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COHOMOLOGICALLY q-COMPLETE ANALYTIC SUBSETS OF OPEN SUBSETS OF $\mathbf{C}^{(\mathbf{N})}$

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Abstract: Let V be a closed analytic subset of an open subset of $\mathbf{C}^{(\mathbf{N})}$ such that the intersection of V with all finite-dimensional coordinate subspaces L_n is cohomologically q-complete. Then $H^i(V, E) = 0$ for every $i > q$ and every holomorphic vector bundle E with finite rank on V. If $L_n \cap V$ is assumed to be smooth and q-complete for all $n > 0$, then $H^i(V, E) = 0$ for every $i > q$ even if E has fibers isomorphic to a Banach space.

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1. Introduction

Let $\mathbf{C}^{(\mathbf{N})}$ denote the countable direct sum of copies with the natural locally convex topology (or, if you prefer) any complex vector space with countable and infinite algebraic dimension and the inductive topology. Let $i_n : \mathbf{C}^n \to \mathbf{C}^{(\mathbf{N})}$ and $j_n : \mathbf{C}^n \to \mathbf{C}^{n+1}$ denotes the natural inclusions (i.e. $i_n((z_1, \dots, z_n)) = (z_1, \dots, z_n, 0, \dots)$ and $i_n((z_1, \dots, z_n)) = (z_1, \dots, z_n, 0)$.

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Definition 1. Fix an integer $q \geq 0$. Let U be an open subset of $\mathbf{C}^{(\mathbf{N})}$ and V a closed subset of U. As in [3], Definition 2.1, we will say that V is a closed analytic subset of U if for every integer $n > 0$ the set $i_n^{-1}(V)$ is a closed analytic subset of $i_n^{-1}(U)$. We will say that the closed analytic set V is q-complete (resp. cohomologically q-complete) if if for every integer $n > 0$ the set $i_n^{-1}(V)$ is q-complete (resp. cohomologically q-complete).

In Definition 1 we used the convention that 0-complete is equivalent to Stein; hence our q-completeness would be called $(q+1)$-completeness in [1].

Remark 1. By [1] q-complete implies cohomologically q-complete.

Proposition 1. *Let U be an open subset of $\mathbf{C}^{(\mathbf{N})}$, V a cohomologically q-complete analytic subset of U and E a holomorphic vector bundle with finite rank on V. Then $H^i(V, E) = 0$ for every $i > q$. In particular we have $H^i(V, \mathcal{O}_V) = 0$ for every $i > q$.*

Proof. Let $\mathcal{I}_n$ be the ideal sheaf of $i_n^{-1}(V)$ in $i_{n+1}^{-1}(V)$. Since $\mathcal{I}_n \otimes i_{n+1}^*(E)$ is coherent and $i_{n+1}^{-1}(V)$ is cohomologically q-complete, we have $H^i(i_{n+1}^{-1}(V), \mathcal{I}_n \otimes i_n^*(E)) = 0$ (see [1]). This equality for $i = q+1$ gives the surjectivity of the restriction map $\rho_n : H^q(i_{n+1}^{-1}(V), i_{n+1}^*(E)) \to H^q(i_n{-}1(V), i_n^*(E))$. Similarly, $H^i(i_n{-}1(V), i_n^*(E)) = 0$ for every $i > 0$. The surjectivity of all maps ρ_n implies that the so-called Mittag-Leffler condition is satisfied and hence taking an inductive limit we conclude. □

Theorem 1. *Let U be an open subset of $\mathbf{C}^{(\mathbf{N})}$ and V a smooth q-complete analytic subset of U, i.e. we assume $i_n^{-1}(V)$ smooth for every $n > 0$. Let E be a holomorphic vector bundle on U with fibers isomorphic to a Banach space. Then $H^i(V, E) = 0$ for every $i > q$.*

Proof. Let $\mathcal{I}_n$ be the ideal sheaf of $i_n^{-1}(V)$ in $i_{n+1}^{-1}(V)$. Hence $\mathcal{I}_n$ is a holomorphic line bundle (even holomorphically trivial, but we do not need this observation). Hence $\mathcal{I}_n \otimes i_{n+1}^*(E)$ is a holomorphic vector bundle with fibers isomorphic on the q-complete manifold $i_{n+1}(V)$. By [2] we have $H^i(i_{n+1}^{-1}(V), \mathcal{I}_n \otimes E) = 0$ for every $i > q$. Consider the following exact sequence on $i_{n+1}^{-1}(V)$:

$$0 \to \mathcal{I}_n \otimes i_{n+1}^*(E) \to i_{n+1}^*(E) \to i_n^*(E) \to 0 \tag{1}$$

Since $H^{q+1}(i_{n+1}^{-1}(V), \mathcal{I}_n \otimes E) = 0$, the restriction map

$$H^q(i_{n+1}^{-1}(V), i_{n+1}^*(E))) \to H^q(i_n^{-1}(V), i_n^*(E))$$

is surjective. Hence the Mittag-Leffler condition is satisfied and we conclude as in the proof of Proposition 1. □

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