ON QUANTUM PHYSICS AND
NON-COMMUTATIVE GEOMETRY

Lucian M. Ionescu

Department of Mathematics
Illinois State University
1504, N. Hershey Rd., Bloomington, IL 61704, USA
e-mail: LMIones@ilstu.edu

Abstract: The *quantum-event (prime ideal in a category) noncommutative-point* alternative to *classical-event (commutative prime ideal) point* is suggested.

Ideals in additive categories, prime spectra and representation of quivers are considered as mathematical tools appropriate to model quantum mechanics. The space-time framework is to be reconstructed from the spectrum of the path category of a quiver.

1. Introduction

Topological quantum field theory (see Witten [33], Atiyah [2], Crane [13], Crane and Yetter [14], Crane and Frenkel [10], etc.) is a rigorous mathematical framework for the Feynman path integral formalism, expressed in categorical language. It is a "categorification" of quantum physics, in the broader sense of the term Crane and Yetter [15], Baez et al [7], Ionescu [19, 20]. Together with the progress in quantum groups and geometric quantization, there is enough evidence to say that, at a deeper level of understanding, what is referred to in the physical realm as quantum phenomena, corresponds in mathematics to noncommutative geometry and, as stated in Rosenberg [29], the viewpoint orig-

inally due to Grothendieck and expressed in Manin [23], p.83, is that *"To do geometry you really do not need a space. All you need is a category of sheaves on this would-be space."*

Our intention in this essay, is to bring together some ideas and categorical tools, and to state some questions and comments on the categorification of the formalism of quantum mechanics in relation to non-commutative geometry.

1.1. Space. Of What Kind?

Since 1930's general relativity, the physical theories based on geometric methods evolved by adding *internal degrees of freedom* to the *external degrees of freedom* (adding "structure" to geometric points) due to the presence of a space-time model, in an attempt to account for the other fundamental interactions.

In Kaluza-Klein theories the additional dimensions were still *external dimensions*. Electromagnetism as a $U(1)$-Yang-Mills theory has one additional *internal dimension*, nonabelian YM theory still add extra *internal dimensions*, etc. One may "extrapolate" and "predict" that quantum physics of the fundamental interactions should evolve towards a no-external-all-internal dimensions type of theory. More precisely, we think of quantum field theory being modeled as an abstract theory from which space-time, not anymore a fundamental concept, is reconstructed not necessarily as a limit, but rather as a moduli space from a categorical structure, as part of a *reduction principle*.

In this abstract theory the basic concept is not the *space-time event* (classical event) as a potential occurrence or "existence", but rather the *elementary quantum process* (quantum event), as a correlation or "interaction". The *interaction* itself, should be the primary concept, and not built on top of a preexisting structure.

In mathematical terms this should correspond to the natural evolution from a "set-theoretical approach" with emphasis on "elements", and with additional structure as secondary data, to a categorical approach with its emphasis on *morphisms* (relations / correlations), and with the structure given through the category itself.

At the physical level, one may identify this tendency as part of the process of deemphasizing configuration spaces, and studying Poisson manifolds rather then cotangent spaces. More then that, one should perhaps *model spaces of evolutions* in a direct manner, rather then to represent them based on a state space (i.e. as integral curves).

1.2. Gravity. Too Classical?

The incompatibility between general relativity and quantum mechanics is well-known and accounted for by the 50-60 years of unconvincing attempts to "quantize gravity" (Porrati [27], Butterfield et al [8]). There must be a deep reason for the incompatibility between general relativity, a classical-phenomenological-macroscopical theory, and quantum physics, a fundamental and highly experimental theory, and for the unsuccessful attempts to unify gravity with the other fundamental interactions. Perhaps general relativity is an effective theory, as for example Fermi theory, which is not "...cloaked in the beautiful geometric structure of General Relativity, that may lead us astray in our search for a solution to the problem of nonrenormalizability..." Porrati [27], p. 74.

We would like to question our phenomenological understanding of space-time, and recall that there are quantum gravity programs which reject "the manifold conception of space-time" Butterfield et al [8], p. 4. (see also Ng [26], p. 2). As stated in Connes [12], p. 4, "... quantum mechanics showed that indeed the parameter space, or phase space of the mechanical system given by a single atom fails to be a manifold".

