

ON A NEW IMPROVEMENT OF  
MATHIEU-BERG'S INEQUALITY

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**Abstract:** In this paper, by using Euler-Maclaurin summation formula, we give a new improvement of Mathieu-Berg inequality.

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1. Introduction

Let  $s(c)$  be the Mathieu series as (see [2])

$$s(c) := \sum_{n=1}^{\infty} \frac{2n}{(n^2 + c^2)^2} \quad (c \in R). \quad (1.1)$$

The inequality of the form

$$\frac{1}{c^2 + 1/2} < s(c) < \frac{1}{c^2} \quad (c > 0), \quad (1.2)$$

is usually called Mathieu-Berg inequality, which was guessed by E. Mathieu in 1890 and proved by L. Berg in 1952 (see [1]). Since then, a good few mathematicians have had various refinements of it. The representative work of them is the result of the form as (see [5]):

$$s(c) = \sum_{i=0}^{k-1} (-1)^i \frac{B_{2i}}{c^{2i+2}} + \delta_k(c) \quad (c > 0), \quad (1.3)$$

where  $\delta_k(c) = (-1)^k \frac{2^{2k}-1}{2^{2k}} [1 + \theta_k \frac{2^{2k}+1}{2^{2k}} \binom{2k}{k}] \frac{B_{2k}}{c^{2k+2}}$  ( $|\theta_k| < 1$ ),  $B_k$  are Bernoulli numbers,  $B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, \dots$ . In particular, for  $k=1,2$ , reducing (1.3), we have

$$\frac{1}{c^2} - \frac{1}{6c^4} - \frac{23}{256c^6} < s(c) < \frac{1}{c^2} - \frac{1}{6c^4} + \frac{7}{256c^6}; \quad (1.4)$$

$$\frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} - \frac{153}{2048c^8} < s(c) < \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} + \frac{57}{2048c^8}. \quad (1.5)$$

For the remainder term  $\delta_k(c)$ , D.C. Russell [3] gave the following integral form:

$$\delta_k(c) = \frac{(-1)^{k-1}}{c^{2k}} \int_0^\infty \left( \frac{x}{e^x - 1} \right)^{2k-1} \cos(cx) dx. \quad (1.6)$$

In this paper, by using (1.4), (1.5) and Euler-Maclaurin summation formula, a sharp result of Mathieu-Berg inequality is obtained which is an improvement of (1.2).

## 2. Some Lemmas

**Lemma 2.1.** *If  $f(x) \in C^2[2, \infty)$ ,  $f'(x) < 0$ , and  $f(\infty) = 0$ , then we have*

$$0 < \int_2^\infty P_3(x) f(x) dx < \frac{1}{64} f(2), \quad (2.1)$$

where  $P_j(x)$  ( $j = 1, 2, \dots$ ) are Bernoulli function (see [4]).

*Proof.* Basing on the periodicity of Bernoulli function and observing that  $|B_4| = \max |P_4(x)|$ , we have

$$\int_2^\infty (|B_4| - |P_4(x)|) |f'(x)| dx > 0.$$

Since  $f'(x) < 0$ , and  $f(\infty) = 0$ , then  $f(x) \downarrow 0$  ( $x \rightarrow \infty$ ), and  $f(x) > 0$ . Since  $B_4 < 0$ , we obtain

$$\begin{aligned} \int_2^\infty P_4(x) f'(x) dx &\leq \int_2^\infty |P_4(x)| |f'(x)| dx < |B_4| \int_2^\infty |f'(x)| dx \\ &= -B_4 f(2). \end{aligned}$$

By using the relation  $P_4'(x) = 4P_3(x)$ , and  $P_4(2) = B_4$ , we have

$$\begin{aligned} \int_2^\infty P_3(x)f(x)dx &= \frac{1}{4} \int_2^\infty f(x)dP_4(x) \\ &= \frac{1}{4} \left[ -B_4f(2) - \int_2^\infty P_4(x)f'(x)dx \right] > 0. \end{aligned} \quad (2.2)$$

