \quad x = \frac{X}{P}, \quad y = \frac{Y}{P},$$

(P is the fin pitch) together with the Biot numbers $B_{11} = \alpha_b P/\lambda_w$, $B_{21} = \alpha_a P/\lambda_w$, $B_{22} = \alpha_a P/\lambda_f$, and aspect ratios $v = H/P$, $\gamma = S/P$, $\delta = D/P$, $\gamma_w = W/2P$, a typical dimensionless model describing heat conductive heat

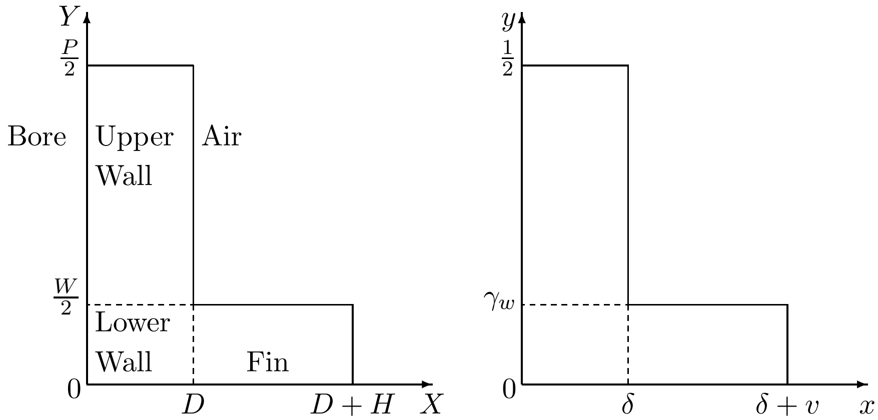

Figure 1: (a) Conjugate geometry (b) Dimensionless geometry

transfer in the wall and fin may be written down (Figure 1(b)),

$$\nabla^2\theta_w = 0, \quad 0 < x < \delta,\ 0 < y < 1/2, \tag{1}$$

$$\nabla^2\theta_f = 0, \quad \delta < x < \delta + v,\ 0 < y < \gamma_w. \tag{2}$$

Heat transfer across external surfaces (bore/wall, wall symmetry, upper fin, fin tip, and inter-fin space) is described by

$$\frac{\partial\theta_w}{\partial x} = B_{11}\left(\theta_w - 1\right), \quad x = 0,\ 0 \le y \le 1/2, \tag{3}$$

$$\frac{\partial\theta_w}{\partial y} = 0, \quad 0 < x \le \delta,\ y = 1/2, \tag{4}$$

$$\frac{\partial\theta_f}{\partial y} = -B_{22}\theta_f, \quad \delta < x \le \delta + v,\ y = \gamma_w, \tag{5}$$

$$\frac{\partial\theta_f}{\partial x} = -B_{22}\theta_f, \quad x = \delta + v,\ 0 \le y \le \gamma_w, \tag{6}$$

$$\frac{\partial\theta_w}{\partial x} = -B_{21}\theta_w, \quad x = \delta,\ \gamma_w \le y \le 1/2. \tag{7}$$

With perfect contact on the wall-fin interface, then

$$\lambda_w\frac{\partial\theta_w}{\partial x} = \lambda_f\frac{\partial\theta_f}{\partial x}, \quad \theta_w = \theta_f, \quad x = \delta,\ 0 \le y \le \gamma_w, \tag{8}$$

and symmetry along $y = 0$ implies

$$\frac{\partial \theta_w}{\partial y} = 0, \quad 0 \le x \le \delta; \quad \frac{\partial \theta_f}{\partial y} = 0, \quad \delta \le x \le \delta + v. \tag{9}$$

3. Conservative Averaging Method

CAM [3] constructs an approximate analytical solution to a two-dimensional problem by solving an associated one-dimensional problem obtained by averaging in a direction parallel to one of the coordinate axes (usually the x-axis or the y-axis). The one-dimensional problem will satisfy 'on average' the conservation law that governs the original problem. A solution is obtained that approximately describes the two-dimensional field. One advantage of the method in terms of the present problem is that numerical difficulties associated with the singular re-entrant corner are avoided.

