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OPTIMAL FILTERING FOR LINEAR SYSTEMS WITH STATE DELAY

Michael V. Basin[1 §], Jesus Rodriguez-Gonzalez[2]
Rodolfo Martinez-Zuniga[3]
[1,2]Department of Physical and Mathematical Sciences
Autonomous University of Nuevo Leon
Apartado Postal 144-F, C.P. 66450, San Nicolas de los Garza
Nuevo Leon, MEXICO
[1]e-mail: mbasin@fcfm.uanl.mx
[2]e-mail: jgrg17@yahoo.com.mx
[3]Department of Electrical and Mechanical Engineering
Autonomous University of Coahuila
Calle Barranquilla, S/N, Col. Guadalupe
Apartado Postal 189, C.P. 25750, Monclova
Coahuila, MEXICO
e-mail: rodolfomart62@hotmail.com

Abstract: In this paper, the optimal filtering problem for linear systems with state delay over linear observations is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate and the error variance. As a result, the optimal estimate equation similar to the traditional Kalman-Bucy one is derived; however, it is impossible to obtain a system of the filtering equations, that is closed with respect to the only two variables, the optimal estimate and the error variance, as in the Kalman-Bucy filter. The resulting system of equations for determining the error variance consists of a set of equations, whose number is specified by the ratio between the current filtering horizon and the delay value in the state equation and increases as the filtering horizon tends to infinity. In the example, performance of the designed

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§Correspondence author

optimal filter for linear systems with state delay is verified against the best Kalman-Bucy filter available for linear systems without delays.

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1. Introduction

The optimal filtering problem for linear system states and observations without delays was solved in 1960's [12], and this closed form solution is known as the Kalman-Bucy filter. However, the related optimal filtering problem for linear states with delay has not been solved in a closed form, regarding as a closed form solution a closed system of a finite number of ordinary differential equations for any finite filtering horizon. The optimal filtering problem for time delay systems itself did not receive so much attention as its control counterpart, and most of the research was concentrated on the filtering problems with observation delays (the papers [2, 13, 10, 16] could be mentioned to make a reference). On the other hand, the duality of the control and filtering problems for linear systems implies that the optimal state estimation for the system with state delays is related to the optimal LQR problem for systems with state delays, which was extensively studied using various approaches (see [9, 6, 3, 20, 1] and references therein). There also exists a considerable bibliography related to the robust control and filtering problems for time delay systems (such as [8, 17]). Comprehensive reviews of theory and algorithms for time delay systems are given in [15, 14, 18, 8, 7, 5].

In this paper, the optimal filtering problem for linear systems with state delay over linear observations is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate and the error variance [19]. As a result, the optimal estimate equation similar to the traditional Kalman-Bucy one is derived. However, it is impossible to obtain a system of the filtering equations, that is closed with respect to the only two variables, the optimal estimate and the error variance, as in the Kalman-Bucy filter. Thus, the resulting system of equations for determining the error variance consists of a set of equations, whose number is specified by the ratio between the current filtering horizon and the delay value in the state equation and increases as the filtering horizon tends to infinity.

Finally, performance of the designed optimal filter for linear systems with state delay is verified in the illustrative example against the best Kalman-Bucy filter available for linear systems without delays. The simulation results show a definite advantage of the designed optimal filter in regard to proximity of the estimate to the real state value and its asymptotic convergence.

The paper is organized as follows. Section 2 and Section 3 present the filtering problem statement for a linear system state with delay over linear observations and its solution, respectively. In Section 4, performance of the obtained optimal filter for linear systems with state delay is verified in the illustrative example against the best filter available for linear systems without delays. The simulation results show asymptotic convergence of the estimate given by the obtained optimal filter for linear systems with state delay to the real system state as time tends to infinity, whereas the conventional Kalman-Bucy estimates calculated without delay adjustment do not converge.

