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SOME NEW IMPROVEMENT OF THE HARDY-HILBERT INEQUALITY WITH APPLICATIONS

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Abstract: This paper deals with some new improvement of the Hardy-Hilbert integral inequality. As an application, an extension and a refinement of Widder Theorem are presented.

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1. Introduction

We first state the famous Hardy-Hilbert integral inequality (see Hardy et al [4]).

Let $f(x), g(x) \geq 0$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $p > 1$. If $f(x) \in L^p(0,\infty)$ and $g(x) \in L^q(0,\infty)$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y}\,dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)\,dx\right)^{\frac{1}{q}}, \tag{1.1}$$

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where the constant factor $\dfrac{\pi}{\sin(\pi/p)}$ involved in (1.1) is the best possible.

This inequality has many useful and important applications in analysis and mathematical physics.

In recent years, several authors including Yang and Debnath [2], Kuang [5], and Yang [1] have studied the Hardy-Hilbert integral inequality. On the other hard, Xu and Guo [7], Gao and Yang [8], Yang and Debnath [2] have given some improvements and generalizations of this inequality (1.1) with double series. In spite of this progress, there is a need for some new improvements of inequality (1.1). So this paper deals with some new improvements of the Hardy-Hilbert integral inequality. As an application, and extension and a refinement of Widder Theorem (see [9]) are given.

In order to make the paper self contained, it is necessary to introduce the following notations:

$$(f^r, g^s) = \int\limits_0^\infty f^r(x) g^s(x)\, dx\,,\ \|f\|_p = \left(\int\limits_0^\infty f^p(x)\, dx\right)^{\frac{1}{p}} \quad \text{and } \|f\|_2 = \|f\|\,.$$

Next we define a function S_r, by

$$S_r(f,h) = \|f\|_r^{-r/2}(f^{r/2}, h)\,, \tag{1.2}$$

where h is a variable unit vector. For the general case, the vector h will be chosen properly such that the specific problems discussed are simplified. Clearly, $S_r(f,h) = 0$ when the vector (function) h selected is orthogonal to $f^{r/2}$.

2. Main Results

In order to prove the main results, we need the following lemma.

Lemma 2.1. *Let $f(x), g(x) \geq 0$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $p > 1$. If $0 < \|f\|_p < +\infty$ and $0 < \|g\|_q < +\infty$, then*

$$(f,g) \leq \|f\|_p \|g\|_q \left(1 - \frac{(S_p - S_q)^2}{1 - S_p S_q}\right)^m\,, \tag{2.1}$$

where $m = \min\left(\dfrac{1}{p}, \dfrac{1}{q}\right)$, $S_p = S_p(f,h)$ and $S_q = S_q(g,h)$ are defined by (1.2). The equality in (2.1) holds if and only if $f^{p/2}$ and $g^{p/2}$ are linearly dependent; or h is a linear combination of $f^{p/2}$ and $g^{p/2}$, and $S_p S_q = 0$ but $S_p \neq S_q$.

The proof of this lemma is given by Hu [6]. Hence, it is omitted here.

We also introduce the following notations:

$$A=\frac{2q}{2q-1}\int_0^1 x^{1-1/2p}f^{p/2}(x)\,dx \text{ and } B=\frac{2q}{2q+1}\int_0^1 x^{1/2p}g^{q/2}(x)\,dx\,. \tag{2.2}$$

Theorem 2.1. *Under the assumptions as in Lemma 2.1, we have*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy<\left(\frac{\pi}{\sin(\pi/p)}\right)\|f\|_p\|g\|_q\left(1-\frac{m\beta}{1-\alpha}\right), \tag{2.3}$$

where $m=\min\left(\frac{1}{p},\frac{1}{q}\right)$, $\alpha=\frac{AB\sin(\pi/p)}{\pi}\left(\|f\|_p^p\|g\|_q^q\right)^{-1/2}$ *and* $\beta=\frac{\sin(\pi/p)}{\pi}\left(A(\|f\|_p^{-p/2}-B(\|g\|_q^{-q/2}\right)^2$, *where* A *and* B *are defined in (2.2).*