To apply CAM the domain of the assembly under consideration is subdivided into three regions:

$$D_1 = \{(x,y): \ \delta \le x \le \delta + v, \ 0 \le y \le \gamma_w\} \quad \text{Fin}$$
$$D_2 = \{(x,y): \ 0 \le x \le \delta, \ \gamma_w \le y \le 1/2\} \quad \text{Upper Wall}$$
$$D_3 = \{(x,y): \ 0 \le x \le \delta, \ 0 \le y \le \gamma_w\} \quad \text{Lower Wall}$$

3.1. Fin

The dimensionless fin temperature is taken to be

$$\theta_f(x,y) = f_0(x) + \left[e^{y/\gamma_w} - 1\right] f_1(x) + \left[1 - e^{-y/\gamma_w}\right] f_2(x). \tag{10}$$

f_1 and f_2 are found in terms of f_0 from conditions (5) and (9),

$$f_1(x) = -\frac{B_{22}\gamma_w}{2\left[\sinh 1 + B_{22}\gamma_w\left(\cosh 1 - 1\right)\right]} f_0(x), \quad f_2(x) = -f_1(x),$$

and equation (10) then becomes

$$\theta_f(x,y) = \frac{\sinh 1 + B_{22}\gamma_w\left[\cosh 1 - \cosh(y/\gamma_w)\right]}{\sinh 1 + B_{22}\gamma_w\left[\cosh 1 - 1\right]} f_0(x). \tag{11}$$

The average temperature at any cross-section along the fin is

$$\overline{\theta}_f(x) = \frac{1}{\gamma_w}\int_0^{\gamma_w}\theta_f(x,y)\,dy \tag{12}$$

$$= \frac{\sinh 1 + B_{22}\gamma_w/e}{\sinh 1 + B_{22}\gamma_w\left[\cosh 1 - 1\right]} f_0(x). \tag{13}$$

To obtain f_0 we divide equation (2) by γ_w and integrate with respect to y from 0 to γ_w, using the definition (12), to give

$$\frac{d^2\overline{\theta}_f}{dx^2} + \frac{1}{\gamma_w}\left.\frac{\partial\theta_f}{\partial y}\right|_{y=0}^{y=\gamma_w} = 0. \tag{14}$$

Using equations (11) and (13) this equation may be expressed in terms of f_0,

$$\frac{d^2 f_0}{dx^2} - m_f^2 f_0 = 0, \quad \delta \le x \le \delta + v, \tag{15}$$

with $m_f^2 = B_{22}\sinh 1/\left[\gamma_w\left(\sinh 1 + B_{22}\gamma_w/e\right)\right]$. The tip condition (6) is now

$$\frac{df_0}{dx} = -B_{22} f_0, \quad x = \delta + v. \tag{16}$$

The analytical solution to equations (15) and (16) is

$$f_0(x) = c_1\left[e^{-m_f x} + q_f e^{m_f x}\right], \quad \delta \le x \le \delta + v, \tag{17}$$

where $q_f = \left[\left(m_f - B_{22}\right)/\left(m_f + B_{22}\right)\right]e^{-2m_f(\delta+v)}$. c_1 is an arbitrary constant determined from the wall-fin interface condition (8) (Appendix C).

3.2. Upper Wall

Combining the (assumed) upper wall temperature

$$\theta_u(x,y) = g_0(y) + \left[e^{x/\delta} - 1\right]g_1(y) + \left[1 - e^{-x/\delta}\right]g_2(y), \tag{18}$$

with conditions (3) and (7), gives g_1 and g_2 in terms of g_0,

$$g_1(y) = -\xi_u g_0(y) + \Theta_u, \quad g_2(y) = -g_1(y) + \delta B_{11}\left[g_0(y) - 1\right],$$

where

$$\xi_u = \frac{\delta}{2}\left[\frac{B_{21} + \delta B_{11}B_{21} + \left(1 - \delta B_{21}\right)B_{11}/e}{\sinh 1 + \delta B_{21}\left(\cosh 1 - 1\right)}\right],$$

$$\Theta_u = \frac{\delta B_{11}}{2}\left[\frac{\delta B_{21} + \left(1 - \delta B_{21}\right)/e}{\sinh 1 + \delta B_{21}\left(\cosh 1 - 1\right)}\right].$$