2. Filtering Problem for Linear State with Delay

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be independent Wiener processes. The partially observed F_t-measurable random process $(x(t), y(t))$ is described by a delay differential equation for the system state

$$dx(t) = (a_0(t) + a(t)x(t-h))dt + b(t)dW_1(t), \quad x(t_0) = x_0, \tag{1}$$

with the initial condition $x(s) = \phi(s)$, $s \in [t_0 - h, t_0]$, and a differential equation for the observation process:

$$dy(t) = (A_0(t) + A(t)x(t) + B(t)dW_2(t), \tag{2}$$

where $x(t) \in R^n$ is the state vector, $y(t) \in R^m$ is the observation process, $\phi(s)$ is a mean square piecewisecontinuous Gaussian stochastic process (see [19] for definition) given in the interval $[t_0 - h, t_0]$ such that $\phi(s)$, $W_1(t)$, and $W_2(t)$ are independent. The system state $x(t)$ dynamics $y(t)$ depends on the delayed state $x(t-h)$, where h is the delay shift, which actually makes the system state space infinite-dimensional (see, for example, [18]). The vector-valued function $a_0(s)$ describes the effect of system inputs (controls and disturbances). It is assumed that $A(t)$ is a nonzero matrix and $B(t)B^T(t)$ is a positive definite matrix. All coefficients in (1)–(2) are deterministic functions of appropriate dimensions.

The estimation problem is to find the estimate of the system state $x(t)$ based on the observation process $Y(t) = \{y(s), 0 \leq s \leq t\}$, which minimizes the Euclidean 2-norm

$$J = E[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))] \,,$$

at each time moment t. In other words, our objective is to find the conditional expectation

$$m(t) = \hat{x}(t) = E(x(t) \mid F_t^Y).$$

As usual, the matrix function

$$P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y]$$

is the estimate variance.

The proposed solution to this optimal filtering problem is based on the formulas for the Ito differentials of the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$, the error variance $P(t)$, and other bilinear functions of $x(t) - m(t)$ (cited after [19]) and given in the following section.

3. Optimal Filter for Linear State with Delay

In the situation of a state delay, the optimal filtering equations could be obtained using from the formula for the Ito differential of the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$ (see [19])

$$dm(t) = E(\varphi(x) \mid F_t^Y)dt + E(x[\varphi_1 - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y) \times \big(B(t)B^T(t)\big)^{-1}(dy(t) - E(\varphi_1(x) \mid F_t^Y)dt), \quad (3)$$

where $\varphi(x)$ is the drift term in the state equation equal to $\varphi(x) = a_0(t) + a(t)x(t-h)$ and $\varphi_1(x)$ is the drift term in the observation equation equal to $\varphi_1(x) = A_0(t) + A(t)x(t)$. Upon performing substitution into (3) and noticing that $E(x(t-h_i) \mid F_t^Y) = E(x(t-h_i) \mid F_{t-h_i}^Y) = m(t-h_i)$ for any $h > 0$, the estimate equation takes the form

$$dm(t) = (a_0(t) + a(t)m(t-h))dt + E(x(t)[A(t)(x(t) - m(t))]^T \mid F_t^Y) \times \big(B(t)B^T(t)\big)^{-1}(dy(t) - (A_0(t) + A(t)m(t)dt) = (a_0(t) + a(t)m(t-h))dt + P(t)A^T(t)\big(B(t)B^T(t)\big)^{-1}(dy(t) - (A_0(t) + A(t)m(t))dt). \quad (4)$$

The obtained form of the optimal estimate equation is similar to the Kalman filter one, except for the term $a(t)m(t-h)$. To compose a closed system of the filtering equations, the equation for the variance matrix $P(t)$ can be obtained using the formula for the Ito differential of the variance $P(t) = E((x(t) -$

