International Journal of Pure and Applied Mathematics
Volume 12 No. 1 2004, 23-31

UNIQUENESS OF POSITIVE RADIAL SOLUTIONS FOR $\triangle u + f(u) = 0$ ON THE ANNULUS

Samuel Jator[1], Zachariah Sinkala[2 §]

[1]Department of Mathematics
AustinPeay State University
Clarksville, TN 37044, USA
e-mail: JatorS@apsu.edu

[2]Department of Mathematical Sciences
Middle Tennessee State University
Murfreesboro, TN 37132, USA
e-mail: zsinkala@mtsu.edu

Abstract: In this paper we prove uniqueness of positive solutions to the semilinear equation $\triangle u + f(u) = 0$ subject to the Dirichlet boundary condition on an annulus in $\Re^n, n \geq 3$. The result applies to a wide class of nonlinear functions f, for example , nonlinear functions of type $f(u) = \sum_{k=1}^{\nu} a_k u^{p_k}, 1 = p_1 < p_2 < ... < p_\nu = p$, $a_1 < 0, a_k \leq 0$ for $2 \leq k \leq \nu - 1$ and $a_\nu > 0$, and $\sum_{k=2}^{\nu-1} a_k + a_\nu \alpha^{p-1} > 0$ and $\sum_{k=2}^{\nu} a_k > 0$. This type class of f also includes an important model case $f(u) = -u + u^p, p > 1$. The result is proved by reducing to initial-boundary problem for the ordinary differential equation $u'' + \frac{n-1}{r} u' + f(u) = 0$ and using a shooting method.

AMS Subject Classification: 34B16, 34B16
Key Words: Dirichlet boundary conditions, radial solution, shooting method

1. Introduction

Let $0 < a < b \leq \infty$. Let $\Omega = \{x \in \Re^n : a < |x| < b\}$ be an annulus in $\Re^n, n \geq 3$. The main purpose of this paper is to prove uniqueness of radial solutions to the

Received: December 19, 2003

§Correspondence author

Dirichlet boundary value problem

$$\begin{aligned} &\triangle u + f(u) = 0 \text{ in } \Omega, \\ &u > 0 \text{ in } \Omega, \\ &u = 0 \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where $f : [0, \infty) \to \Re$ satisfies:

(i) $f \in C^1([0, \infty)) \cap C^2((0, \infty)), f(0) = 0, f'(0) < 0.$

(ii) for some $\alpha > 0, f(u) < 0$ for $0 < u < \alpha$,
$f(u) > 0$ for $u > \alpha$ and $f'(\alpha) > 0$.

(iii) if $u > \alpha$ then $f''(u) > 0$.

Note that the Dirichlet condition at $b = \infty$ is interpreted as $u \in L^2(\Omega)$.

Theorem 1.1. *Let $0 < a < b \leq \infty$. Let $\Omega = \{x \in \Re^n : a < |x| < b\}$ be an annulus $\Re^n, n \geq 3$. Then for any f satisfying conditions (i),(ii) and (iii), the Dirichlet problem* (1) *admits at most one radial solution.*

Problem (1) for the case, where the nonlinear function $f(u) = -u + u^p, p > 1$ in a finite ball $\Omega = \{x \in \Re^n : |x| < b\}$ has been well studied in a large number of papers, pioneered by the work of Coffmann [1] in 1972. The uniqueness problem has been completely solved in this case, for example see the work by Peletier and Serrin [6], McLeod and Serrin [5] and Kwong [3].

In the annulus case, problem (1) with nonlinearity $f(u) = -u + u^p, p > 1$ is more difficult since positive radial solutions of (1) in annulus are not monotonous in the radial direction. In 1996, Coffman [2] proved for the case $n = 3$ and $1 < p \leq 3$. Yadava [9] proved for $n = 3, 4$ and $1 < p \leq \frac{n}{n-2}$ and $n \in \{5, 6, 7, 8\}$ and $1 < p \leq p_0(n) < \frac{n}{n-2}$. In 2003, Tang [8] proved for $n \geq 3$ and $p > 1$.