Equation (18) takes the form

$$\theta_u(x,y) = \left\{1 + \delta B_{11}\left[1 - e^{-x/\delta}\right] - 2\xi_u\left[\cosh(x/\delta) - 1\right]\right\} g_0(y)$$

$$+ \quad 2\Theta_u\left[\cosh(x/\delta) - 1\right] - \delta B_{11}\left[1 - e^{-x/\delta}\right], \tag{19}$$

and an average upper-wall temperature is defined as

$$\overline{\theta}_u(y) = \frac{1}{\delta}\int_0^\delta \theta_u(x,y)\,dx \tag{20}$$

$$= \phi_u g_0(y) + \psi_u, \quad \gamma_w \le y \le 1/2. \tag{21}$$

$\phi_u = 1 + \delta B_{11}/e - 2\xi_u\left[\sinh 1 - 1\right]$, $\psi_u = 2\Theta_u\left[\sinh 1 - 1\right] - \delta B_{11}/e$. g_0 is the solution of a second-order differential equation. On dividing equation (1) by δ and integrating with respect to x from 0 to δ, using definition (20), we obtain

$$\frac{d^2\overline{\theta}_u}{dy^2} + \frac{1}{\delta}\left.\frac{\partial\theta_u}{\partial x}\right|_{x=0}^{x=\delta} = 0, \quad \gamma_w \le y \le 1/2. \tag{22}$$

Combining this equation with equations (19) and (21) gives

$$\frac{d^2 g_0}{dy^2} - m_u^2 g_0 = -m_u^2 k_u, \quad \gamma_w \le y \le 1/2, \tag{23}$$

where

$$m_u^2 = \frac{\delta B_{11}\left(1 - \frac{1}{e}\right) + 2\xi_u \sinh 1}{\delta^2\phi_u}, \quad k_u = \frac{\delta B_{11}\left(1 - \frac{1}{e}\right) + 2\Theta_u \sinh 1}{\delta B_{11}\left(1 - \frac{1}{e}\right) + 2\xi_u \sinh 1}.$$

In terms of g_0 the boundary condition (4) can be written

$$\frac{dg_0}{dy} = 0, \quad y = 1/2. \tag{24}$$

The solution to equations (23) and (24) is

$$g_0(y) = c_2^*\left[e^{-m_u y} + e^{-m_u} e^{m_u y}\right] + k_u, \quad \gamma_w \le y \le 1/2, \tag{25}$$

$$= c_2\left[e^{-m_u(y-\gamma_w)} + q_u e^{m_u(y-\gamma_w)}\right] + k_u, \tag{26}$$

where $q_u = e^{-m_u\gamma}$, $\gamma_w = (1-\gamma)/2$ and c_2 is an arbitrary constant, determined by matching analytical solutions obtained for the three regions (D_1, D_2 and D_3) defined earlier (Appendix C).

3.3. Lower Wall

The solution for the lower wall region is taken to be

$$\theta_l(x,y) = h_0(x) + \left[e^{y/\gamma_w} - 1\right] h_1(x) + \left[1 - e^{-y/\gamma_w}\right] h_2(x). \tag{27}$$

The symmetry condition (9) requires $h_2(x) = -h_1(x)$ and therefore

$$\theta_l(x,y) = h_0(x) + 2\left[\cosh(y/\gamma_w) - 1\right] h_1(x). \tag{28}$$

For $0 \le x \le \delta$ a lower-wall average temperature is defined as

$$\overline{\theta}_l(x) = \frac{1}{\gamma_w}\int_0^{\gamma_w} \theta_l(x,y)\, dy \tag{29}$$

$$= h_0(x) + 2\left[\sinh 1 - 1\right] h_1(x). \tag{30}$$

It should be noted that there remain two undetermined functions, h_0 and h_1, since just one boundary condition has been enforced. At this point Buikis [3] considered a lower-wall solution along the x-axis and an average temperature with respect to x. Here we divide equation (1) by γ_w and integrate with respect to y from 0 to γ_w, using definition (29), to obtain

$$\frac{d^2\overline{\theta}_l}{dx^2} + \left.\frac{\partial \theta_l}{\partial y}\right|_{y=0}^{y=\gamma_w} = 0. \tag{31}$$