We consider that the classical algebraic-geometry-physics point of view:

commutative prime ideal ↔ space-time point ↔ classical event

should be replaced by:

prime ideal in a category ↔ NC-point ↔ quantum event.

An abstract C(ategorical)QFT, based on representations of quivers or abstract categories, and abandoning the manifold approach to space-time (compare Butterfield et al [8]), should generalize TQFTs based on a cobordism category (see also Fukaya [18], Example 2.2 and Remark 2.3, p. 103). The "space-time" should be reconstructed as a prime spectrum of a category, as part of a "correspondence principle", and should not be considered a primary concept.

As an "example" to give substance to this idea, one may consider the Feynman diagrams as representations of quivers bounding a certain basic quiver specific to the quantum experiment being modeled, and determined by the "geometry" and preparation stages of the experiment. The allowable Feynman diagrams bounded by the basic quiver should be related to the cut-off procedures applied to the experimental data.

The divergency of the Feynman integrals defined on the reconstructed space, and due to the *unnecessary abundance of topological paths*, should be reduced

according to a "Hodge-principle" (existence of a "canonical representative", e.g. harmonic form) applied to homotopy classes of paths.

The physical counterpart would be the "particle-wave duality", where the quantum process (say involving an electron) behaves "like a particle" in an "irreducible subprocess", means that the calculation can be classical, on a geometric path (the "Hodge representative"), while the overall amplitude is a result of the interference due to the nontrivial homotopy structure of the basic quiver (integration on the homotopy classes), so that overall the quantum process behaves "like a wave", i.e. exhibits interference.

Due to the believe that quantum phenomena should be modeled by noncommutative geometry, we consider as a long-term goal to develop a "finer" axiomatic theory of quantum mechanics based on spectra of categories of representations of quivers, rather then based on Hilbert spaces and operators, in a natural correspondence with noncommutative geometry (categorical analog of string theory?). It is well-agreed, Connes [12], p. 13, that "C^*-algebras give the required framework for noncommutative topology". We advocate here the apriori quantum interpretation for the structure of C^*-algebras (Section 2.3), when used to model quantum phenomena. The byproduct is that, by labeling diagrams with Lagrangian densities, one obtains a discrete analog of a Riemannian or Lagrangian manifold, consistent to Connes [12], p. 4: "... in a discrete manifold the principle of metric relations is already contained in the concept of the manifold, ...".

A unification of the fundamental interactions (without gravity if understood only by macroscopic evidence), is then perhaps conceivable not by focusing on finding the "overall gauge group", but a unification at a categorical level, and obtaining "gravity" by reduction at the classical level (on the "moduli space"), as a byproduct of the unification.

To support this view, we refer to the result of Connes-Kreimer [11], that "... the combinatorics of Feynman graphs gives rise to a Hopf algebra...", and opening a new direction of study of the classical particle physics principle of correspondence: particles - representations, interactions-representation theory.

2. NC Geometry

Assuming that the "categorification" and "Spec" functors should commute, we translate the definition of an affine scheme.

2.1. Ideals and the Prime Spectrum

Ideals in additive categories are discussed in Ionescu [21], and in the references cited therein.

Let $\mathcal{A}$ be a pre-additive category, i.e. a category such that for any two objects A, B in $\mathcal{A}$, $\text{Hom}(A, B)$ is an object in the category $\mathcal{A}b$ of Abelian groups.

A (bilateral) *ideal* in $\mathcal{A}$ is a class of morphisms (subfunctor, Ionescu [20], p. 2), stable under addition and composition whenever defined. More precisely, a left ideal I in $\mathcal{A}$, is a family of Abelian subgroups $I(A, X)$ of $\text{Hom}(A, X)$ indexed by $A, X \in \mathcal{A}$, and such that the family of groups is stable under left composition with arbitrary morphisms $\phi : X \to Y$:

$$I(A, \phi) = Hom(A, \phi)_{|I(A,X)}$$

$$A \underset{I(B)}{\overset{I(A)}{\rightrightarrows}} \Downarrow I(f) \quad \mathcal{A}b$$

$$A \underset{I(A,Y)}{\overset{I(A,X)}{\rightrightarrows}} X \qquad I(B,X) \xrightarrow{\cdot \circ f} I(A,X)$$
$$\downarrow \phi \qquad \phi \circ \cdot \downarrow \qquad \downarrow \phi \circ \cdot$$
$$Y \qquad I(B,Y) \xrightarrow[\cdot \circ f]{} I(A,Y)$$

The double arrow denotes a set of morphisms, and the left diagram is not necessarily "commutative": composing ϕ with morphisms from $I(A, X)$ may result in a proper subset of $I(A, Y)$.