The main idea in our approach is based on the equation $u'(r) = \beta v'(r)e^{K(r)}$. In particular, we let $u = u(r)$, $r = |x|$ be a radial solution of (1). If $u'(a) = \beta$, then $\beta > 0$ by Hopf Boundary Lemma. Hence, u is also the unique solution to the initial value problem of the ordinary differential equation

$$\begin{aligned} &u'' + \tfrac{n-1}{r}u' + f(u) = 0, \\ &u(a) = 0, \;\; u'(a) = \beta > 0, \end{aligned} \tag{2}$$

where f satisfies conditions (i) , (ii) and (iii). We denote the solution by $u(r, \beta)$ and $v = \frac{\partial u(r,\beta)}{\partial \beta}$ solves

$$\begin{aligned} &v'' + \tfrac{n-1}{r}v' + f'(u)v = 0\,, \\ &v(a) = 0, \;\; v'(a) = 1\,. \end{aligned} \tag{3}$$

We define ζ as in Tang [8] and its derivative by

$$\begin{aligned}\zeta(r) &= r^n[u'v' + f(u)v] + (n-2)r^{n-1}u'v,\\ \zeta'(r) &= 2r^{n-1}f(u)v,\end{aligned} \tag{4}$$

where ζ' is the derivative of ζ.

This paper is organized as follows: In Section 2 we recall some basic properties of radial solutions. In particular, we shall mention that, in the interval (a, b), a radial solutions u of equation (1) has exactly one critical point, say $r = c$, at which necessarily u takes a maximum value and $\eta = u(c) > \alpha$. In Section 3, we shall recall a vanishing property for the variation v, namely, v is positive on $(a, c]$ and vanishes at a point, say $r = \tau$, in (c, b).

As usual, whether or not v has another zero beyond τ is the most crucial and difficult part for the proof of uniqueness. But in this paper, we produce a very simple proof that v has a unique zero at $r = \tau$. This simple proof is given in Section 4. Finally, the results in this paper extends results of Tang [8]. The critical information in Tang [8] results is the assumption that $uf'(u) - f(u) > 0$, for all $r \in (a, b)$. In our results we do not require this assumption. We apply our main result on problem(1) with $f(u) = \sum_{k=1}^{\nu} a_k u^{p_k}$, where $1 = p_1 < p_2 < \ldots p_\nu = p$, $a_1 < 0, a_k \leq 0$ for $2 \leq k \leq \nu - 1$ and $a_\nu > 0$,

$$\sum_{k=2}^{\nu-1} a_k + a_\nu \alpha^{p-1} > 0 \text{ and } \sum_{k=2}^{\nu} a_k > 0.$$

In particular, the description of f above includes, for example $f(u) = -u - u^2 + 2u^3$ and $f(u) = -u + u^p, p > 1$. But $f(u) = -u - u^2 + 2u^3$ does not satisfy $uf'(u) - f(u) > 0$, for all $r \in (a, b)$. Therefore Tang [8] results cannot be applied on the example, $f(u) = -u - u^2 + 2u^3$.

2. Basic Properties of Solutions

Theorem 2.1. *Let $0 < a < b \leq \infty$, and $\Omega = \{x \in \Re^n : a < |x| < b\}$ be an annulus in $R^n, n \geq 3$. Then for any f satisfying conditions (i), (ii) and (iii) the Dirichlet problem(1) admits at most one radial solution.*

Theorem 2.1 follows if we can show that there is at most one $\beta > 0$ such that the solution of equation (2) satisfies, for

$$b < \infty, u(r) > 0 \text{ for } r \in (a, b) \text{ and } u(b) = 0;$$

by Hopf Boundary Lemma, we have $u'(b) < 0$. For $b = \infty, u(r) > 0$ for $r > a$ and $\lim_{r\to\infty} u(r) = 0$. In the second case $u(r)$ decays to zero exponentially, see McLeod and Serrin [5], and Coffman [2]:

$$\lim_{r\to\infty} u(r)e^{\frac{r}{2}} = \lim_{r\to\infty} u'(r)e^{\frac{r}{2}} = 0. \tag{5}$$

By Dirichlet boundary condition, $u(r)$ has critical points in (a, b); In fact it has one such point, at which u takes the maximum value. This can be proved by standard argument using energy function,

$$E(r) = \frac{u'^2(r)}{2} + F(u(r)), \text{ where } F(u) = \int_0^u f(s)ds, \tag{6}$$

using the conditions of f. Note that

$$E'(r) = u'u'' + f(u)u' = -\frac{n-1}{r}u'^2(r) \leq 0.$$

We shall denote this critical point by c. Hence

$$u'(r) > 0 \text{ for } r \in (a, c) \text{ and } u'(r) < 0 \text{ for } r \in (c, b). \tag{7}$$

Lemma 2.2. *Let $u = u(r)$ be a radial solution of problem(1). Let $\eta = u(c)$ be the maximum of u over (a, b). Then $F(\eta) > 0$ and $\eta > \alpha$.*

Proof. Since

$$E'(r) \leq 0,$$

then $E(r)$ is a non-increasing function.