Combining this equation with equations (28) and (30) generates a differential equation involving both h_0 and h_1,

$$\frac{d^2 h_0}{dx^2} + 2\left[\sinh 1 - 1\right]\frac{d^2 h_1}{dx^2} + \frac{2}{\gamma_w^2}\sinh 1\, h_1 = 0, \quad 0 \le x \le \delta. \tag{32}$$

3.4. Matching Solutions

The constants c_1 (eqn(17)) and c_2 (eqn 26)), and functions h_0 and h_1 (eqn 32)) may be determined by enforcing temperature and flux continuity on the wall-wall interface,

$$\theta_l = \theta_u, \quad \frac{\partial \theta_l}{\partial y} = \frac{\partial \theta_u}{\partial y}, \quad 0 < x \le \delta,\ y = \gamma_w, \tag{33}$$

together with the remaining boundary conditions (8) and (3) at the wall-fin interface and lower-wall-bore surface, respectively. The analysis is detailed in

Appendices A, B and C, and results in the expressions

$$h_0(x) = c_3P(x) + c_2Q(x) + R(x), \quad h_1(x) = \frac{-h_0(x) + c_2T(x) + S(x)}{2\left[\cosh 1 - 1\right]},$$

$$c_1 = \frac{\phi_4 + c_2\phi_3 + c_3P(\delta)}{\phi_2}, \qquad c_2 = -\psi_1 - \psi_2 c_3,$$

$$c_3 = \frac{-\psi_1\left[\phi_u\left(q_u+1\right) - I_Q - \lambda_0\right] + k_u\phi_u + \psi_u - I_R}{I_P + \psi_2\left[\phi_u\left(q_u+1\right) - I_Q - \lambda_0\right]},$$

$$\psi_1 = \frac{\lambda_3\phi_2 + \lambda_1\phi_4}{\lambda_2\phi_2 + \lambda_1\phi_3}, \qquad \phi_2 = \frac{d_P\phi_2 + \lambda_1 P(\delta)}{\lambda_2\phi_2 + \lambda_1\phi_3}.$$

The functions P, Q, R, S and T are given in Appendix A and Appendix B. From equations (19), (26) and (28), the average wall temperature (with respect to y) is

$$\begin{aligned}
\overline{\theta}_w(x) &= 2\int_0^{1/2} \theta_w(x,y)\,dy \\
&= 2\left[\int_0^{\gamma_w} \theta_l(x,y)\,dy + \int_{\gamma_w}^{1/2} \theta_u(x,y)\,dy\right] \qquad (34)\\
&= 2\left[1 + \delta B_{11}\left[1 - e^{-x/\delta}\right] - 2\xi_u\left[\cosh(x/\delta) - 1\right]\right]\xi_3 \\
&+ \gamma\left[2\Theta_u\left[\cosh(x/\delta) - 1\right] - \delta B_{11}\left[1 - e^{-x/\delta}\right]\right] \\
&+ 2\gamma_w\left[h_0(x) + 2\left[\sinh 1 - 1\right]h_1(x)\right], \qquad (35)
\end{aligned}$$

where $\xi_3 = c_2\left[q_u\left[e^{m_u\gamma/2} - 1\right] - e^{-m_u\gamma/2} + 1\right]/m_u + k_u\gamma/2$.

4. Discussion

To illustrate the nature of the solutions developed, a mild steel assembly ($\lambda = 45$, in W/mK) is considered [9]. The geometry of the assembly ($D = 0.00368$, $H = 0.0127$, $W = 0.00089$, $S = 0.00539$, in m) is combined with surface heat transfer coefficients $\alpha_a = 533$ and $\alpha_b = 1343$ (in W/m^{2}K) and external temperatures $U_b = 121$ and $U_a = 26$ (in °C).

