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A q-EXTENSION OF THE GENERALIZED HERMITE POLYNOMIALS WITH THE CONTINUOUS ORTHOGONALITY PROPERTY ON $\mathbb{R}$

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Abstract: In this paper we study in detail a q-extension of the generalized Hermite polynomials of Szegő. A continuous orthogonality property on $\mathbb{R}$ with respect to the positive weight function is established, a q-difference equation and a three-term recurrence relation are derived for this family of q-polynomials.

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1. Introduction

The generalized Hermite polynomials were introduced by Szegő [12] as

$$\begin{aligned} H_{2n}^{(\mu)}(x) &:= (-1)^n\, 2^{2n}\, n!\, L_n^{(\mu-1/2)}(x^2), \\ H_{2n+1}^{(\mu)}(x) &:= (-1)^n\, 2^{2n+1}\, n!\, x\, L_n^{(\mu+1/2)}(x^2)\ , \end{aligned} \tag{1.1}$$

where $\mu > -1/2$, $L_n^{(\alpha)}(x)$ are the Laguerre polynomials,

$$\begin{aligned} L_n^{(\alpha)}(z) &:= \frac{(\alpha+1)_n}{n!}\, {}_1\mathrm{F}_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| z \right) \\ &= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{z^k}{k!}, \end{aligned} \tag{1.2}$$

and $(a)_n = \Gamma(a+n)/\Gamma(a)$, $n = 0, 1, 2, \dots$, is the shifted factorial. Observe that the zero value of the parameter μ in (1.1) corresponds to the ordinary Hermite polynomials $H_n(x)$, i.e., $H_n^{(0)}(x) = H_n(x)$.

The generalized Hermite polynomials (1.1) are orthogonal with respect to the weight function $|x|^{2\mu} e^{-x^2}$, $x \in \mathbb{R}$, i.e.,

$$\begin{aligned} \int_{-\infty}^{\infty} & H_n^{(\mu)}(x) H_m^{(\mu)}(x) |x|^{2\mu} e^{-x^2}\, dx \\ &= 2^{2n} \left[\frac{n}{2}\right]!\, \Gamma\left(\left[\frac{n+1}{2}\right] + \mu + \frac{1}{2} \right) \delta_{nm}, \end{aligned} \tag{1.3}$$

where $[x]$ denotes the greatest integer not exceeding x. They satisfy a three-term recurrence relation

$$2x H_n^{(\mu)}(x) = H_{n+1}^{(\mu)}(x) + 2(n + 2\mu\, \theta_n) H_{n-1}^{(\mu)}(x), \qquad n \ge 0, \tag{1.4}$$

and a second-order differential equation

$$\left[x \frac{d^2}{dx^2} + 2(\mu - x^2) \frac{d}{dx} + 2nx - 2\mu\, \theta_n x^{-1} \right] H_n^{(\mu)}(x) = 0, \quad n \ge 0, \tag{1.5}$$

with $\theta_n := n - 2[n/2]$ (see Szegő [12], Chihara [4]). A detailed discussion of other properties of $H_n^{(\mu)}(x)$ can be found in Markett [9], Rosenblum [11].

The reason for interest in studying the generalized Hermite polynomials (1.1) is twofold. Pure mathematically they are of interest as an explicit example of the complete orthonormal set in $L^2_\mu(\mathbb{R})$, the Hilbert space of Lebesgue measurable functions $f(x)$, $x \in \mathbb{R}$, with

$$||f||_\mu := \left(\int_{-\infty}^{\infty} |f|^2 \, |x|^{2\mu} \, dx \right)^{1/2} < \infty \, . \tag{1.6}$$

Hence one can build the Bose-like oscillator calculus in terms of these polynomials, which generalizes the well-known calculus, based on the quantum-mechanical harmonic oscillator in physics (see, for example, Rosenblum [11]). So we try to make one step further by considering a generalization of the classical Hermite polynomials $H_n(x)$ with two additional parameters, μ and q.

The aim of this paper is to investigate in detail a q-extension of the generalized Hermite polynomials (1.1) with the continuous orthogonality property on $\mathbb{R}$ (the case of discrete orthogonality requires a different technique, see, for example, Berg et al [3]). In Section 2 we introduce this family $\{\mathcal{H}_n^{(\mu)}(x;q)\}$ in terms of the q-Laguerre polynomials and find a relevant q-difference equation for it. In Section 3 the continuous orthogonality property for $\{\mathcal{H}_n^{(\mu)}(x;q)\}$ with respect to the positive weight function on $\mathbb{R}$ is explicitly formulated. Section 4 is devoted to the derivation of a three-term recurrence relation for this family of q-polynomials.