$m(t))(x(t)-m(t))^T \mid F_t^Y)$ (cited again after [19]):

$$\begin{aligned} dP(t) = (E((x(t)-m(t))\varphi^T(x) \mid F_t^Y) \\ + E(\varphi(x)(x(t)-m(t))^T) \mid F_t^Y) + b(t)b^T(t) \\ - E(x(t)[\varphi_1 - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y)\big(B(t)B^T(t)\big)^{-1} \\ \times E([\varphi_1 - E(\varphi_1(x) \mid F_t^Y)]x^T(t) \mid F_t^Y))dt \\ + E((x(t)-m(t))(x(t)-m(t))[\varphi_1 - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y) \\ \times \big(B(t)B^T(t)\big)^{-1}(dy(t) - E(\varphi_1(x) \mid F_t^Y)dt). \end{aligned}$$

Here, the last term should be understood as a 3D tensor (under the expectation sign) convoluted with a vector, which yields a matrix. Upon substituting the expressions for φ and φ_1, the last formula takes the form

$$\begin{aligned} dP(t) = (E((x(t)-m(t))x^T(t-h)a^T(t) \mid F_t^Y) \\ + E(a(t)x(t-h)(x(t)-m(t))^T) \mid F_t^Y) + b(t)b^T(t) \\ - E(x(t)(x(t)-m(t))^T \mid F_t^Y)A^T(t)\big(B(t)B^T(t)\big)^{-1} \\ \times [A(t)E((x(t)-m(t))x^T(t)) \mid F_t^Y))dt \\ + E((x(t)-m(t))(x(t)-m(t))(A(t)(x(t)-m(t))^T \\ \times \big(B(t)B^T(t)\big)^{-1}(dy(t) - (A_0(t) + A(t)m(t))dt). \end{aligned}$$

Taking into account that the matrix $P_1(t) = E((x(t)-m(t))(x(t-h))^T \mid F_t^Y)$ in the first two right-hand side terms of the last formula is not equal to $P(t) = E(x(t)(x(t)-m(t))^T \mid F_t^Y)$, the equation for $P(t)$ should be represented as

$$\begin{aligned} dP(t) = (P_1(t)a^T(t) + a(t)P_1^T(t) + b(t)b^T(t) \\ - P(t)A^T(t)\big(B(t)B^T(t)\big)^{-1}A(t)P(t))dt \\ + E((x(t)-m(t))(x(t)-m(t))(x(t)-m(t)) \mid F_t^Y)A^T(t) \\ \times \big(B(t)B^T(t)\big)^{-1}(dy(t) - (A_0(t) + A(t)m(t))dt). \end{aligned}$$

The last term in this formula contains the conditional third central moment $E((x(t)-m(t))(x(t)-m(t))(x(t)-m(t)) \mid F_t^Y)$ of $x(t)$ with respect to observations, which is equal to zero, because $x(t)$ is conditionally Gaussian, in view of Gaussianity of the noises and the initial condition and linearity of the state and observation equations. Thus, the entire last term is vanished and the following variance equation is obtained

$$dP(t) = (P_1(t)a^T(t) + a(t)P_1^T(t) + b(t)b^T(t) - P(t)A^T(t)\big(B(t)B^T(t)\big)^{-1}A(t)P(t))dt. \quad (5)$$

However, the obtained system (4), (5) is not yet a closed system with respect to the variables $m(t)$ and $P(t)$, since the variance equation (5) depends on the unknown matrix $P_1(t)$. Thus, the equation for the matrix $P_1(t)$ should be obtained proceeding from its definition as $E((x(t) - m(t))(x(t-h))^T \mid F_t^Y)$, which implies that $P_1(t) = E(x(t)(x(t-h))^T \mid F_t^Y) - m(t)(m(t-h))^T$. Based on the equation (1) for $x(t)$ (and, therefore, for $x(t-h)$) and the equation (5) for $m(t)$ (and, therefore, $m(t-h)$) the following formula is obtained for the Ito differential of $P_1(t)$

