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ON A REPRESENTATION OF SOLUTION OF DIRAC DIFFERENTIAL EQUATION SYSTEMS WHICH HAVE DISCONTINUITY IN INTERVAL

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Abstract: In this article, the existence of transformation operators was proved for a class of Dirac operators which have discontinuity conditions inside and some properties of kernel of this transformation operator was investigated.

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1. Introduction

For solving of the inverse problems for Dirac differential operators – regular or singular – the transformation operators have a special place.

The main aim of the present paper is to prove the solution of boundary value problem L which has the type of tranformation operator. By using this expression, direct and inverse problems of spectral theory can be investigated for Dirac operators. Analogous problem [1] have been investigated in finite interval.

Boundary-value problems with discontinuty conditions arise in different branches of mathematics, mechanics, radio, electronics, geophysics and other fields of natural science and technology. For example, discontinuous conditions

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inside an interval are connected with discontinuous or nonsmooth properties of media (see [10], [8]). Inverse problems of this type are connected with the investigation of discontinuous solutions of some nonlinear equations in mathematical phyiscs.

For the classical Sturm-Liouville operators, Schrödinger equation and hyperbolic equations, direct and inverse problems are studied fairly completely (see [9], [3] and references therein).

For Dirac differential equation, direct and inverse problems have been investigated enough (see [9], [11], [12], [3], [4], [5], [1] and coresponding references). The presence of discontinuity conditions inside an interval introduces qualitative changes in the investigation of such problems. Some aspects of direct and inverse problems for differential operators with discontinuity conditions were studied in [6], [13], [7], [2], [14], [1].

Let us a get system of Dirac differential equations with canonical form in semi axis,

$$B\frac{dy}{dx} + \Omega(x)y = \lambda y, \quad 0 < x < \pi\,, \tag{1.1}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$p(x)$, $q(x) \in L_2(0,\infty)$ and λ is a parameter.

In this study, an expression of boundary value problem generated by boundary conditions

$$y_1(0) = 0\,, \tag{1.2}$$

$$y_1(\pi) = 0\,, \tag{1.3}$$

and

$$y(a-0) = Ay(a+0)\,, \tag{1.4}$$

have been given, where $0 < a < \pi$, $\quad A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\alpha \neq 1$ is real number.

These conditions have been given like special as seen from the conditions (1.2) and (1.3). However, boundary-value conditions which are given more general can deal with instead of the matrice A, and any self-adjoint matrice satisfying the condition $\det A = 1$.

2. Representation of Solution

Let $Y_0(0,\lambda)$ be a solution of matrice equation (1.1) corresponding to the case of $\Omega(x)\equiv 0$ satisfying $Y_0(0,\lambda)=I$ (I unite matrice) and discontinuity conditions (1.4).

In this case, for the function $Y_0(x,\lambda)$, it is obvious that

$$Y_0(x,\lambda)=\begin{cases} e^{-\lambda Bx}, & 0<x<a\,, \\ e^{-\lambda B(x-a)}A^{-1}e^{-\lambda Ba}, & a<x<\pi\,. \end{cases} \tag{2.1}$$

Let $Y(x,\lambda)$ be a solution of matrice equation (1.1) satisfying the initial condition $Y(0,\lambda)=I$ and discontinuity conditions (1.4).

Let us show that the function $Y(x,\lambda)$ as

$$Y(x,\lambda)=Y_0(x,\lambda)+\int\limits_{-x}^{x}K(x,t)e^{-\lambda Bt}dt\,, \tag{2.2}$$

where $K(x,t)$ is a seconder matrice function. It is obvious that the solution $Y(x,\lambda)$ satisfies the integral equation

$$Y(x,\lambda)=Y_0(x,\lambda)+\int\limits_{0}^{x}Y_0(x,\lambda)Y_0^{-1}(t,\lambda)B\Omega(t)Y(t,\lambda)dt \tag{2.3}$$

to satisfy the of function $Y(x,\lambda)$ given as the type of (2.1), it is necessary that the equality

$$\int\limits_{-x}^{x}K(x,t)e^{-\lambda Bt}dt=\int\limits_{0}^{x}Y_0(x,\lambda)Y_0^{-1}(t,\lambda)B\Omega(t)Y_0(t,\lambda)dt$$
$$+\int\limits_{0}^{x}Y_0(x,\lambda)Y_0^{-1}(t,\lambda)B\Omega(t)\int\limits_{-t}^{t}K(t,s)e^{-\lambda Bs}dsdt \tag{2.4}$$

must be satisfied. On the constrary that if the matrice function $K(x,t)$ satisfies the equality (2.4), the function $Y(x,\lambda)$ satisfies the equation (2.2).