	Buikis [3]		Present
x/δ	Upper	Lower	Upper/Lower
0.0	72.53	75.90	73.33
0.1	72.03	75.33	72.83
0.2	71.57	74.60	72.38
0.3	71.16	73.71	71.97
0.4	70.78	72.65	71.61
0.5	70.45	71.41	71.27
0.6	70.14	69.97	70.97
0.7	69.87	68.32	70.70
0.8	69.62	66.45	70.45
0.9	69.40	64.33	70.23
1.0	69.20	61.94	70.02

Table 1: The wall temperature at $0 \leq x \leq \delta$, $y = \gamma_w$

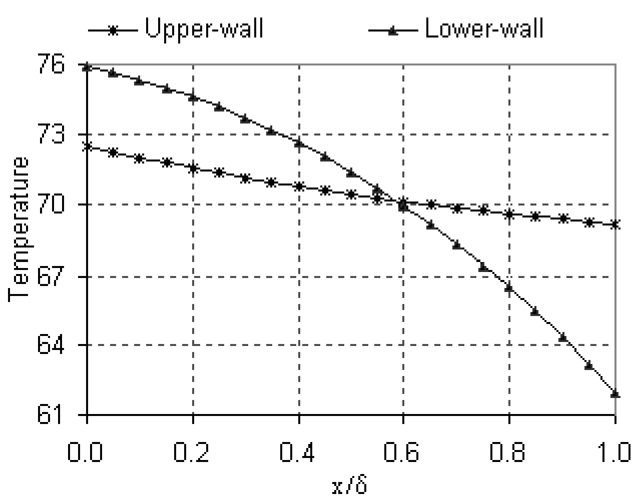

Figure 2: Temperature (in °C) on the upper/lower wall interface [3]

A comparison of the wall temperature obtained by Buikis [3] with the present work (Table 1) shows that to obtain θ_u and θ_l it is not sufficient to simply match the average temperatures in the upper and lower wall regions (Figure 2) at $y = \gamma_w$. Table 2 and Table 3 show that the temperature of the entire assembly is affected by the choice of matching criterion enforced along $y = \gamma_w$. In certain cases, such as the data used in this study, a 1D model may suffice

(Figure 3). In cases where dimensional effects are pronounced (requiring a 2D model), expressions (10), (18) and (27) may well be of practical use in avoiding, as they do, the difficulty associated with the re-entrant corner at $x = \delta$, $y = \gamma_w$ exhibited by numerical methods (e.g. finite differences).

	x							
y	0.000	0.195	0.391	0.586	0.586	1.260	1.934	2.608
0.500	74.168	72.706	71.652	70.887				
0.357	74.075	72.611	71.556	70.791				
0.214	73.797	72.326	71.268	70.505				
0.071	73.330	71.847	70.784	68.416				
0.071	73.330	71.847	70.784	68.416	68.416	48.132	38.813	35.842
0.047	73.349	71.861	70.785	68.504	68.475	48.163	38.831	35.855
0.024	73.360	71.869	70.786	68.623	68.509	48.181	38.841	35.863
0.000	73.363	71.872	70.786	68.753	68.519	48.186	38.844	35.866

Table 2: Temperature (in °C) for the assembly

	x							
y	0.000	0.195	0.391	0.586	0.586	1.260	1.934	2.608
0.500	73.400	71.918	70.857	70.096				
0.357	73.304	71.820	70.757	69.997				
0.214	73.016	71.524	70.458	69.700				
0.071	72.532	71.027	69.957	69.202				
0.071	75.900	73.378	68.893	61.941	69.202	48.542	39.050	36.024
0.047	75.821	73.296	68.809	61.857	69.262	48.574	39.069	36.038
0.024	75.773	73.247	68.760	61.807	69.296	48.591	39.079	36.046
0.000	75.757	73.231	68.743	61.790	69.307	48.597	39.082	36.048

Table 3: Assembly temperature (in °C) from Buikis [3]

5. Conclusions

A semi-analytical solution based upon the conservative averaging method is presented that ensures temperature and flux continuity at all points in both the primary surface and the fin, and therefore overcomes a weakness of the original technique [3].

A further advantage is that singularities at re-entrant corners pose no difficulties by virtue of having an analytic solution as opposed to a purely numerical solution in which, for example, matching techniques can be used [1].

None-the-less, the applicability of the method to (i) domains of a general geometry and (ii) higher (three) dimensional problems still needs to be assessed.