The approach in Rosenberg [30] to define the spectrum of an Abelian category leads to a reconstruction theorem extending a know result of Gabriel. Since we consider that the primary concepts are the morphisms we will adopt another approach. We will consider the categorification of the usual definition of a prime ideal Atiyah et al [4], Jategaonkar [22] (p. 2). For alternative definitions see Baumgartner [5] (p. 191), Sun et al [31] (p. 140), Botha et al [6] (p. 1173).

An ideal P is a *prime ideal* iff for any composible morphisms f and g, $f \circ g \in P$ implies either $f \in P$ or $g \in P$. The *prime spectrum* of $\mathcal{A}$ is the class of prime ideals.

2.2. Representations of Quivers and Algebras

Between sets and categories, a *quiver* Q (or *precategory*) is a "small category" without a composition of morphisms. A *morphism of quivers* is a "functor"

without the condition regarding the composition of morphisms, since there is none yet defined.

The category of quivers $\mathcal{Q}v$ relates to the (2-)category of small categories $\mathcal{C}at$ similar to how sets relates to monoids. There is a forgetful functor $U : \mathcal{C}at \to \mathcal{Q}v$, with a left adjoint $\mathcal{P}$, giving the free category generated by a given quiver:

$$\operatorname{Hom}_{\mathcal{C}at}(\mathcal{P}(Q), \mathcal{C}) \equiv \operatorname{Hom}_{\mathcal{Q}v}(Q, U(\mathcal{C}))$$

$\mathcal{P}(Q)$ is also called the *path category* generated by Q.

Let $\mathcal{C}$ be an arbitrary category and k a ring, e.g. $\mathbb{Z}$. Consider the category $\mathcal{A} = k\mathcal{C}$ with the same objects as $\mathcal{C}$ and with morphisms the free k-modules generated by the morphisms of the category $\mathcal{C}$. Then $\mathcal{A}$ is a k-linear category, and in particular additive, in an obvious way.

To an arbitrary k-linear small category (additive, if $k = \mathbb{Z}$), one may associate a k-algebra (algebra), called its *matrix algebra* (see Mitchell [24], p. 33). The matrix algebra of the free k-linear category associated to the path category generated by a quiver Q is the *path algebra* kQ. It is a categorification of the group ring construction, corresponding to the "multi-object" case. Explicitly, on generators, the product of two paths is zero, unless they produce a concatenated new path Ringel [28] (p. 146).

Note that any additive category $\mathcal{A}$ is a quotient of a path category $\mathcal{P}(U\mathcal{A})$ by the kernel (ideal in $\mathcal{P}(U\mathcal{A})$) of the path evaluation functor (via composition).

Alternatively, in order to relate with the geometric side (state-sum models and all that), the elements of $C_\bullet(\mathcal{A}) = \mathcal{P} \circ U(\mathcal{A})$ may be viewed as singular simplices, i.e. functors from the standard simplex (semi simplicial category) Δ_n to $\mathcal{A}$ (Ionescu [20], p. 8). In this way morphisms may be thought of as 1-simplices, and the path algebra, as a k-module, is just the (graded) free k-module generated by singular simplices of (categorical) singular homology.

A *representation of quivers* $F : Q \to U(k - \mathrm{mod})$ is a morphism of quivers valued in a category of modules. It extends uniquely to a functor $F : \mathcal{P}(Q) \to k - \mathrm{mod}$. If $k = \mathbb{Z}$, then it is a $\mathcal{P}(Q)$-module Street [32] (p. 140), Baumgartner [5] (p. 171). A morphism of representations of quivers is a restriction (with respect to morphisms) of a natural transformation of functors.

Regarding the quantum mechanics formalism, considering representations of quivers rather then representations of C^*-algebras, involves a gain in the initial structure involved.

2.3. Algebras and Auslender-Reiten Quivers

The idea of generalizing the ordinary "commutative" geometry by studying non-commutative algebras goes back to von Neumann and his contemporaries Witten [34] (p. 253). The inconvenience is that such an algebra might not have enough ideals ("global" ideal, contrasted to categorical ideals). It is a "pointless geometry".