Case 1. For $b < \infty$, we have

$$E(b) = \frac{u'^2(b)}{2} + F(u(b)).$$

Since $u(b) = 0$ then $F(u(b)) = 0$. We get

$$E(b) = \frac{u'^2(b)}{2} > 0.$$

Since $E(r)$ is a non-increasing function and $E(b) > 0$ then $E(c) = \frac{u'^2(c)}{2} + F(u(c)) > 0$. Then $F(\eta) > 0$. Let $\eta \leq \alpha$. By condition (ii) $f(u(r)) < 0$ for all $r \in [a, b]$. Then $F(\eta) < 0$ and this contradicts the fact that $F(\eta) > 0$. Thus, $\eta > \alpha$.

Case 2. For $b = \infty$, we have $E(r) \to 0$ as $r \to \infty$. Thus $E(r) > 0$ for any $a \leq r < b$. In particular, $E(c) = F(u(c)) = F(\eta) > 0$, resulting $\eta > \alpha$. □

3. Vanishing Properties of the Variation

In this section, we prove the existence of the first zero of v beyond a. Our proof is a generalization of the proof given by Tang [8]. Earlier work done on Location of the first zero of v beyond a see work done by Coffman [2] and Yadava [9] in the Dirichlet case and special cases of f.

Lemma 3.1. *Let $u = u(r, \beta)$ be a radial solution of problem (1). Then there exists $\tau \in (c, b)$ such that the variation $v = v(r, \beta)$ vanishes at τ and remains positive for $r \in (a, \tau)$.*

Proof. Since u has a unique critical point $r = c$, and $\eta = u(c) > \alpha$ by Lemma 2.2. There exists

$$a_1 \in (a, c) \text{ and } b_1 \in (c, b)\,,$$

$$u(a_1) = \alpha, \quad u(b_1) = \alpha, \quad u'(a_1) > 0 \quad \text{and} \quad u'(b_1) < 0.$$

We must show that v must vanish somewhere in (a_1, b_1). Suppose for contradiction that v has the same sign, say $v > 0$, over (a_1, b_1). Let

$$\varrho(r) = r^{n-1}[f'(u)u'v - f(u)v'].$$

Note that $\varrho(r)$ is the Wronskian of u' and v', $\varrho(r) = v''u' - u''v'$.

Using the fact that

$$u(a_1) = u(b_1) = \alpha, u'(a_1) > 0, u'(b_1) < 0.$$

Then

$$\varrho(r) = r^{n-1}[f'(u)u'v - f(u)v'],$$

$$\varrho(a_1) = a_1^{n-1}[f'(\alpha)u'(a_1)v(a_1)] \geq 0$$

and

$$\varrho(b_1) = b_1^{n-1}[f'(\alpha)u'(b_1)v(b_1)] \leq 0.$$

But by a differentiation of $\varrho(r)$ and for $r \in (a_1, b_1)$, $u(r) > \alpha$, we get

$$\varrho'(r) = r^{n-1}f''(u)vu'^2 > 0\,,$$

for all $r \in (a_1, b_1)$, except at $r = c$, where $\varrho'(r) = 0$, we have a contradiction. Therefore, v must vanish at some point in (a_1, b_1).

Let τ be the first zero of v beyond $r = a$ then $\tau < b_1$. Since $v(a) = 0$ and $v(\tau) = 0$, then v must have critical points in (a, τ) (by Rolle Theorem); Let $d \in (a, \tau)$ be such that $v'(d) = 0$, and $v' > 0$ in (a, d) then $d > a_1$.

Now, we show that $\tau > c$. Note that since $\varrho(a_1) \geq 0$ and $\varrho(r) > 0$ in (a_1, τ),

$$\varrho(d) = d^{n-1} f'(u(d)) u'(d) v(d).$$

Suppose $d \geq c$. Using the fact that $f'(\alpha) > 0$ and $f'(u)$ is increasing for $u > \alpha$, we have $f'(u(d)) > 0$ and $u'(d) \leq 0$. Then $\varrho(d) \leq 0$. This contradicts the fact $\varrho(r) > 0$ in (a_1, τ). Then $d < c$. Thus the first critical point of v appears in (a_1, c). The function

$$\begin{aligned} \zeta(d) &= d^n [u'(d)v'(d) + f(u(d))v(d)] + (n-2) d^{n-1} u'(d) v(d) \\ &= d^n [f(u(d))v(d)] + (n-2) d^{n-1} u'(d) v(d) > 0. \end{aligned}$$