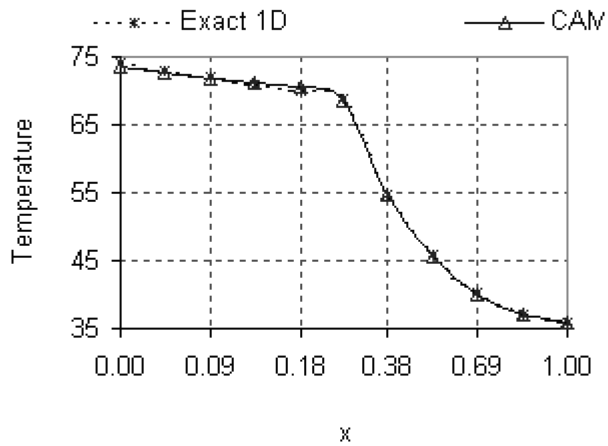

Figure 3: Temperature profiles from a 1D analytical solution [10] and the present 2D CAM solution (averaged with respect to y)

The complex nature of the analytical derivation process for the simple conjugate problem presented here does not bode well for general geometries.

Nomenclature

Symbol	Units	Description
B		Biot number
D	m	Wall depth
H	m	Fin height
$P = S + W$	m	Fin pitch
S	m	Fin spacing
U	°C	Temperature
v		Dimensionless fin height
W	m	Fin width
X, Y	m	Coordinates
x,y		Dimensionless coordinates
Greek		
α	W/m^{2}K	Surface heat transfer coefficient
λ	W/mK	Thermal conductivity
δ		Dimensionless wall depth
γ		Dimensionless fin spacing
θ		Dimensionless temperature
$2\gamma_w$		Dimensionless fin width
Subscript		
a		Air side
b		Bore side
f		Fin
l		Lower wall
u		Upper wall
w		Wall

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Appendix A: Upper/Lower Interface Temperature

We obtain h_1 in terms of h_0 by forcing temperature continuity, equation(33) at $y = \gamma_w$. Equations (19) and (26) (upper wall) and equation (28) (lower wall) give

$$h_1(x) = \frac{-h_0(x) + c_2 T(x) + S(x)}{2\left[\cosh 1 - 1\right]}, \tag{36}$$

$$\theta_l(x,y) = \frac{\cosh 1 - \cosh(y/\gamma_w)}{\cosh 1 - 1} h_0(x) + \left[\cosh(y/\gamma_w) - 1\right] \frac{c_2 T(x) + S(x)}{\cosh 1 - 1}, \tag{37}$$

where

$$\begin{aligned} T(x) &= (q_u + 1)\left[1 + \delta B_{11}\left[1 - e^{-x/\delta}\right] - 2\xi_u\left[\cosh(x/\delta) - 1\right]\right] \\ S(x) &= k_u\left[1 + \delta B_{11}\left[1 - e^{-x/\delta}\right] - 2\xi_u\left[\cosh(x/\delta) - 1\right]\right] \\ &+ 2\Theta_u\left[\cosh(x/\delta) - 1\right] - \delta B_{11}\left[1 - e^{-x/\delta}\right]. \end{aligned}$$

Appendix B: Bore - Lower Wall Surface

Using equation (36) to replace h_1 in equation (32), an ODE arises for h_0,

$$\frac{d^2h_0}{dx^2} - \beta_0^2 h_0 = E_1 e^{-x/\delta} + E_2 e^{x/\delta} + E_3, \tag{38}$$

where $\beta_0^2 = e \sinh 1/\gamma_w^2$ and

$$E_1 = c_2 D_{11} + D_{21} + E_2,\ E_2 = \frac{c_2 D_{12} + D_{22}}{2},\ E_3 = c_2 D_{13} + D_{23},$$

$$D_{11} = \frac{B_{11}\phi_1 (q_u + 1)}{\delta\gamma_w^2}, \quad D_{21} = \frac{B_{11}\phi_1 (k_u - 1)}{\delta\gamma_w^2},$$

$$D_{12} = \frac{2\xi_u \phi_1 (q_u + 1)}{\delta^2\gamma_w^2}, \quad D_{22} = \frac{2\phi_1 (\xi_u k_u - \Theta_u)}{\delta^2\gamma_w^2},$$

$$D_{13} = -e a_0 (q_u + 1) \frac{\beta_0^2}{e}, \quad D_{23} = [-a_0 k_u + 2\Theta_u + \delta B_{11}] \beta_0^2,$$

$$a_0 = 1 + \delta B_{11} + 2\xi_u, \quad \phi_1 = e\left[\left(\gamma_w^2 + \delta^2\right)\sinh 1 - \gamma_w^2\right].$$