2. Generalized Hermite Polynomials

It is known from Hahn [7], Exton [5], and Moak [10] that the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ are explicitly given as

$$\begin{aligned} L_n^{(\alpha)}(x;q) :=& \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \, {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array} \middle| \, q, -q^{n+\alpha+1}\, x \right) \\ =& \frac{1}{(q;q)_n} \, {}_2\phi_1\left(\begin{array}{c} q^{-n}, -x \\ 0 \end{array} \middle| \, q, q^{n+\alpha+1} \right) , \end{aligned} \tag{2.1}$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{j=0}^{n-1}(1 - aq^j)$, $n = 1, 2, \ldots$, is the q-shifted factorial, and

$$
\begin{aligned}
&{}_r\phi_p \left(\begin{matrix} q^{-n}, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_p \end{matrix} \middle| q \,,\, z \right) \\
&= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (a_2;q)_k \cdots (a_r;q)_k}{(b_1;q)_k (b_2;q)_k \cdots (b_p;q)_k} \frac{z^k}{(q;q)_k} \left[(-1)^k q^{k(k-1)/2} \right]^{p-r+1}
\end{aligned}
\tag{2.2}
$$

is the basic hypergeometric polynomial of degree n in the variable z (throughout this paper, we will employ the standard notations of the q-special functions theory, see Gasper et al [6] or Andrews et al [2]). The q-Laguerre polynomials (2.1) satisfy two kinds of orthogonality relations, an absolutely continuous one and a discrete one. The former orthogonality relation, in which we are interested in the present paper, is given by

$$
\int_0^\infty \frac{x^\alpha}{E_q(x)} L_m^{(\alpha)}(x;q)\, L_n^{(\alpha)}(x;q)\, dx = d_n^{-1}(\alpha)\, \delta_{mn}\,, \quad \alpha > -1\,, \tag{2.3}
$$

where $E_q(x)$ is the Jackson q-exponential function,

$$
E_q(z) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n = (-z;q)_\infty, \tag{2.4}
$$

and the normalization constant $d_n(\alpha)$ is equal to

$$
d_n(\alpha) = \frac{1}{\pi} \sin \pi(\alpha+1) \frac{q^n\,(q;q)_n}{(q^{\alpha+1};q)_n} \frac{(q;q)_\infty}{(q^{-\alpha};q)_\infty}\,. \tag{2.5}
$$

The q-Laguerre polynomials (2.1) are defined in such a way that in the limit as $q \to 1$ they reduce to the ordinary Laguerre polynomials $L_n^{(\alpha)}(x)$, i.e.,

$$
\lim_{q \to 1} L_n^{(\alpha)}((1-q)x;q) = L_n^{(\alpha)}(x). \tag{2.6}
$$

We can now define, in complete analogy with the relationship (1.1), a q-extension of the generalized Hermite polynomials $H_n^{(\mu)}(x)$ of the form

$$
\begin{aligned}
\mathcal{H}_{2n}^{(\mu)}(x;q) &:= (-1)^n\,(q;q)_n\, L_n^{(\mu-1/2)}(x^2;q), \\
\mathcal{H}_{2n+1}^{(\mu)}(x;q) &:= (-1)^n\,(q;q)_n\, x\, L_n^{(\mu+1/2)}(x^2;q)\ ,
\end{aligned}
\tag{2.7}
$$

which are orthogonal on the real line $\mathbb{R}$. Indeed, since

$$\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n\,, \tag{2.8}$$

with the aid of (2.6) one readily verifies that

$$\lim_{q\to 1}(1-q)^{-n/2}\,\mathcal{H}_n^{(\mu)}(\sqrt{1-q}\,x;q) = 2^{-n}\,H_n^{(\mu)}(x). \tag{2.9}$$

Observe also that the zero value of the parameter μ in (2.7) corresponds to polynomials $\mathcal{H}_n(x;q) \equiv \mathcal{H}_n^{(0)}(x;q)$. The sequence $\{\mathcal{H}_n(x;q)\}$ can be expressed either in terms of the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$, $\alpha = \pm 1/2$ (as it obvious from definition (2.7) itself), or through the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ of type II:

$$\mathcal{H}_n(x;q^2) = q^{n(n-1)/2}\,\tilde{h}_n(x;q)\,. \tag{2.10}$$

A detailed discussion of the properties of the polynomials $\mathcal{H}_n(x;q)$ can be found in our previous paper Álvarez-Nodarse et al [1] on this subject.