In view of that fact that (see [1, p. 83])

$$P_3(x) = \sum_{v=0}^3 \binom{3}{v} B_v x^{3-v} \quad (x \in [0, 1]),$$

and

$$\int_k^{k+1} P_3(x)dx = 0 \quad (k = 1, 2, \dots),$$

we obtain

$$\begin{aligned} \int_2^\infty P_3(x)f(x)dx &= \sum_{k=2}^\infty \int_k^{k+1} P_3(x)(f(x) - f(k+1))dx \\ &= \sum_{k=2}^\infty \left\{ \int_k^{k+1/2} P_3(x)(f(x) - f(k+1))dx \right. \\ &\quad \left. + \int_{k+1/2}^{k+1} P_3(x)(f(x) - f(k+1))dx \right\} \\ &= \sum_{k=2}^\infty (f(k) - f(k+1)) \int_k^{k+1/2} P_3(x)dx + \sum_{k=2}^\infty \alpha_k \\ &= \sum_{k=2}^\infty (f(k) - f(k+1)) \int_0^{1/2} P_3(x)dx + \sum_{k=2}^\infty \alpha_k \\ &= f(2) \sum_{v=0}^3 \binom{3}{v} B_v \int_0^{1/2} x^{3-v}dx + \sum_{k=2}^\infty \alpha_k = \frac{1}{64}f(2) + \sum_{k=2}^\infty \alpha_k, \end{aligned}$$

where  $\alpha_k$  is defined by

$$\alpha_k := \int_k^{k+1/2} P_3(x)(f(x) - f(k))dx - \int_{k+1/2}^{k+1} P_3(x)(f(k+1) - f(x))dx.$$

Note that  $f(x) \downarrow 0 \quad (x \rightarrow \infty)$ , we have  $\alpha_k < 0$ , and  $\sum_{k=2}^\infty \alpha_k < 0$ . Hence we have

$$\int_2^\infty P_3(x)f(x)dx < \frac{1}{64}f(2). \quad (2.3)$$

In view of (2.2) and (2.3), we have (2.1). The lemma is proved.  $\square$

**Lemma 2.2.** Define the function  $I(x)$  as

$$I(x) := \frac{2}{(1+x^2)^2} + \frac{1}{4+x^2} + \frac{11}{6(4+x^2)^2} + \frac{47}{12(4+x^2)^3} - \frac{5}{(4+x^2)^4} \quad (x \in R). \quad (2.4)$$

Then:

(i) if  $|c| > 1$ , we have

$$\frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} + \frac{57}{2048c^8} < \frac{1}{c^2 + 1/6}; \quad (2.5)$$

(ii) if  $|c| \leq 1$ , we have

$$I(c) < \frac{1}{c^2 + 1/6}. \quad (2.6)$$

*Proof.* (i) When  $|c| > 1$ , inequality (2.5) is equivalent to the following

$$\begin{aligned} 1 - \left[ \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} + \frac{57}{2048c^8} \right] (c^2 + 1/6) \\ = \left( \frac{11c^4}{180} - \frac{2053c^2}{92160} - \frac{57}{12288} \right) \frac{1}{c^8} > 0. \end{aligned} \quad (2.5)'$$

Obviously (2.5)' is always valid when  $|c| > 1$ . Hence (2.5) is true.

(ii) If  $|c| \leq 1$ , replacing  $1/(4+c^2)$  by  $y$  in (2.6), we obtain after simplifications,

$$g(y) := 88 - 1393y + 8010y^2 - 21249y^3 + 12420y^4 < 0, \quad y \in [1/5, 1/4]. \quad (2.6)'$$

Since  $g^{(4)}(y) > 0$ ,  $g^{(3)}(y)$  is increasing in  $[1/5, 1/4]$ , we can find that  $\max\{g^{(3)}(y); 1/5 \leq y \leq 1/4\} = g^{(3)}(1/4) < 0$ , whence  $g^{(3)}(y) < 0$ . It follows that  $g^{(2)}(y)$  is decreasing in  $[1/5, 1/4]$ . We find also that  $\max\{g^{(2)}(y); 1/5 \leq y \leq 1/4\} = g^{(2)}(1/5) < 0$ , whence  $g^{(2)}(y) < 0$ . Hence  $g'(y)$  is decreasing in  $[1/5, 1/4]$ . Since  $g'(1/5) < 0$ , then  $g'(y) < 0$ , and  $g(y)$  is increasing in  $[1/5, 1/4]$ . At last we find that  $g(1/5) < 0$ , whence  $g(y) < 0$ . Consequently (2.6)' is true; so is (2.6). The lemma is proved.  $\square$

**Lemma 2.3.** Define the function  $J(x)$  as

$$J(x) := \frac{2}{(1+x^2)^2} + \frac{1}{4+x^2} + \frac{11}{6(4+x^2)^2} + \frac{65}{48(4+x^2)^3} + \frac{8}{(4+x^2)^4} - \frac{16}{(4+x^2)^5} \quad (x \in R). \quad (2.7)$$