Proof. Let

$$F=\frac{f(x)}{(x+y)^{1/p}}\left(\frac{x}{y}\right)^{\frac{1}{pq}} \text{ and } G=\frac{g(y)}{(x+y)^{1/q}}\left(\frac{y}{x}\right)^{\frac{1}{pq}}\,.$$

Using the inequality (2.1), we estimate the left hand side of (2.3) as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy$$
$$=\int_0^\infty\int_0^\infty FG\,dxdy\le\|F\|_p\|G\|_q\left(1-\frac{(S_p-S_q)^2}{1-S_pS_q}\right)^m\,. \tag{2.4}$$

It is easy to deduce that

$$\begin{aligned}\|F\|_p&=\left(\int_0^\infty\int_0^\infty F^p\,dxdy\right)^{1/p}=\left(\frac{\pi}{\sin(\pi/p)}\right)^{1/p}\|f\|_p \text{ and}\\ \|G\|_q&=\left(\int_0^\infty\int_0^\infty G^q\,dxdy\right)^{1/q}=\left(\frac{\pi}{\sin(\pi/p)}\right)^{1/q}\|g\|_q\,.\end{aligned} \tag{2.5}$$

Let us define a function h by

$$h(x,y)=\begin{cases}\sqrt{\frac{12}{7}x(x+y)}\,, & 0\le x,y\le 1\,,\\ 0\,, & x,y\ge 1\,.\end{cases}$$

Then $\|h\| = \left(\int\limits_0^\infty \int\limits_0^\infty h^2(x,y)\,dxdy\right)^{1/2} = 1$.

It is easy to show that

$$\left(F^{p/2}, h\right) = \int\limits_0^\infty \int\limits_0^\infty F^{p/2} h\,dxdy = A \quad \text{and}$$
$$\left(G^{q/2}, h\right) = \int\limits_0^\infty \int\limits_0^\infty G^{q/2} h\,dxdy = B\,,$$

where A and B are defined by (2.2). Hence we have

$$\begin{aligned} S_p &= \left(F^{p/2}, h\right) \|F\|_p^{-p/2} = \left(\frac{\pi}{\sin(\pi/p)}\right)^{1/2} A\|f\|_p^{-p/2} \quad \text{and} \\ S_q &= \left(G^{q/2}, h\right) \|G\|_q^{-q/2} = \left(\frac{\pi}{\sin(\pi/p)}\right)^{1/2} B\|g\|_q^{-q/2}\,. \end{aligned} \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), and then using the inequality $(1 - x)^m < 1 - mx$, where $0 < x < 1$ and $0 < m < 1$, gives inequality (2.3) after simplifications.

The proof of the theorem is completed. □

When $p = 2$, the following result follows from (2.3).

Corollary 2.1. *If* $0 < \|f\| < +\infty$ *and* $0 < \|g\| < +\infty$, *then*

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \pi\|f\|\|g\|\left(1 - \frac{R}{1-S}\right), \tag{2.7}$$

where

$$R = \frac{8}{\pi}\left(\frac{1}{3}\|f\|^{-1}\int\limits_0^1 x^{3/4} f(x)\,dx - \frac{1}{5}\|g\|^{-1}\int\limits_0^1 x^{1/4} g(x)\,dx\right)^2,$$

and

$$S = \left(\frac{16}{15\pi}\right)\|f\|^{-1}\|g\|^{-1}\int\limits_0^1 x^{3/4} f(x)\,dx \int\limits_0^1 x^{1/4} g(x)\,dx\,.$$

Corollary 2.2. *If* $0 < \|f\| < +\infty$, *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y}\,dxdy < \pi\|f\|^2\left(1-\frac{r}{1-s}\right), \tag{2.8}$$

where

$$r = \frac{8}{\pi\|f\|}\left(\frac{1}{3}\int_0^1 x^{3/4}f(x)\,dx - \frac{1}{5}\int_0^1 x^{1/4}f(x)\,dx\right)^2,$$

and

$$s = \frac{16}{15\pi\|f\|^2}\int_0^1 x^{3/4}f(x)\,dx \int_0^1 x^{1/4}f(x)\,dx\,.$$

The inequalities (2.7) and (2.8) are obviously an improvement of Hilbet integral inequality.