We point out that there is a well-known correspondence between the one object, "total" algebra, point of view and the multi-objects point of view, where additive categories are just "rings with several objects" Mitchell [24] (groupoids = multi-object groups, etc.). For a more detailed discussion of this correspondence, see Ringel [28], Assem et al [1], Martins et al [25] and Bleher [9].

We only note that to a finite dimensional algebra R over an algebraic closed field k, one may associate the Alexander-Reiten quiver, having vertices (objects) isomorphism classes of indecomposable R-modules (i.e. $\mathrm{Spec}(R - \mathrm{mod})$), and with arrows representing a basis in a certain group of irreducible maps Ringel [28] (p. 181). If R is a basic algebra Ringel [28] (p. 147), then R is of the form $k\Gamma/I$ with Γ a quiver and I an ideal in $k\Gamma$.

2.4. Points or Paths?

Feynman path integral formalism leads to mathematical difficulties, one may argue, due to the philosophy "*space determines the paths*". For example, once a typical "space-time " is given, the set of "acceptable paths" is determined (topological, or smooth, piecewise smooth, etc.), leading to an unnecessary large space for integration in the Lagrangian formalism.

Consistent with the emphasis in quantum processes and NC-geometry on:

quantum event $< - >$ interaction/correlation $< - >$ path/morphism

it is legitimate to require having the "points" and "paths" given at the same time, as a category for instance. The structure of the category encoded in morphisms and their composition $\circ$, has a causal character ("local time"), while a monoidal structure $\otimes$ may be thought of as modeling the "spatial structure".

3. Quantum Mechanics: an Example

3.1. Interference Experiment

We will consider the elementary interference experiment with electrons, as explained in Feynman [17] (p. 37-5):
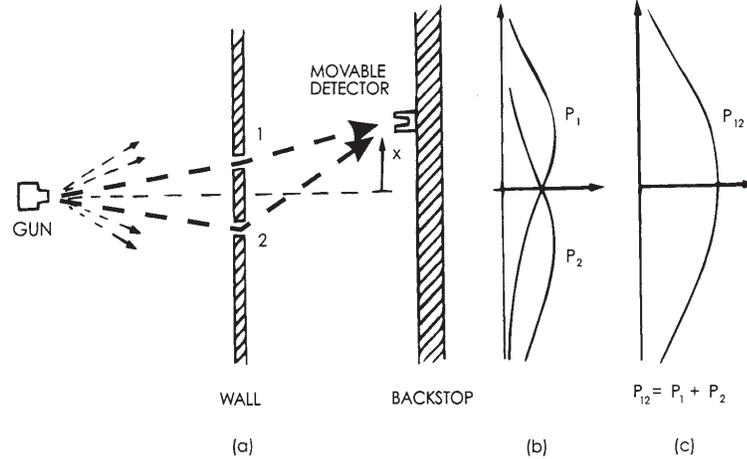
from which the "first principles of quantum mechanics" are derived.

"An experiment is one in which all of the initial and final conditions of the experiment are completely specified." What we will call an event is, in general, just a specific set of initial and final conditions. (Feynman [17] p. 37-10)

A categorical mind might choose to interpret the initial and final conditions as objects, more specifically as source and target (e.g. TQFTs: Witten [33], Atiyah [2, 3]). We will consider the following:

Principle. To any *quantum experiment* there is an *associated quiver* corresponding to the "geometry" of the experiment, including the source(s), preparation stages and finally, the measurement devices.

A perturbative approach would consider quivers (diagrams) containing it, and should be thought of as a *bounding quiver*. A filtering of the experimental data (cut-off procedure), would alter the class of allowable quivers corresponding to this bounding quiver.

For the above experiment, it is reasonable to prescribe the following quiver Q and representation S:

A gauge group $G = U(1)$ interpreted as a one object category $\mathcal{G} = \mathcal{T}_G$, and an action functor $\mathcal{S}$ as a representation of the quiver (valued in the additive category $\mathbb{C}\mathcal{G}$), are also associated:

$$\mathcal{S}(A) = *_G, \qquad \mathcal{S}(\gamma) = e^{i\mathcal{L}(\gamma)},$$

with $\mathcal{L}$ a geometric Lagrangian $\mathcal{L}(\gamma) = \text{length}(\gamma)$.