Then:

(i) if  $c^2 > 5/3$ , we have

$$\frac{1}{c^2 + 5/12} < \frac{1}{c^2} - \frac{1}{6c^4} - \frac{23}{256c^6}; \tag{2.8}$$

(ii) if  $c^2 \leq 5/3$ , we have

$$J(c) > \frac{1}{c^2 + 12/5}. \tag{2.9}$$

*Proof.* (i) When  $c^2 > 5/3$ , inequality (2.8) is equivalent to the following

$$c^4 - \frac{367c^2}{576} - \frac{115}{768} > 0. \tag{2.8}'$$

Obviously (2.8)' is valid for all  $c$  ( $c^2 > 5/3$ ). Hence (2.8) is true.

(ii) Replacing  $1/(4+c^2)$  by  $y$  in (2.9), we obtain easily after simplifications,

$$h(y) := 3 - \frac{1084y}{48} + \frac{10765y^2}{48} - \frac{63642y^3}{48} + \frac{203709y^4}{48} - 8952y^5 + 6192y^6 > 0 \quad (y \in [\frac{3}{17}, \frac{1}{4}]). \tag{2.9}'$$

In fact,

$$h^{(4)}(y) = 203709/2 - 1074240y + 2229120y^2,$$

$$h^{(4)}(3/17) = -18298.3027 < 0,$$

and  $h^{(4)}(1/4) = -27385.5 < 0$ , then  $h^{(4)}(y) < 0$ , and  $h^{(3)}$  is decreasing in  $[3/17, 1/4]$ . We can find that  $\max\{h^{(3)}(y); y \in [3/17, 1/4]\} = h^{(3)}(3/17) = -2624.37853 < 0$ , then  $h^{(3)}(y) < 0$ , and  $h^{(2)}$  is decreasing in  $[3/17, 1/4]$ . We find that  $h^{(2)}(3/17) = -397.41057 < 0$ , then  $h^{(2)}(y) < 0$ , and  $h'$  is decreasing in  $[3/17, 1/4]$ . Since  $h'(3/17) = -11.0578809 < 0$ , then  $h'(y) < 0$ , and  $h$  is increasing in  $[3/17, 1/4]$ . At last, we find that  $h(1/4) = 18.64689127 > 0$ , whence  $h(y) > 0$  and (2.9)' is true; so is (2.9). The lemma is proved.  $\square$

### 3. Main Result and Remarks

**Theorem 3.1.** *Let  $s(c)$  be Mathieu series defined by (1.1). Then we have*

$$\frac{1}{c^2 + 5/12} < s(c) < \frac{1}{c^2 + 1/6} \quad (c \in R), \quad (3.1)$$

where the constant  $1/6$  is the best possible.

*Proof.* Let  $f(x)$  be a function which is continuous and  $q$ th differential. Using Euler-Maclaurin formula (see [5]), we have

$$\begin{aligned} \sum_{k=n+1}^m f(k) &= \int_n^m f(x)dx + \sum_{k=1}^q \frac{(-1)^k}{k!} B_k f^{(k-1)}(x)|_n^m \\ &\quad + \frac{(-1)^{q+1}}{q!} \int_n^m P_q(x) f^{(q)}(x)dx, \end{aligned} \quad (3.2)$$

where  $m > n$ ,  $n, m \in N$ . Let  $q = 3$ ,  $n = 2$ , and  $m \rightarrow \infty$ . we can obtain

$$\sum_{k=2}^{\infty} f(k) = \int_2^{\infty} f(x)dx + \frac{1}{2}f(2) - \frac{1}{12}f'(2) + \frac{1}{6} \int_2^{\infty} P_3(x)f^{(3)}(x)dx. \quad (3.3)$$

Assume that  $f(x) := 2x/(x^2 + c^2)^2$ . By computation, we find the following results:

$$\begin{aligned} \int_2^{\infty} f(x)dx &= \frac{1}{4 + c^2}; \quad f(2) = \frac{4}{(4 + c^2)^2}; \\ f'(2) &= \frac{2}{(4 + c^2)^2} - \frac{32}{(4 + c^2)^3}; \\ f^{(3)}(x) &= \frac{480c^2}{(x^2 + c^2)^4} - \left[ \frac{120}{(x^2 + c^2)^3} + \frac{384c^4}{(x^2 + c^2)^5} \right]. \end{aligned}$$