$$\begin{aligned} dP_1(t) = {} & E((a_0(t) + a(t)x(t-h))(x(t-h))^T \mid F_t^Y)dt \\ & - (a_0(t)(m(t-h))^T + a(t)m(t-h)(m(t-h))^T)dt \\ & + E(x(t)(a_0(t-h) + a(t-h)x(t-2h))^T \mid F_t^Y)dt \\ & - (m(t)a_0^T(t-h) + m(t)(a(t-h)m(t-2h))^T)dt \\ & + \frac{1}{2}(b(t)b^T(t-h) + b(t-h)b^T(t))dt \\ & - P(t)A^T(t)(B(t)B^T(t))^{-1}(B(t)B^T(t-h)) \\ & \times (B(t-h)B^T(t-h))^{-1}A(t-h)P(t-h)dt, \end{aligned}$$

where the third order term, which is equal to zero in view of conditional Gaussianity of $x(t)$ (as well as in the equation (5) for $P(t)$), is omitted. Upon denoting $P_2(t) = E((x(t)-m(t))(x(t-2h))^T \mid F_t^Y)$, the last equation takes the form

$$\begin{aligned} dP_1(t) = {} & (a(t)P(t-h) + P_2(t)a^T(t-h))dt \\ & + \frac{1}{2}(b(t)b^T(t-h) + b(t-h)b^T(t))dt \\ & - P(t)A^T(t)(B(t)B^T(t))^{-1} \\ & \times (B(t)B^T(t-2h))(B(t-2h)B^T(t-2h))^{-1}A(t-h)P(t-h)dt. \quad (6) \end{aligned}$$

Adding the equation (6) to the system (4), (5) does not result yet in a closed system of the filtering equations, since the equation (6) depends on the

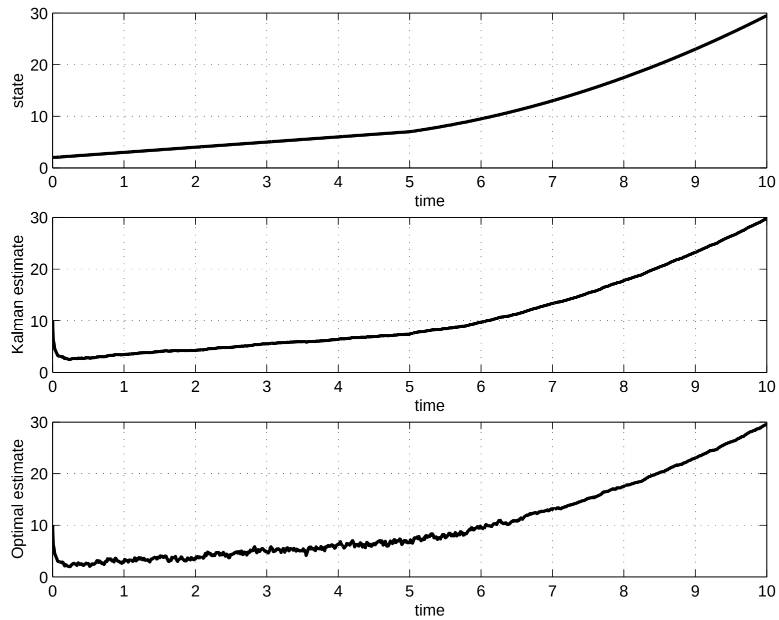

Figure 1: Graphs of the reference state variable $x(t)$ and the estimates $m_K(t)$ and $m(t)$ on the entire simulation interval $[0, 10]$

unknown matrix $P_2(t)$. The equation for the matrix $P_2(t)$ is obtained directly form the equation (6) by changing h to $2h$ in the definition of $P_1(t)$:

$$\begin{aligned} dP_2(t) = (a(t)P_1(t-h) + P_3(t)a^T(t-2h))dt \\ + \frac{1}{2}(b(t)b^T(t-2h) + b(t-2h)b^T(t))dt \\ - P(t)A^T(t)(B(t)B^T(t))^{-1}(B(t)B^T(t-2h)) \\ \times (B(t-2h)B^T(t-2h))^{-1}A(t-2h)P(t-2h)dt. \quad (7) \end{aligned}$$