We know that $a_1 < d < \tau < b_1$, using $\zeta'(r) = 2r^{n-1} f(u) v$, and integrating both sides from d to τ, we get

$$\zeta(\tau) = \zeta(d) + 2 \int_d^\tau r^{n-1} f(u) v dr > 0\,, \tag{8}$$

from which it follows that $\tau > c$. □

4. Proof of Theorem

Since $f(0) = 0$ and v vanishes in the interval (a, b), the uniqueness of radial solutions in a finite annulus is valid if $v(b) \neq 0$, for example see Lemma 3.1, Tang [7]. Uniqueness for the case $b = \infty$ will follow if we can prove that $v \to -\infty$ as $r \to \infty$, see Lemma 3, McLeod and Serrin [5].

Lemma 4.1. *Let f satisfies conditions (i), (ii) and (iii) and $u = u(r, \beta)$ be a radial solution of problem (1). Then v vanishes exactly once in (a, b). Moreover, if $b < \infty$, then $v(b) < 0$; and if $b = \infty$, then $v \to -\infty$ as $r \to \infty$.*

Proof. We first show that $v < 0$ in (τ, b). We multiply both sides of equations (2) and (3) by v' and u' respectively. Then yields

$$v'u'' + \frac{n-1}{r} u'v' + f(u)v' = 0, \tag{9}$$

$$u'v'' + \frac{n-1}{r} u'v' + f'(u) v u' = 0. \tag{10}$$

Subtract equation (10) from equation (9), yield

$$v'u'' - u'v'' + f(u)v' - f'(u)u'v = 0. \tag{11}$$

Then divide both sides of equation (11)result to

$$\frac{u''}{u'} - \frac{v''}{v'} + \frac{f(u)v' - f'(u)u'v}{u'v'} = 0. \tag{12}$$

Integrating from a to r, $r \in [a, b]$, we have

$$\ln |\frac{u'}{v'\beta}| = K(r), \tag{13}$$

where $K(r) = \int_a^r \frac{f(u)v - f'(u)u'v}{u'v'} ds$. Since $u'(a) = \beta > 0$, $v'(a) = 1$ and $K(a) = 0$ then $K(a) = 0$. Thus

$$u' = \beta v' e^{K(r)} \text{ provided } K(r) \text{ is finite.} \tag{14}$$

Clearly $K(r)$ is finite if $u'(r)$ and $v'(r)$ are not equal zero. From Section 3, we know that $v(\tau) = 0$. If there exists an interval such that $s \in (\tau, \tau + \epsilon)$, $v'(s) > 0$ for some $\epsilon > 0$ then the equation (14) and the fact that $u'(s) < 0$, we get a contradiction. If there exists an interval $(\tau, \tau + \epsilon)$ on which v' is constant then this implies that $\zeta(\tau) = 0$ in the interval $(\tau, \tau + \epsilon)$ by equation (4). This contradict the fact that $\zeta(\tau) > 0$ by (8). Therefore, there exist an interval $(\tau, \tau + \epsilon_1)$ on which v is negative. Suppose τ_1 is the next zero of v. There exist a interval $(\tau_1 - \epsilon, \tau_1)$ on which $v' > 0$. This contradicts equation(14) because u' is negative on this interval. Therefore there is no zero of v larger than τ. In particular $v(b) < 0$.

Second, we show that if $b = \infty$ then $\lim_{r \to b} = -\infty$. We first note that as $r \to \infty$, either $v(r) \to -\infty$ or $v(r) \to 0$, see Lemma 2(b), [4]. Hence, we only need to prove that the second case does not occur, which is in fact obvious, it will contradict equation (14). The proof is complete. □

Example 4.2. *Let f be a general "polynomial",*

$$f(u) = \sum_{k=1}^{\nu} a_k u^{p_k},$$

where $1 = p_1 < p_2 < ...p_\nu = p$, $a_1 < 0, a_k \leq 0$ for $2 \leq k \leq \nu - 1$ and $a_\nu > 0$.

f satisfies conditions (i) and (ii) (see, McLeod [4]). Condition (iii) is satisfied if we assume that

$$\sum_{k=2}^{\nu-1} a_k + a_\nu \alpha^{p-1} > 0 \text{ and } \sum_{k=2}^{\nu} a_k > 0.$$