A q-difference equation for the introduced polynomials $\mathcal{H}_n^{(\mu)}(x;q)$ is, in fact, an easy consequence of the known q-difference equation

$$\begin{aligned} &q^{\alpha}\,(1+x)\,L_n^{(\alpha)}(q\,x;q) + L_n^{(\alpha)}(q^{-1}x;q) \\ &\qquad = \left[1 + q^{\alpha}(1+q^n\,x)\right]\,L_n^{(\alpha)}(x;q) \end{aligned} \tag{2.11}$$

for the q-Laguerre polynomials (see, for example, formula (3.21.6) in Koekoek et al [8]). Indeed, from this q-difference equation and definition (2.7) it follows immediately that

$$\begin{aligned} &q^{\mu-1/2}\,(1+x^2)\,\mathcal{H}_n^{(\mu)}(q^{1/2}\,x;q) + \mathcal{H}_n^{(\mu)}(q^{-1/2}\,x;q) \\ &\qquad = \left[q^{-\theta_n/2} + q^{\mu+(\theta_n-1)/2}\,(1+q^{[n/2]}\,x^2)\right]\,\mathcal{H}_n^{(\mu)}(x;q)\,, \end{aligned} \tag{2.12}$$

where, as before, $\theta_n = n - 2[n/2]$. Taking into account that the dilations $x \to q^{\pm 1}\,x$ are represented by the operators $q^{\pm x\,\frac{d}{dx}}$, that is, $q^{\pm x\,\frac{d}{dx}}\,f(x) = f(q^{\pm 1}x)$, one now readily verifies that the q-difference equation (2.12) coincides with the second-order differential equation (1.5) in the limit as $q \to 1$.

3. Orthogonality Relation

We begin this section with the following theorem:

Theorem 1. *The sequence of the q-polynomials* $\{\mathcal{H}_n^{(\mu)}(x;q)\}$, *which are defined by the relations (2.7), satisfies the orthogonality relation*

$$\int_{-\infty}^{\infty} \mathcal{H}_m^{(\mu)}(x;q)\,\mathcal{H}_n^{(\mu)}(x;q)\,\frac{|x|^{2\mu}\,dx}{E_q(x^2)} = \frac{\pi}{\cos\pi\mu}\,\frac{(q^{1/2-\mu};q)_\infty}{(q;q)_\infty}\,q^{-\frac{n}{2}-\mu\theta_n}\,(q;q)_{[\frac{n}{2}]}\,(q^{\mu+1/2};q)_{[\frac{n+1}{2}]}\,\delta_{mn}\,, \tag{3.1}$$

on the whole real line $\mathbb{R}$ *with respect to the continuous positive weight function* $w(x) = 1/E_q(x^2)$.

Proof. Since the weight function in (3.1) is an even function of the independent variable x and $\mathcal{H}_n^{(\mu)}(-x;q) = (-1)^n\,\mathcal{H}_n^{(\mu)}(x;q)$ by the definition (2.7), the q-polynomials of an even degree $\mathcal{H}_{2m}^{(\mu)}(x;q)$ and of an odd degree $\mathcal{H}_{2n+1}^{(\mu)}(x;q)$, $m,n = 0,1,2,\ldots$, are evidently orthogonal to each other. Consequently, it suffices to prove only those cases in (3.1), when degrees of polynomials m and n are either simultaneously even or odd. Let us consider first the former case. From (2.7) and (2.3) it follows that

$$\int_{-\infty}^{\infty} \mathcal{H}_{2m}^{(\mu)}(x;q)\,\mathcal{H}_{2n}^{(\mu)}(x;q)\,\frac{|x|^{2\mu}\,dx}{E_q(x^2)}$$

$$= (-1)^{m+n}\,(q;q)_m(q;q)_n \int_{-\infty}^{\infty} L_m^{(\mu-1/2)}(x^2;q)\,L_n^{(\mu-1/2)}(x^2;q)\,\frac{|x|^{2\mu}\,dx}{E_q(x^2)}$$