The general solution of equation (38) is

$$h_0(x) = \frac{\delta^2}{1 - \delta^2\beta_0^2}\left[E_1 e^{-x/\delta} + E_2 e^{x/\delta}\right] - \frac{E_3}{\beta_0^2} + c_3 e^{\beta_0 x} + c_4 e^{-\beta_0 x}, \tag{39}$$

where c_3 and c_4 are constants to be determined. Utilising the average temperature $\overline{\theta}_l$, equation (30), the bore-side condition (3) is

$$\frac{dh_0}{dx} = B_{11}(h_0 - 1), \quad x = 0. \tag{40}$$

Combining equations (39) and (40) gives

$$\begin{aligned} c_4 \ &= \ \frac{\beta_0 - B_{11}}{\beta_0 + B_{11}} c_3 + \frac{B_{11}\left(c_2 D_{13} + D_{23} + \beta_0^2\right)}{\beta_0^2 (\beta_0 + B_{11})} \\ &- \ \frac{\delta\left[c_2 D_{11} + D_{21} + \delta B_{11}\left(c_2 (D_{11} + D_{12}) + D_{21} + D_{22}\right)\right]}{(\beta_0 + B_{11})\left(1 - \delta^2\beta_0^2\right)} \end{aligned}$$

and equation (39) takes the form

$$h_0(x) = c_3 P(x) + c_2 Q(x) + R(x), \quad 0 \le x \le \delta, \tag{41}$$

where c_2 and c_3 remain as unknown constants. The functions P, Q and R are defined as

$$\begin{aligned}
P(x) &= \frac{2}{\beta_0 + B_{11}} \left[\beta_0 \cosh(\beta_0 x) + B_{11} \sinh(\beta_0 x)\right], \\
Q(x) &= q_{11} e^{-\beta_0 x} + q_{12} e^{-x/\delta} + q_{13} \cosh(x/\delta) + q_{14}, \\
R(x) &= r_{11} e^{-\beta_0 x} + r_{12} e^{-x/\delta} + r_{13} \cosh(x/\delta) + r_{14},
\end{aligned}$$

with

$$q_{11} = \frac{-a_0 B_{11} \gamma_w^2 \left(q_u + 1\right) \left[e \left(\sinh 1 - 1\right) + 1\right]}{\left(\beta_0 + B_{11}\right) \left[\gamma_w^2 - \delta^2 e \sinh 1\right]},$$

$$q_{12} = \delta B_{11} \xi_2 \left(q_u + 1\right), \quad q_{13} = 2\xi_u \xi_2 \left(q_u + 1\right), \; q_{14} = a_0 \left(q_u + 1\right),$$

$$r_{11} = \frac{B_{11} \left(b_0 - a_0 k_u\right) \left(1 + \xi_2\right)}{\beta_0 + B_{11}},$$

$$r_{12} = \delta B_{11} \xi_2 \left(k_u - 1\right), \; r_{13} = 2\xi_2 \left(\xi_u k_u - \Theta_u\right),$$
$$r_{14} = k_u a_0 - \delta B_{11} - 2\Theta_u,$$

$$\xi_2 = \frac{e \left[\left(\gamma_w^2 + \delta^2\right) \sinh 1 - \gamma_w^2\right]}{\gamma_w^2 - \delta^2 e \sinh 1}, \quad b_0 = 1 + \delta B_{11} + 2\Theta_u.$$

Appendix C: Upper/Lower Interface Flux

For g_0 we consider flux continuity along the upper/lower wall interface as described by equation (33). By replacing θ_u from equation (19) and θ_l from equation (28), the average temperature flux with respect to x is

$$\phi_u \frac{dg_0}{dy} = \frac{2}{\delta \gamma_w} \sinh 1 \int_0^\delta h_1(x)\, dx, \quad y = \gamma_w.$$