The equation (7) for $P_2(t)$ depends on the unknown matrix $P_3(t) = E((x(t)-m(t))(x(t-3h))^T \mid F_t^Y)$, the equation for $P_3(t)$ will depend on $P_4(t) = E((x(t)-m(t))(x(t-4h))^T \mid F_t^Y)$, and so on. Thus, to obtain a closed system of the filtering equations for the state (1) over the observations (2), the following equations for matrices $P_i(t) = E((x(t)-m(t))(x(t-ih))^T \mid F_t^Y)$, $i \geq 1$, should be

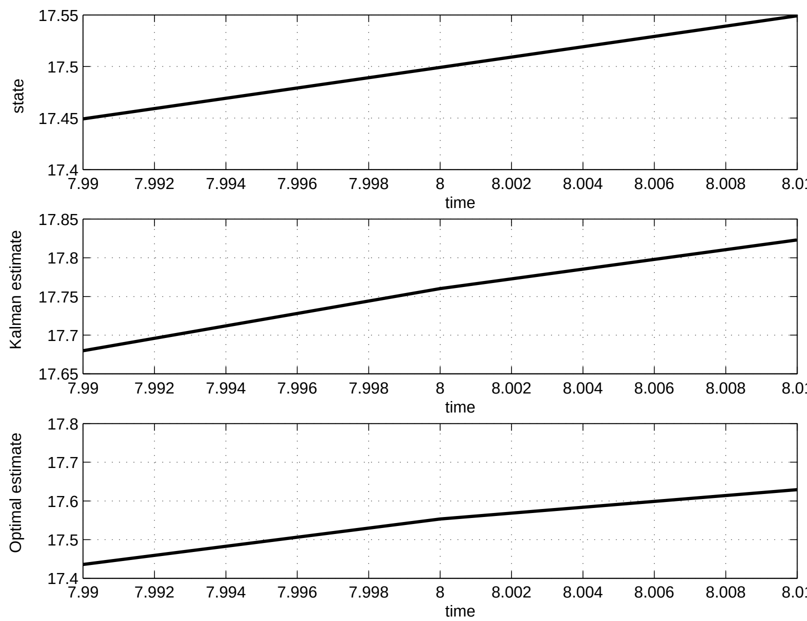

Figure 2: Graphs of the reference state variable $x(t)$ and the estimates $m_K(t)$ and $m(t)$ around the reference time point $T = 8$