Hence by (3.3), we have

$$\begin{aligned} s(c) &= \frac{2}{(1 + c^2)^2} + \sum_{n=2}^{\infty} \frac{2n}{(n^2 + c^2)^2} = \frac{2}{(1 + c^2)^2} + \frac{1}{4 + c^2} + \frac{11}{6(4 + c^2)^2} \\ &\quad + \frac{8}{3(4 + c^2)^3} - \frac{1}{6} \int_2^{\infty} P_3(x) \left[ \frac{120}{(x^2 + c^2)^3} + \frac{384c^4}{(x^2 + c^2)^5} \right] dx \\ &\quad + \frac{1}{6} \int_2^{\infty} P_3(x) \frac{480c^2}{(x^2 + c^2)^4} dx. \end{aligned} \quad (3.4)$$

By Lemma 2.1, we have

$$\begin{aligned}
 0 &< \int_2^\infty P_3(x) \frac{480c^2}{(x^2 + c^2)^4} dx < \frac{480c^2}{64(4 + c^2)^4} = \frac{30}{4(4 + c^2)^3} - \frac{30}{(4 + c^2)^4} \\
 &- \frac{1}{64} \left[ \frac{120}{(4 + c^2)^3} + \frac{384c^4}{(4 + c^2)^5} \right] < - \int_2^\infty P_3(x) \left[ \frac{120}{(x^2 + c^2)^3} + \frac{384c^4}{(x^2 + c^2)^5} \right] dx \\
 &< 0.
 \end{aligned}$$

Consequently, (3.4) can be reduced to  $J(c) < s(c) < I(c)$ , where  $I(x)$  and  $J(x)$  are defined respectively by (2.4) and (2.7). In virtue of (2.8) and (2.9), and in view of the left-hand side of (1.4), it follows that the inequality of the left-hand side of (3.1) is valid. It is know from the right-hand side of (1.5), (2.5) and (2.6) that the right-hand side of (3.1) is also valid.

We now show that for any  $\epsilon > 0$ , there exists  $c_0 > 0$ , such for  $c > c_0$ , that

$$s(c) \geq \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1 + \epsilon}{30c^6} + \frac{57}{2048c^8}. \tag{3.5}$$

Otherwise, by (1.5), we may get an increasing sequence  $\{c_n; c_n \uparrow \infty\}$ , which provides the following inequality

$$\begin{aligned}
 \frac{1}{c_n^2} - \frac{1}{6c_n^4} - \frac{1}{30c_n^6} - \frac{153}{2048c_n^8} &< s(c_n) < \frac{1}{c_n^2} - \frac{1}{6c_n^4} - \frac{1 + \epsilon}{30c_n^6} + \frac{57}{2048c_n^8}; \\
 \left( \frac{\epsilon}{30}c_n^2 - \frac{210}{2048} \right) \frac{1}{c_n^6} &< s(c_n) - \left( \frac{1}{c_n^2} - \frac{1}{6c_n^4} - \frac{1 + \epsilon}{30c_n^6} + \frac{57}{2048c_n^8} \right) < 0.
 \end{aligned}$$

When  $n$  is large enough, we have  $\left( \frac{\epsilon}{30}c_n^2 - \frac{210}{2048} \right) \frac{1}{c_n^6} > 0$ . This is a contradictory. Then (3.5) is valid. Hence we may find a real number  $c$ , such  $c (c > c_0)$  is larger enough, that

$$\begin{aligned}
 1 - s(c) \left( c^2 + \frac{1}{6} + \epsilon \right) &\leq 1 - \left[ \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1 + \epsilon}{30c^6} + \frac{57}{2048c^8} \right] \left( c^2 + \frac{1}{6} + \epsilon \right) \\
 &= - \left( \epsilon - \frac{11 + 36\epsilon}{180c^2} - \dots \right) \frac{1}{c^2}. \tag{3.6}
 \end{aligned}$$

In virtue of (3.6), we obtain that  $s(c) > \frac{1}{c^2 + \epsilon + 1/6}$ , for this  $c$ . Hence the constant  $1/6$  in (3.1) is the best possible. Thus we complete the proof of the theorem.  $\square$

**Remark.** New inequality (3.1) is always valid for any real number  $c$ . Evidently the obtained result (3.1) is a bilateral improvement of (1.2). Moreover it is more succinct than (1.3). Particularly we point out that it is better than (1.3) when  $|c| < 1$ . At last we mention that it is impossible to improve the left-hand side of (3.1) any more, because of the constant  $1/6$  being the best possible.

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