Corollary 2.3. *Let* $f(x) \geq 0$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $p>1$. *If* $0 < \|f\|_p < +\infty$, *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y}\,dxdy < \left(\frac{\pi}{\sin(\pi/p)}\right)\|f\|_p\|f\|_q\left(1-\frac{m\beta_1}{1-\alpha_1}\right), \tag{2.9}$$

where

$$\begin{aligned} m &= \min\left(\frac{1}{p},\frac{1}{q}\right), \\ \alpha_1 &= \frac{AB^*\sin(\pi/p)}{\pi}\left(\|f\|_p^p\|f\|_q^q\right)^{-1/2} \quad \text{and} \\ \beta_1 &= \frac{\sin(\pi/p)}{\pi}\left(A\|f\|_p^{-p/2} - B^*\|f\|_q^{-q/2}\right)^2, \end{aligned} \tag{2.10}$$

where A *is defined by (2.2) and* B^* *is defined by*

$$B = \frac{2q}{2q+1}\int_0^1 x^{1/2p}f^{q/2}(x)\,dx\,.$$

3. Applications

As applications of the above results, we give an extension of Widder Theorem (see [9]), and a refinement of it.

Theorem 3.1. *Let* $a_n \geq 0$, $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $A^*(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $P > 1$. *If* $A(x) \neq 0$ *and* $e^{-x}A^*(x) \in L^P(0, +\infty)$, *then*

$$\int_0^1 A^2(x)\,dx < \left(\frac{\pi}{\sin(\pi/p)}\right) \|f\|_p \|f\|_q \left(1 - \frac{m\beta_1}{\alpha_1}\right), \tag{3.1}$$

where $f(x) = e^{-x}A^*(x)$, $m = \min\left(\frac{1}{p}, \frac{1}{q}\right)$, α_1 *and* β_1 *are defined by (2.10).*

Proof. Consider the relation between $A(x)$ and $A^*(x)$ in the form

$$A(x) = \int_0^\infty e^{-t} A^*(tx)\,dt = \frac{1}{x} \int_0^\infty e^{-s/x} A^*(s)\,ds\,.$$

Hence,

$$\int_0^1 A^2(x)\,dx = \int_0^1 \frac{dx}{x^2} \left(\int_0^\infty e^{-s/x} A^*(s)\,ds\right)^2$$

$$= \int_1^\infty dy \left(\int_0^\infty e^{-sy} A^*(s)\,ds\right)^2 = \int_0^\infty dw \left(\int_0^\infty e^{-sw} f(s)\,ds\right)^2,$$

where $f(x) = e^{-x}A^*(x)$.

By Corollary 2.3, we have

$$\int_0^1 A^2(x)\,dx = \int_0^\infty dw \int_0^\infty e^{-sw} f(s)\,ds \int_0^\infty e^{-tw} f(t)\,dt$$

$$= \int_0^\infty \int_0^\infty \left(\int_0^\infty e^{-(t+s)w}\,dw\right) f(s)f(t)\,dsdt = \int_0^\infty \int_0^\infty \frac{f(s)f(t)}{s+t}\,dsdt$$

$$< \left(\frac{\pi}{\sin(\pi/p)}\right) \|f\|_p \|f\|_q \left(1 - \frac{m\beta_1}{1-\alpha_1}\right),$$

where α_1 and β_1, are defined by (2.10), $m = \min\left(\frac{1}{p}, \frac{1}{q}\right)$.

This completes the proof. □

Corollary 3.1. *With the assumptions as in Theorem 3.1, and $p = 2$, then*

$$\int_0^1 A^2(x)\,dx < \pi \|f\|^2 \left(1 - \frac{r}{1-s}\right), \tag{3.2}$$

where r and s are given in Corollary 2.2.

Remark. If r in (3.2) is replaced by zero, then Widder Theorem (see [9]) is obtained. It follows that inequality (3.2) is a refinement of Widder Theorem, and inequality (3.1) is a generalization of it.

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