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MONODROMY GROUPS AND A THEOREM OF BERTINI FOR COMPLEX BANACH ANALYTIC PROJECTIVE SETS

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Abstract: Here we prove the following result. Let V be a complex Banach space and $X \subset \mathbf{P}(V)$ a closed finitely determined irreducible subset with codimension $m > 0$. Fix any integer $y > m$. Then for a sufficiently general $A \in G(y+1, V)$ the closed analytic subset $(X \cap A)_{red}$ is an irreducible projective variety of dimension $y - m$ and the scheme $X \cap A$ is reduced at a general point of $(X \cap A)_{red}$.

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1. Introduction

Let V be a complex Banach space space and $\mathbf{P}(V)$ the projective space parametrizing all one-dimensional linear subspaces of V. It is well-known that a closed analytic subset X of $\mathbf{P}(V)$ may have bad properties (e.g. any compact and metrizablle topological space has a structure of analytic set for some separable Banach space ([2], Proposition II.1.3). However, if X is locally finitely determined (i.e. given locally by finitely many holomorphic equations in a Banach

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manifold), then it is nice: Sing(X) is a proper closed analytic subset of X ([2], Theorem III.3.1.1), there is a decomponsition of X into irreducible components and it is "algebraic", i.e. the zero-locus of continuous homogeneous polynomials on V. From now on, we will always assume that X is a finitely determined closed analytic subset of $\mathbf{P}(V)$ ([2], Theorem III.2.3.1). We will also assume that X is irreducible, i.e. that the set X_{reg} of its smooth points is connected. For every integer $m \geq 0$ let $G(m+1, V)$ denote the Grassmannian of all $(m+1)$-linear subspaces of V. By Hahn - Banach every $A \in G(m+1, V)$ is closed and it has a closed supplement. We will always see each $A \in G(m+1, V)$ as a closed m-dimensional linear subspace of $\mathbf{P}(V)$.

Theorem 1. *Let V be a complex Banach space and $X \subset \mathbf{P}(V)$ a closed finitely determined irreducible subset with codimension $m > 0$. Fix any integer $y > m$. Then for a sufficiently general $A \in G(y+1, V)$ the closed analytic subset $(X \cap A)_{red}$ is an irreducible projective variety of dimension $y - m$ and the scheme $X \cap A$ is reduced at a general point of $(X \cap A)_{red}$.*

We will prove Theorem 1 at the end of Section 2. To prove it in Section 2 we will prove a theorem concerning a monodromy group associated to X (see Theorem 2).

2. The Monodromy Group

In this section we will always use the notation introduced in the statement of Theorem 1. Call U any open dense subset of $G(m+1, V)$ such that $A \cap \mathrm{sing}(X) = \emptyset$, $\mathrm{card}(A \cap X) = d$ and A intersects tranversally $T_P X$ at each $P \in A \cap X$ for every $A \in U$ (Lemma 2). Consider the incidence corresponence $\Gamma := \{(P, A) : P \in X, A \in U, P \in A\} \subseteq X \times U$ and call $f : \Gamma \to U$ the projection on the second factor. By construction f is a topological covering (e.g. f is proper and it has everywhere invertible differential by the tranversality condition required in the definition of U). Fix any $A_0 \in U$. Since U is connected and locally arc-wise connected, it has a universal covering $g : Y \to U$ and $U \cong Y/\pi_1(Y, A_0)$. The map f corresponds to a finite index subgroup H of $\pi_1(Y, A_0)$. Call H_1 the maximal normal subgroup of $\pi_1(Y, A_0)$ contained in H and set $G := \pi_1(Y, A_0)/H_1$. The finite group G is called the monodromy group or the Galois group of f. We will also call it the monodromy group or the Galois group of the generic zero-dimensional section of X. The group G acts as a permutation group on the set $\{1, \ldots, d\}$ and this action is faithful, i.e. we may see G as a subgroup of the full symmetric group S_d. Since Theorem 1 is trivial if $d = 1$ (i.e. if X is a linear subspace of $\mathbf{P}(V)$), we will always assume

$d \geq 2$.

Theorem 2. *The monodromy group G of X is the full symmetric group S_d.*

Lemma 1. *There is an open and dense subset Ω of $G(m, V)$ such that $M \cap X = \emptyset$ for every $M \in X$.*

Proof. We use induction on m, the case $m = 1$ being trivial: take $\Omega := \mathbf{P}(V) \backslash X$. Assume $m \geq 2$ and that the result true for the integer $m' := m - 1$. Choose a general $A \in G(m-1, V)$, say $A = \mathbf{P}(B)$ and a closud supplement W of B in V. By the inductive assumption we may assume $A \cap X = \emptyset$ Consider the linear projection $u_A : \mathbf{P}(V) \backslash \mathbf{A} \to \mathbf{P}(W)$. Since $A \cap X = \emptyset$, $u_A | X$ is a holomorphic map and $u_A(X)$ is a hypersurface of $\mathbf{P}(W)$. For any $Q \in \mathbf{P}(W)$, $u_A^{-1}(Q) \cup A \in G(m+1, V)$. If $Q \in \mathbf{P}(W) \backslash u_A(X)$, then $(u_A^{-1}(Q) \cup A) \cap X = \emptyset$. □

Lemma 2. *For a "sufficiently general" $A \in G(m+1, V)$ we have $A \cap \mathrm{Sing}(X) = \emptyset$, and for every $P \in X_{reg} \cap A$, A is transversal to the Zariski tangent space $T_P X$ of X at P. Set $d := \mathrm{card}(X)$. We have $d > 0$ and the integers d (called the degree of X) does not depend from the choice of A as above.*