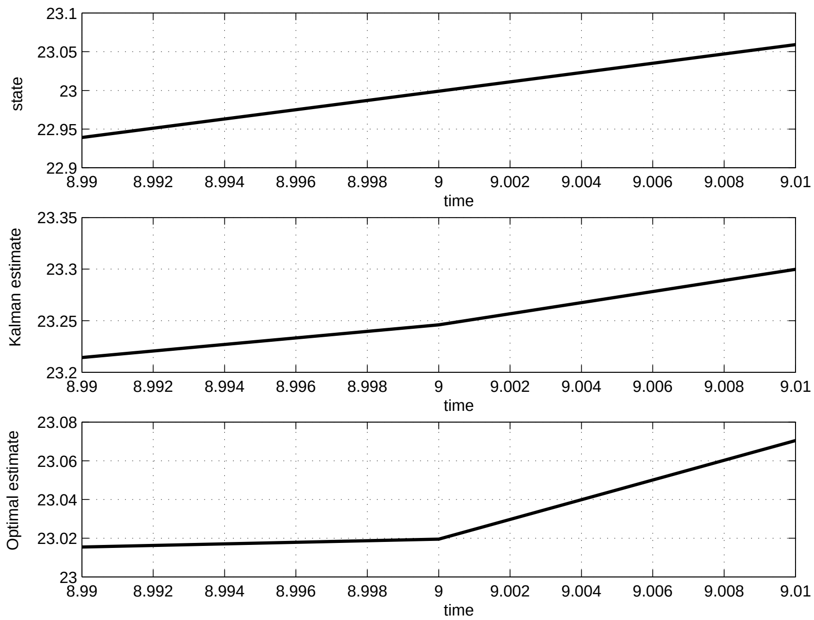

Figure 3: Graphs of the reference state variable $x(t)$ and the estimates $m_K(t)$ and $m(t)$ around the reference time point $T = 9$

included

$$\begin{aligned}dP_i(t) = (a(t)P_{i-1}(t-h) + P_{i+1}(t)a^T(t-ih))dt \\ + \frac{1}{2}(b(t)b^T(t-ih) + b(t-ih)b^T(t))dt \\ - P(t)A^T(t)(B(t)B^T(t))^{-1}(B(t)B^T(t-ih))(B(t-ih)B^T(t-ih))^{-1} \\ \times A(t-ih)P(t-ih)dt. \quad (8)\end{aligned}$$

It should be noted that, for every fixed t, the number of equations in (8), that should be taken into account to obtain a closed system of the filtering equations, is not equal to infinity, since the matrices $a(t)$, $b(t)$, $A(t)$, and $B(t)$ are not defined for $t < t_0$. Therefore, if the current time moment t belongs to the semi-open interval $(kh, (k+1)h]$, where h is the delay value in the equation (1), the number of equations in (8) is equal to k.

The last step is to establish the initial conditions for the system of equations (4), (5), (8). The initial conditions for (4) and (5) are stated as

$$m(s) = E(\phi(s)),\ s \in [t_0 - \tau, t_0) \text{ and } m(t_0) = E(\phi(t_0) \mid F^Y_{t_0}),\ s = t_0, \quad (9)$$

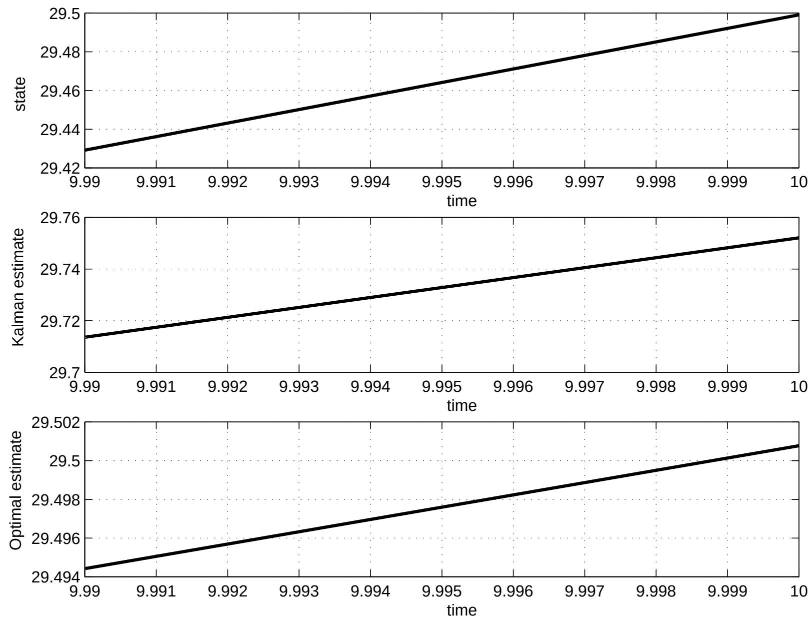

Figure 4: Graphs of the reference state variable $x(t)$ and the estimates $m_K(t)$ and $m(t)$ around the reference time point $T = 10$