With equations (26), (36) and (41), we can write

$$c_2 \left[\phi_u \left(q_u + 1\right) - I_Q - \lambda_0\right] - c_3 I_P = k_u \phi_u - \psi_u + I_R, \tag{42}$$

with

$$
\begin{aligned}
I_P &= \frac{2\left[\beta_0 \sinh(\beta_0\delta) + B_{11}\cosh(\beta_0\delta) - B_{11}\right]}{\delta\beta_0\left(\beta_0 + B_{11}\right)}, \\
I_Q &= \frac{q_{11}}{\delta\beta_0}\left[1 - e^{-\beta_0\delta}\right] + q_{12}\left[1 - 1/e\right] + q_{13}\sinh 1 + q_{14}, \\
I_R &= \frac{r_{11}}{\delta\beta_0}\left[1 - e^{-\beta_0\delta}\right] + r_{12}\left[1 - 1/e\right] + r_{13}\sinh 1 + r_{14}, \\
\lambda_0 &= \frac{m_u\phi_u\gamma_w\left(q_u + 1\right)\left[\cosh 1 - 1\right]}{\sinh 1}.
\end{aligned}
$$

Appendix D: Lower Wall - Fin Interface

From condition (8) (wall-fin flux continuity), on taking the average temperature with respect to y,

$$\frac{1}{\gamma_w}\int_0^{\gamma_w} \lambda_w \frac{\partial\theta_l}{\partial x}\, dy = \frac{1}{\gamma_w}\int_0^{\gamma_w} \lambda_f \frac{\partial\theta_f}{\partial x}\, dy, \quad x = \delta. \tag{43}$$

By enforcing continuity of the average temperature (with respect to y) at the wall-fin interface $x = \delta$, we have

$$\frac{1}{\gamma_w}\int_0^{\gamma_w} \theta_l(\delta, y)\, dy = \frac{1}{\gamma_w}\int_0^{\gamma_w} \theta_f(\delta, y)\, dy. \tag{44}$$

Replacing θ_f and θ_l from equations (11) and (37), equations (43) and (44) give

$$c_1\lambda_1 + c_2\lambda_2 + c_3 d_P = \lambda_3, \tag{45}$$

$$c_1\phi_2 - c_2\phi_3 - c_3 P(\delta) = \phi_4, \tag{46}$$

where

$$\begin{aligned}
\lambda_1 &= -\frac{\lambda_f}{\lambda_w} m_f k_1 e \left[-e^{-m_f\delta} + \xi_1 e^{m_f\delta}\right](\cosh 1 - 1),\\
\lambda_2 &= d_Q + e\,(q_u+1)\,[\sinh 1 - 1]\left[\frac{B_{11}}{e} - \frac{2\xi_u}{\delta}\sinh 1\right],\\
\lambda_3 &= d_Q + e\,[\sinh 1 - 1]\left[k_u\left[\frac{B_{11}}{e} - \frac{2\xi_u}{\delta}\sinh 1\right] - \frac{B_{11}}{e} + \frac{2\Theta_u}{\delta}\sinh 1\right],\\
k_1 &= \frac{\sinh 1 + B_{22}\gamma_w/e}{\sinh 1 + B_{22}\gamma_w\,[\cosh 1 - 1]},\\
d_P &= \frac{2\beta_0}{\beta_0 + B_{11}}\left[\beta_0 \sinh(\beta_0\delta) + B_{11}\cosh(\beta_0\delta)\right],\\
d_Q &= -\beta_0 q_{11} e^{-\beta_0\delta} - \frac{q_{12}}{\delta e} + \frac{q_{13}}{\delta}\sinh 1,\\
d_R &= -\beta_0 r_{11} e^{-\beta_0\delta} - \frac{r_{12}}{\delta e} + \frac{r_{13}}{\delta}\sinh 1,\\
\phi_2 &= e k_1\left(e^{-m_f\delta} + \xi_1 e^{m_f\delta}\right)[\cosh 1 - 1],\\
\phi_3 &= Q(\delta) + e\,[\sinh 1 - 1]\,T(\delta),\ \ \phi_4 \;=\; R(\delta) + e\,[\sinh 1 - 1]\,S(\delta).
\end{aligned}$$

On solving equations (42), (45) and (46), the arbitrary constants c_1, c_2 and c_3 are evaluated.