Proof. Since $\mathrm{Sing}(X)$ is locally contained in a hypersurface of X, the first assertion follows from Lemma 1 (or its inductive proof). Fix $A \in G(m+1, V)$ such that $A \cap \mathrm{Sing}(X) = \emptyset$ and set $\{P_1, \ldots, P_s\} := A \cap X$ with $P_i \neq P_j$ if $i \neq j$. If A is not transversal to X at P_i, i.e. if $A \subset T_{P_i} X$, then we may move A so that near P_i it intects transversally X at more than one point. Since s is finite, we can do that simultaneously for all points $P_1, \ldots, P_s$, concluding the proof. □

Lemma 3. *G is transitive.*

Proof. By the definition of monodromy group the transitivity of G is equivalent to the irreducibility of X. □

Lemma 4. *G is 2-transitive.*

Proof. Since G is transitive, the result is obvious if $d = 2$ and hence we will assume $d \geq 3$. Fix a general $A \in U$ and set $P_1, \ldots, P_d\} := A \cap X$. Set $U' := \{B \in U : P_1 \in B\}$. Since G is transitive, it is sufficient to find a continuous loop $u : [0,1] \to U_1$, $u(0) = u(1) = A$ and a continuous path $v : [0,1] \to X$ such that $v(0) = P_2$ and $v(1) = P_3$. Since X_{reg} is connected and locally path-connected, we may find a path $w : [0,1] \to X_{reg}$ joining P_2 with P_3. We may also find such a path with the condition $w(t) \neq P_1$ for every $t \in [0,1]$.

Hence for any fixed $t \in [0,1]$ the points P_1 and $w(t)$ are linearly independent, i.e. they span a line D_t. By the proof of Lemma 2 (just modifying the path w to go around any bad value) we may even assume that for every $t \in [0,1]$ $D_t \cap \mathrm{Sing}(X) = \emptyset$ and that D_t intersects tranversally X_{reg}. By the definition of U_1, we have $P_1 \in D_t$ for every t and hence w is the path which proves the 2-transitivity of G. $\square$

Remark 1. The proof of Lemma 1 gives that if X is not contained in a closed hyperplane of $\mathbf{P}(V)$, then G is $(m+1)$-transitive and in particular $d \geq m+1$.

Lemma 5. *G contains a simple transposition.*

Proof. As in the classical (i.e. the finite-dimensional case) it is suffient to find $A \in G(m+1,V)$ such that $A \cap \mathrm{Sing}(X) = \emptyset$, $\mathrm{card}((A \cap X)_{red}) = d-1$, say $(A \cap X)_{red} = \{P_1, \ldots, P_{d-1}\}$, and A intersecting transversally X at each P_i with $i > 1$. Taking a linear projection from a general $B \in G(m,V)$ we reduce to the case $m = 1$. Take a general $P \in X_{reg}$ and a general line $L \in T_PX$ containing P. We need to show that L intersects transversally $X \backslash \{P\}$ at exactly $d-2$ points and that it has intersection multiplicity 2 at P. First, we will check the second assertion, i.e. we will check that for a general $P \in X_{reg}$ there is a line D such that $P \in D \subset I_PX$, $D \cap X$ is finite, and the connected component of the length d zero-dimensional scheme $D \cap X$ has length 2. Take a general plane E containing P. Thus $E \cap X$ is a plane curve (a priori with some multiple component) and $P \in (E \cap X)_{reg}$. Since X is not a cone with vertex containing P, we may find P and E so that the unique component F, of $(E \cap X)_{red}$ containing P is not a line. Take as D the tangent line to $E \cap X$ at P. For a general $Q \in F$ the the tangent line T_QF has multiplicity two with F at P and hence we may take T_QF as D if we choose Q instead of P as general point of X. Now we will check that for a general $P \in X_{reg}$ and a general line D such that $P \in D \cap T_PX$ we have $D \cap \mathrm{Sing}(X) = \emptyset$ and D intersects tranversally $X_{reg} \backslash \{P\}$. Take the plane A as above. If $\deg(F) > 2$, then a general tangent line to F is not multitangent and we obtain even the second part just using a general $Q \in F$. Now assume $\deg(F) = 2$, but that the curve $A \cap X$ has no multiple component. In particular all other components of $E \cap X$ are different from F. Since in characterstic zero every projective variety is reflexive, this implies that a general tangent line T_QF is not tangent to any other component of $X \cap A$ and hence we conclude in this case. Now assume that $E \cap X$ has a multiple component T. Since G is 2-transitive, this implies that F must occur with multiplicity at least two in $X \cap E$, contradicting the assumption $E \nsubseteq T_PX$, i.e. the smoothness of the scheme $E \cap X$ at P. □

Proof of Theorem 2. $G = S_d$ because G is 2-transitive (Lemma 4) and it contains a simple transposition (Lemma 5). □

Proof of Theorem 1. Fix sufficiently general $B \in G(m+1, V)$ and $A \in G(m+1, V)$ such that $B \subset A$. Thus the scheme $X \cap B$ is reduced and formed by d points. Since $X \cap B$ and every $(y-m)$-irreducible component of $(X \cap A)_{red}$ contains at least one point of $B \cap X$, the scheme $X \cap A$ is generically reduced. By Theorem 2 we easily obtain that either each irreducible component of $(X \cap A)_{red}$ contains exactly one point of $B \cap X$ or there is only such irreducible component. Hence it is sufficient to rule out the first possibility, i.e. that $(A \cap X)_{red}$ is the union of d distinct $(y-m)$-linear spaces. Since $d \geq 2$ and X is irreducible, we easily find a $E \in G(m+2, V)$ such that $(E \cap X)_{red}$ contains two points P, Q, in the same irreducible component of $(E \cap X)_{red}$, but such that the line $\langle \{P, Q\} \rangle$ through them is not contained in X, ruling out the first possibility and hence concluding the proof. □

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