and

$$P(t_0) = E[(x(t_0) - m(t_0)(x(t_0) - m(t_0))^T \mid F_{t_0}^Y]. \tag{10}$$

The initial conditions for matrices $P_i(t) = E((x(t)-m(t))(x(t-ih))^T \mid F_t^Y)$ should be stated as functions in the intervals $[t_0 + (i-1)h, t_0 + ih]$, since the i-th of the equations (8) depends on functions with the arguments delayed by ih and the definition of $P_i(t)$ itself assumes dependence on $x(t-ih)$. Thus, the initial conditions for the matrices $P_i(t)$ in (8) are stated as

$$P_i(s) = E((x(s) - m(s))(x(s-ih) - m(s-ih))^T \mid F_s^Y),$$
$$s \in [t_0 + (i-1)h,\ t_0 + ih]\,. \tag{11}$$

The obtained system of the filtering equations (4), (5), (8) with the initial conditions (10)–(12) presents the optimal solution to the filtering problem for the linear state with delay (1) over the linear observations (2). A considerable advantage of the designed filter is a finite number of the filtering equations for any fixed filtering horizon, although the state space of the delayed system (1) is infinite-dimensional.

Remark. The convergence properties of the obtained optimal estimate (4) are given by the standard convergence theorem (see, for example, [11]): if in the system (1), (2) the pair $(a(t), b(t))$ is uniformly completely controllable and the pair $(a(t), A(t)$ is uniformly completely observable, then the error of the obtained optimal filter (4), (5), (8) is uniformly asymptotically stable. As usual, the uniform complete controllability condition is required for assuring non-negativeness of the variance matrix $P(t)$ (5) and may be omitted, if the matrix $P(t)$ is non-negative in view of its intrinsic properties. The uniform complete controllability and observability conditions for a linear system with delay (1) and observations (2) can be found in [18].

4. Example

This section presents an example of designing the optimal filter for a linear state with delay over linear observations and comparing it to the best filter available for a linear state without delay, that is the Kalman-Bucy filter [12].

Let the unobserved state $x(t)$ with delay be given by

$$\dot{x}(t) = x(t-5), \quad x(s) = \phi(s), \ s \in [-5, 0], \tag{12}$$

where $\phi(s) = N(0,1)$ for $s \leq 0$, and $N(0,1)$ is a Gaussian random variable with zero mean and unit variance. The observation process is given by

$$y(t) = x(t) + \psi(t), \tag{13}$$

where $\psi(t)$ is a white Gaussian noise, which is the weak mean square derivative of a standard Wiener process (see [19]). The equations (12) and (13) present the conventional form for the equations (1) and (2), which is actually used in practice [4].

The filtering problem is to find the optimal estimate for the linear state with delay (12), using direct linear observations (13) confused with independent and identically distributed disturbances modeled as white Gaussian noises. Let us set the filtering horizon time to $T = 10$. Since $10 \in (1 \times 5, 2 \times 5]$, where 5 is the delay value in the state equation (12), the only first of the equations (8), along with the equations (4) and (5), should be employed.

The filtering equations (4), (5), and the first of the equations (8) take the following particular form for the system (12), (13)

$$\dot{m}(t) = m(t-5) + P(t)[y(t) - m(t)], \tag{14}$$

with the initial condition $m(s) = E(\phi(s)) = 0$, $s \in [-5, 0)$ and $m(0) = E(\phi(0) \mid y(0)) = m_0$, $s = 0$;

$$\dot{P}(t) = 2P_1(t) - P^2(t), \tag{15}$$

with the initial condition $P(0) = E((x(0) - m(0))^2 \mid y(0)) = P_0$; and

$$\dot{P}_1(t) = 2P(t-5) + P_2(t) - P(t)P(t-5), \tag{16}$$

with the initial condition $P_1(s) = E((x(s) - m(s))(x(s-5) - m(s-5)) \mid F_s^Y)$, $s \in [0, 5]$; finally, $P_2(s) = E((x(s) - m(s))(x(s-10) - m(s-10)) \mid F_s^Y)$, $s \in [5, 10]$. The particular forms of the equations (12) and (14) and the initial condition for $x(t)$ imply that $P_1(s) = P_0$ for $s \in [0, 5]$ and $P_2(s) = P_0$ for $s \in [5, 10]$.