Let $u > \alpha$. If A is a subset of $[a,b]$ for each $r \in A$ is such that $u(r) > 1$ then

$$f''(u) = \sum_{k=2}^{\nu-1} a_k p_k (p_k - 1) u^{p_k - 2} + a_\nu p(p-1) u^{p-2}$$
$$= \frac{1}{u^2} \sum_{k=2}^{\nu-1} \left(a_k p_k (p_k - 1) u^{p_k} + \frac{a_\nu}{\nu - 2} p(p-1) u^p \right).$$

Since $p_k \le p$ for all $k = 2, \ldots \nu$, we get

$$f''(u) \ge \frac{1}{u^2} \sum_{k=2}^{\nu-1} \left(a_k p(p-1) u^{p_k} + \frac{a_\nu}{\nu - 2} p(p-1) u^p \right)$$
$$\ge \frac{p(p-1)}{u^2} \sum_{k=2}^{\nu-1} \left(a_k u^p + \frac{a_\nu}{\nu - 2} u^p \right)$$
$$= \frac{p(p-1)u^p}{u^2} \sum_{k=2}^{\nu-1} \left(a_k + \frac{a_\nu}{\nu - 2} \right)$$
$$= \frac{p(p-1)u^p}{u^2} \sum_{k=2}^{\nu} a_k > 0.$$

For $r \in [a,b] - A$, we have $u(r) \le 1$. Then

$$f''(u) = \sum_{k=2}^{\nu-1} a_k p_k (p_k - 1) u^{p_k - 2} + a_\nu p(p-1) u^{p-2}$$
$$= \frac{1}{u^2} \sum_{k=2}^{\nu-1} \left(a_k p_k (p_k - 1) u^{p_k} + \frac{a_\nu}{\nu - 2} p(p-1) u^p \right).$$

Since $p_k \le p$ for all $k = 2, \ldots \nu$, we get

$$f''(u) \ge \frac{1}{u^2} \sum_{k=2}^{\nu-1} \left(a_k p(p-1) u^{p_k} + \frac{a_\nu}{\nu - 2} p(p-1) u^p \right)$$
$$\ge \frac{p(p-1)}{u^2} \sum_{k=2}^{\nu-1} \left(a_k u + \frac{a_\nu}{\nu - 2} u^p \right)$$
$$= \frac{p(p-1)}{u} \sum_{k=2}^{\nu-1} \left(a_k + \frac{a_\nu}{\nu - 2} u^{p-1} \right).$$

Since $u > \alpha$, we have

$$f''(u) \geq \frac{p(p-1)u^p}{u^2} \sum_{k=2}^{\nu-1} \left(a_k + a_\nu \alpha^{p-1}\right) > 0.$$

References

[1] C.V. Coffman, Uniqueness of the ground state solution for $\triangle u - u + u^3 = 0$ and a variational characterization of other solutions, *Arch. Rational Mech. Anal.*, **46** (1972), 81-95.

[2] C.V. Coffman, Uniqueness of the positive radio solution on an annulus of the Dirichlet problem for $\triangle u - u + u^3 = 0$, *Differential Equations*, **128** (1996), 379-386.

[3] M.K. Kwong, Uniqueness of positive solutions $\triangle u - u + u^p = 0$ in $\Re^n$, *Arch. Rational Mech. Anal.*, **105** (1989), 243-266.

[4] K. McLeod, Uniqueness of positive radial solutions $\triangle u + f(u) = 0$ in $\Re^n$, *Trans. Amer. Math. Soc.*, **339** (1993), 495-505.

[5] K. McLeod, J. Serrin, Uniqueness of positive radial solutions $\triangle u + f(u) = 0$ in $\Re^n$, *Arch. Rational Mech. Anal.*, **99** (1987), 115-145.

[6] L.A. Peletier, J. Serrin, Uniqueness of positive solutions of semilinear equations in $\Re^n$, *Arch. Rational Mech. Anal.*, **81** (1983), 181-197.

[7] E. Tang, Uniqueness of positive radial solutions of $\triangle u + f(|x|, u) = 0$, *Differential Integral Equations*, **11** (1998), 725-743.

[8] M. Tang, Uniqueness of positive solutions $\triangle u - u + u^p = 0$ on an annulus, *J. Differential Equations*, **189** (2003), 148-160.

[9] S.L. Yadava, Uniqueness of positive radial solutions of a semilinear Dirichlet problem in an annulus, *Proc. Roy. Soc. Edinburgh A*, **130** (2000), 1417-1428.