$$= 2(-1)^{m+n}\,(q;q)_m(q;q)_n \int_0^{\infty} L_m^{(\mu-1/2)}(x^2;q)\,L_n^{(\mu-1/2)}(x^2;q)\,\frac{x^{2\mu}\,dx}{E_q(x^2)}$$

$$= (-1)^{m+n}(q;q)_m(q;q)_n \int_0^{\infty} L_m^{(\mu-1/2)}(y;q)\,L_n^{(\mu-1/2)}(y;q)\,\frac{y^{\mu-1/2}\,dy}{E_q(y)}$$

$$= (q;q)_n^2\,d_n^{-1}(\mu-1/2)\,\delta_{mn}\,,$$

where the normalization constant $d_n(\alpha)$ is defined in (2.5). Thus

$$\int_{-\infty}^{\infty} \mathcal{H}_{2m}^{(\mu)}(x;q)\,\mathcal{H}_{2n}^{(\mu)}(x;q)\frac{|x|^{2\mu}\,dx}{E_q(x^2)} = \frac{\pi}{\cos\pi\mu}\,\frac{(q^{1/2-\mu};q)_\infty}{(q;q)_\infty}\,q^{-n}\,(q;q)_n\,(q^{\mu+1/2};q)_n\,\delta_{mn}\,. \quad (3.2)$$

Likewise, one finds that in the latter case

$$\int_{-\infty}^{\infty} \mathcal{H}_{2m+1}^{(\mu)}(x;q)\,\mathcal{H}_{2n+1}^{(\mu)}(x;q)\,\frac{|x|^{2\mu}\,dx}{E_q(x^2)}$$

$$= 2(-1)^{m+n}\,(q;q)_m\,(q;q)_n \int_0^{\infty} L_m^{(\mu+1/2)}(x^2;q)\,L_n^{(\mu+1/2)}(x^2;q)\,\frac{x^{2(\mu+1)}\,dx}{E_q(x^2)}$$

$$= (-1)^{m+n}\,(q;q)_m\,(q;q)_n \int_0^{\infty} L_m^{(\mu+1/2)}(y;q)\,L_n^{(\mu+1/2)}(y;q)\,\frac{y^{\mu+1/2}\,dy}{E_q(y)}$$

$$= (q;q)_n^2\,d_n^{-1}(\mu+1/2)\,\delta_{mn}.$$

Consequently,

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathcal{H}_{2m+1}^{(\mu)}(x;q)\,\mathcal{H}_{2n+1}^{(\mu)}(x;q)\,\frac{|x|^{2\mu}\,dx}{E_q(x^2)}\\ &= \frac{\pi}{\cos\pi\mu}\,\frac{(q^{1/2-\mu};q)_\infty}{(q;q)_\infty}\,q^{-n-\mu-1/2}\,(q;q)_n\,(q^{\mu+1/2};q)_{n+1}\,\delta_{mn}\,.\end{aligned} \quad (3.3)$$

Putting (3.2) and (3.3) together results in the orthogonality relation (3.1).

The positivity of Jackson q-exponential function $E_q(x^2)$ for $x \in \mathbb{R}$ and $q \in (0,1)$ is obvious from its definition (2.4): for it is represented as an infinite sum of positive terms (or an infinite product of positive factors). This completes the proof. □

To conclude this section, we note the obvious fact that in the limit as $q \to 1$ the (3.1) reduces to the orthogonality relation (1.3) for the generalized Hermite polynomials (1.1). This follows immediately from the limit relations (2.8) and (2.9), upon using the fact that

$$\lim_{q\to 1} E_q((1-q)z) = e^z\,. \quad (3.4)$$

Also, in the event the parameter μ is zero, then the (3.1) coincides with the orthogonality relation for the polynomials (2.10), derived in Álvarez-Nodarse et al [1].

4. Recurrence Relation

In this section we derive a three-term recurrence relation for the q-extension of the generalized Hermite polynomials (2.7). Since an arbitrary family of orthogonal polynomials $p_n(x)$ satisfies a recurrence relation of the form (see Chihara [4, p.19])

$$(a_n x + b_n)p_n(x) = p_{n+1}(x) + c_n p_{n-1}(x)\,, \qquad n \geq 0\,, \tag{4.1}$$

one needs to find coefficients a_n, b_n, and c_n, which correspond to the case under discussion.