When an event can occur in several alternative ways, the probability am-plitude for the event is the sum of the probabilities amplitudes for each way considered separately. (Feynman [17] p. 37-10)

The alternative ways for an interaction, or "paths", should be viewed as morphisms (e.g. cobordisms in TQFTs Atiyah [2, 3]), and the amplitude is:

$$\int_{\text{Hom}_{\mathcal{P}}(S,T)} \mathcal{S}(\gamma)\mathcal{D}\gamma \quad = \quad \int_{\text{Spec}(\mathcal{P})} \mathcal{S}(P)\mathcal{D}P$$

where $\mathcal{P} = \mathbb{C}\mathcal{P}(Q)$ is the $\mathbb{C}$-linear category of paths generated by Q. Here the integration is just the summation over the "possible ways" (morphisms) constituting the "event".

Alternatively, the amplitude can be viewed as an integral over the prime spectrum of the path category, and consisting of the two maximal ideals:

$$P_1 = \mathbb{C}\{S \xrightarrow{S1} 1, \ 1 \xrightarrow{1T} T, \ 1T \circ S1\},$$

$$P_2 = \mathbb{C}\{S \xrightarrow{S2} 2, \ 2 \xrightarrow{2T} T, \ 2T \circ S2\},$$

with the appropriate definition of the amplitude on each "NC-point". This latter may be thought of as the result of enriching the category $\mathcal{P}$ in a category of Lagrange spaces, or in a category of complexes in a similar way the Fukaya categories are defined (see Fukaya [18] and references therein).

3.2. Quantum Versus Cosmological Event

Although stated before, we would like to stress the following two levels. One is the classical (macroscopical) phenomenological level:

EVENT $\leftrightarrow$ Space-Time POINT $\leftrightarrow$ (commutative ring) prime IDEAL

and the other is modern (quantum) fundamental level:

Quantum INTERACTION $\leftrightarrow$ NC POINT $\leftrightarrow$ (categorical) prime IDEAL

3.3. Particle-Wave Duality

The modeling of the interference experiment as a quantum process and studying the "electron" (or rather the phenomenon "electron in this experimental setup") was sketched in Section 3, by associating the basic quiver and its representation (action).

It can be interpreted as considering the four *elementary processes* on which the "electron" behaves like a particle, and consequently defining the action $S(\gamma)$ as corresponding to a geometric path, as if operating a "homotopical" reduction of the space of all Feynman paths. The overall behavior of the "electron", i.e. the result of the interference due to a non-trivial π_1, is the "behavior like a wave", within this experiment. In this way the "particle-wave duality" principle is a natural consequence of the theory.

4. Conclusions

Whether it is "cohomological physics" or "categorical physics", it benefitted from more and more abstract mathematical models.

In this essay we have tried to raise some implicit questions, through the introduction of some ideas and goals:

1) Formulation of quantum physics in categorical language, in terms of representations of quivers.

2) The mathematical model should be relevant to non-commutative geometry.

3) **The primary concepts are Feynman diagrams**, viewed as representations of quivers. Paths (morphisms) should be considered part of the primary data, along with kernel operators attached via representations of quivers.

The reduction to homotopy classes should be viewed as a principle establishing the correspondence with the traditional approach to Feynman integrals.

4) Space-time is no longer a basic concept in quantum physics, and should be reconstructed as a moduli space, rather then as a "classical limit".

5) "Particle-wave" duality principle should be a natural consequence of the theory. The "wave-like behavior" (interference) occurs as a consequence of

the multitude of "homotopy classes" of paths. The "particle-like behavior" is the result of modeling a subprocess as a classical propagation along a classical (minimal/geometrical) path.

6) Gravity should be obtained at a classical level as a byproduct of a unification of the three fundamental interactions not by enlarging the gauge group, but by enriching the categorical structure of the theory.

To conclude, as an alternative to the manifold approach to space-time, representations of quivers should replace the classical functions on configuration spaces, and paths being thought of as encoding the possible transitions of the system.

The "reality" should be less thought of as a classical-limit, where one may forget its "true quantum nature", and evaluations of partition and n-point functions should rather be considered as "quantum computing", where the mathematical quantum model is used both as a theoretical computational device to be "built" into a specific quantum computing device, as well as a theory which should be tested on such a quantum computer Deutsch et al [16].

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