The estimates obtained upon solving the equations (14)–(16) are compared to the conventional Kalman-Bucy estimates satisfying the following filtering equations for the linear state with delay (12) over linear observations (13), where the variance equation is a Riccati one and the equations for matrices $P_i(t), i \geq 1$, are not employed:

$$\dot{m_K}(t) = m_K(t-5) + P_K(t)[y(t) - m_K(t)], \tag{17}$$

with the initial condition $m_K(s) = E(\phi(s)) = 0$, $s \in [-5, 0)$ and $m_K(0) = E(\phi(0) \mid y(0)) = m_0$, $s = 0$;

$$\dot{P_K}(t) = 2P_K(t) - P_K^2(t), \tag{18}$$

with the initial condition $P_K(0) = E((x(0) - m(0))^2 \mid y(0)) = P_0$.

Numerical simulation results are obtained solving the systems of filtering equations (14)–(16) and (17)–(18). The obtained values of the estimates $m(t)$ and $m_K(t)$ satisfying (14) and (17) respectively are compared to the real values of the state variable $x(t)$ in (12).

For each of the two filters (14)–(16) and (17)–(18) and the reference system (12) involved in simulation, the following initial values are assigned: $x_0 = 2$, $m_0 = 10$, $P_0 = 100$. Gaussian disturbances $\psi_1(t)$ and $\psi_2(t)$ in (9) are realized using the built-in *MatLab* white noise function.

The following graphs are obtained: graphs of the reference state variable $x(t)$ for the system (12); graphs of the Kalman-Bucy filter estimate $m_K(t)$ satisfying the equations (17)–(18); graphs of the optimal delayed state filter estimate $m(t)$ satisfying the equations (14)–(16). The graphs of all those variables are shown on the entire simulation interval from $T = 0$ to $T = 10$ (Figure 1), and around the reference time points: $T = 8$ (Figure 2), $T = 9$ (Figure 3), and

$T = 10$ (Figure 4). It can also be noted that the error variance $P(t)$ converges to zero, since the optimal estimate (14) converges to the real state (12).

The following values of the reference state variable $x(t)$ and the estimates $m(t)$ and $m_K(t)$ are obtained at the reference time points: for $T = 8$, $x(8) = 17.5$, $m(8) = 17.55$, $m_K(8) = 17.76$; for $T = 9$, $x(9) = 23.0$, $m(9) = 23.02$, $m_K(9) = 23.24$; for $T = 10$, $x(10) = 29.5$, $m(10) = 29.5$, $m_K(0) = 29.75$.

Thus, it can be concluded that the obtained optimal filter for a linear state with delay over linear observations (14)–(16) yield definitely better estimates than the conventional Kalman-Bucy filter. Subsequent discussion of the obtained simulation results can be found in Section 5.

5. Conclusions

The simulation results show that the values of the estimate calculated by using the obtained optimal filter for a linear state with delay over linear observations are noticeably closer to the real values of the reference variable than the values of the Kalman-Bucy estimates. Moreover, it can be seen that the estimate produced by the optimal filter for a linear state with delay over linear observations asymptotically converges to the real values of the reference variable as time tends to infinity, although the reference system (12) itself is unstable. On the contrary, the conventionally designed (non-optimal) Kalman-Bucy estimates do not converge to the real values. This significant improvement in the estimate behavior is obtained due to the more careful selection of the filter gain matrix using the multi-equational system (15)–(16), which compensates for unstable dynamics of the reference system, as it should be in the optimal filter. Although this conclusion follows from the developed theory, the numerical simulation serves as a convincing illustration.

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