Before starting this derivation we note that in what follows it proves convenient to use the following form

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{q^{k(k+\alpha)}}{(q^{\alpha+1};q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x)^k \tag{4.2}$$

of the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$, which comes from the first line in definition (2.1), upon using the relation

$$\frac{(q^{-n};q)_k}{(q;q)_k} = (-1)^k q^{k(k-1)/2-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q . \tag{4.3}$$

Let us first consider the case when n in (4.1) is even. Then from (2.7) and (4.2) we find that

$$\begin{aligned} &\mathcal{H}_{2n+1}^{(\mu)}(x;q) + c_{2n}(q)\,\mathcal{H}_{2n-1}^{(\mu)}(x;q) \\ &= (-1)^n\, x\,(q^{\mu+3/2};q)_n \sum_{k=0}^{n} \frac{q^{k(k+\mu+1/2)}}{(q^{\mu+3/2};q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x^2)^k \\ &+ (-1)^{n-1}\, c_{2n}(q)\, x\,(q^{\mu+3/2};q)_{n-1} \sum_{k=0}^{n-1} \frac{q^{k(k+\mu+1/2)}}{(q^{\mu+3/2};q)_k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-x^2)^k . \end{aligned} \tag{4.4}$$

The next step is to employ the relation

$$(1-q^{\alpha})\,(q^{\alpha+1};q)_n = (1-q^{n+\alpha})\,(q^{\alpha};q)_n \tag{4.5}$$

in order to rewrite the quotient $(q^{\mu+3/2};q)_n/(q^{\mu+3/2};q)_k$ from the first term in the right side of (4.4) as

$$\frac{(q^{\mu+3/2};q)_n}{(q^{\mu+3/2};q)_k} = \frac{1-q^{n+\mu+1/2}}{1-q^{k+\mu+1/2}}\,\frac{(q^{\mu+1/2};q)_n}{(q^{\mu+1/2};q)_k}\,. \tag{4.6}$$

In the second term in the right side of (4.4) one can use the evident relation $(q^{\mu+3/2};q)_{n-1} = (q^{\mu+1/2};q)_n/(1-q^{\mu+1/2})$ and the same formula (4.5) for the factor $(q^{\mu+3/2};q)_k$. We recall also the property of the q-binomial coefficient

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \frac{1-q^{n-k}}{1-q^n}\begin{bmatrix} n \\ k \end{bmatrix}_q\,. \tag{4.7}$$

Putting this all together, we obtain

$$\mathcal{H}^{(\mu)}_{2n+1}(x;q) + c_{2n}(q)\,\mathcal{H}^{(\mu)}_{2n-1}(x;q) = (-1)^n\,x\,(q^{\mu+1/2};q)_n \times$$
$$\sum_{k=0}^{n}\frac{q^{k(k+\mu+1/2)}}{(q^{\mu+1/2};q)_k}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-x^2)^k}{1-q^{k+\mu+1/2}}\left\{1-q^{n+\mu+1/2} - c_{2n}(q)\frac{1-q^{n-k}}{1-q^n}\right\}. \tag{4.8}$$

The right-hand side of (4.8) should match with

$$\mathcal{H}^{(\mu)}_{2n}(x;q) = (-1)^n\,(q^{\mu+1/2};q)_n\sum_{k=0}^{n}\frac{q^{k(k+\mu-1/2)}}{(q^{\mu+1/2};q)_k}\begin{bmatrix} n \\ k \end{bmatrix}_q (-x^2)^k\,, \tag{4.9}$$

multiplied by $a_{2n}(q)\,x + b_{2n}(q)$. This means that the coefficient $c_{2n}(q)$ can be found from the equation

$$1-q^{n+\mu+1/2} - c_{2n}(q)\frac{1-q^{n-k}}{1-q^n} = d_n(q)\,q^{-k}\,(1-q^{k+\mu+1/2})\,, \tag{4.10}$$

where $d_n(q)$ is some k-independent factor. It is not difficult to verify that the only solution of the equation (4.10) is $c_{2n}(q) = 1-q^n$ and $d_n(q) = q^n$. Thus

$$\mathcal{H}^{(\mu)}_{2n+1}(x;q) + (1-q^n)\mathcal{H}^{(\mu)}_{2n-1}(x;q) = q^n\,x\,\mathcal{H}^{(\mu)}_{2n}(x;q)\,. \tag{4.11}$$

Similarly, in the case of an odd n from (4.8) we have

$$\begin{aligned}\mathcal{H}^{(\mu)}_{2n+2}(x;q) &+ c_{2n+1}(q)\mathcal{H}^{(\mu)}_{2n}(x;q)\\ &= (-1)^{n+1}\,(q^{\mu+1/2};q)_{n+1}\sum_{k=0}^{n+1}\frac{q^{k(k+\mu-1/2)}}{(q^{\mu+1/2};q)_k}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q (-x^2)^k\\ &+\,(-1)^n\,c_{2n+1}(q)\,(q^{\mu+1/2};q)_n\sum_{k=0}^{n}\frac{q^{k(k+\mu-1/2)}}{(q^{\mu+1/2};q)_k}\begin{bmatrix} n \\ k \end{bmatrix}_q (-x^2)^k\,.\end{aligned} \tag{4.12}$$

In this case it is even easier to find the coefficient $c_{2n+1}(q)$. Indeed, one will obtain from (4.12) an expression $[a_{2n+1}(q)\,x + b_{2n+1}(q)]\,\mathcal{H}^{(\mu)}_{2n+1}(x;q)$ only if the two constant terms in (4.12) with $k = 0$ cancel each other. This means that the $c_{2n+1}(q)$ should satisfy the equation

$$(q^{\mu+1/2};q)_{n+1} - (q^{\mu+1/2};q)_n\, c_{2n+1}(q) \equiv (q^{\mu+1/2};q)_n[1 - q^{n+\mu+1/2} - c_{2n+1}(q)] = 0\,. \quad (4.13)$$

Consequently, $c_{2n+1}(q) = 1 - q^{n+\mu+1/2}$ and, therefore, by (4.12) one obtains

$$\mathcal{H}^{(\mu)}_{2n+2}(x;q) + (1 - q^{n+\mu+1/2})\mathcal{H}^{(\mu)}_{2n}(x;q) = (-1)^{n+1}\,(q^{\mu+1/2};q)_{n+1} \times \sum_{k=0}^{n} \frac{q^{(k+1)(k+\mu+1/2)}}{(q^{\mu+1/2};q)_{k+1}} \left\{ \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q - \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \right\} (-x^2)^k$$

$$= q^{n+\mu+1/2}\, x\, \mathcal{H}^{(\mu)}_{2n+1}(x;q)\,, \quad (4.14)$$

upon using the q-Pascal identity

$$\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q - \begin{bmatrix} n \\ m+1 \end{bmatrix}_q = q^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q\,, \quad (4.15)$$

for the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$. From (4.11) and (4.14) it thus follows that the q-polynomials $\mathcal{H}^{(\mu)}_n(x;q)$ satisfy a three-term recurrence relation of the form ($\theta_n := n - 2[n/2]$)

$$\mathcal{H}^{(\mu)}_{n+1}(x;q) + (1 - q^{n/2+\mu\theta_n})\,\mathcal{H}^{(\mu)}_{n-1}(x;q) = q^{n/2+\mu\theta_n}\, x\, \mathcal{H}^{(\mu)}_n(x;q)\,. \quad (4.16)$$

With the aid of (2.9) one now readily verifies that the (4.16) coincides with the three-term recurrence relation (1.4) for the generalized Hermite polynomials $H^{(\mu)}_n(x)$ in the limit as $q \to 1$.

5. Concluding Remarks

We conclude this exposition with the following remark. It is well-known that the Hermite functions $H_n(x)\,e^{-x^2/2}$ (or the wave functions of the linear harmonic oscillator in quantum mechanics) are eigenfunctions of the Fourier integral transform (with respect to the kernel $e^{\mathrm{i}xy}$) with eigenvalues i^n. One can introduce the generalized Fourier transform operator

$$\mathcal{F}_\mu f(x) := c_\mu^{-1} \int_{-\infty}^{\infty} e_\mu(-\mathrm{i}xt)\, f(t)\, |t|^\mu \, dt\,, \tag{5.1}$$

with a kernel

$$e_\mu(-\mathrm{i}x) := \frac{c_\mu}{2}\, \frac{J_{\mu-1/2}(x) - \mathrm{i}\, J_{\mu+1/2}(x)}{x^{\mu-1/2}}, \tag{5.2}$$

where the constant $c_\mu := 2^{\mu+1/2}\,\Gamma(\mu{+}1/2)$ and $J_\alpha(x)$ is the Bessel function. The generalized Hermite polynomials (1.1) are the eigenfunctions of the generalized Fourier transform operator (5.1), see Rosenblum [11]. It is of interest to find a q-extension of (5.1) and (5.2). This study is under way.

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