We shall transform the right hand side of the equality (2.4) such that it be will similar to the left hand side of this equality. First, let us assume the following expressions

$$K_{\pm}(x,t)=\frac{1}{2}\left[K(x,t)\pm BK(x,t)B\right]\,.$$

It is seen clearly from the expressions of the matrice functions $K_+(x,t)$ and $K_-(x,t)$ that these functions have the properties;

$$\begin{aligned} K(x,t) &= K_+(x,t) - K_-(x,t), \\ BK_+(x,t) &= \frac{1}{2}\left[BK(x,t) - K(x,t)B\right] = -K_+(x,t)B, \\ BK_-(x,t) &= \frac{1}{2}\left[BK(x,t) + K(x,t)B\right] = K_-(x,t)B. \end{aligned}$$

If we set:

$$Y_0(x,\lambda)Y_0^{-1}(t,\lambda) = \begin{cases} e^{-\lambda B(x-t)}, & 0<t<x<a\,, \\ \alpha^+ e^{-\lambda B(x-t)} + \alpha^- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{-\lambda B(2a-x-t)}, & t<a<x\,, \\ e^{-\lambda B(x-t)}, & a<t<x\,, \end{cases}$$

where $\alpha^{\pm} = \frac{1}{2}\left(\frac{1}{\alpha} \pm \alpha\right)$ and the expression of the functions $K_{\pm}(x,t)$ and if we transform right hand side of the equality (2.4) such that it will be similar to the expression in the left hand side, we get the following system of integral equations for the matrice function $K_+(x,t)$ and $K_-(x,t)$:

I. For $t<x<a$:

$$\begin{aligned} K_+(x,t) &= \frac{1}{2}B\Omega\left(\frac{x+t}{2}\right) + \int\limits_{(x+t)/2}^{x} B\Omega(\xi)K_-(\xi,t+x-\xi)d\xi\,, \\ K_-(x,t) &= \int\limits_{(x+t)/2}^{x} B\Omega(\xi)K_+(\xi,t+x-\xi)d\xi\,. \end{aligned}$$

II. For $x>a,\ -x<t<x-2a$:

$$\begin{aligned} K_+(x,t) = {} & \frac{1}{2}\alpha^+ B\Omega\left(\frac{x+t}{2}\right) + \alpha^+ \int\limits_{(x+t)/2}^{a} B\Omega(\xi)K_-(\xi,t-\xi+x)d\xi \\ & +\alpha^- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int\limits_{(2a-x-t)/2}^{a} B\Omega(\xi)K_+(\xi,t+x+\xi-2a)d\xi \\ & + \int\limits_{a}^{x} B\Omega(\xi)K_-(\xi,t+x-\xi)d\xi\,, \end{aligned}$$

$$K_-(x,t) = \alpha^+ \int\limits_{(x+t)/2}^{a} B\Omega(\xi)K_+(\xi, t-\xi+x)d\xi$$

$$-\alpha^- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int\limits_{(2a-x-t)/2}^{a} B\Omega(\xi)K_-(\xi, t+x+\xi-2a)d\xi$$

$$+\int\limits_{a}^{x} B\Omega(\xi)K_+(\xi, t+x-\xi)d\xi\,.$$

III. For $x > a,\ x-2a < t < 2a-x$:

$$\begin{aligned} K_+(x,t) &= \frac{\alpha^+}{2}B\Omega\left(\frac{x+t}{2}\right) + \alpha^+ \int\limits_{(x+t)/2}^{a} B\Omega(\xi)K_-(\xi, t-\xi+x)d\xi \\ &+\alpha^- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int\limits_{(2a-x-t)/2}^{a} B\Omega(\xi)K_+(\xi, t+x+\xi-2a)d\xi \\ &+\int\limits_{a}^{x} B\Omega(\xi)K_-(\xi, t+x-\xi)d\xi, \end{aligned}$$

$$\begin{aligned} K_-(x,t) &= \frac{1}{2}\alpha^- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B\Omega\left(\frac{t-x+2a}{2}\right) \\ &+\alpha^+ \int\limits_{(x+t)/2}^{a} B\Omega(\xi)K_+(\xi, t-\xi+x)d\xi \\ &+\alpha^- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int\limits_{(2a-x-t)/2}^{a} B\Omega(\xi)K_-(\xi, t+x+\xi-2a)d\xi \\ &+\int\limits_{a}^{x} B\Omega(\xi)K_+(\xi, t+x-\xi)d\xi. \end{aligned}$$

IV. For $x > a,\ 2a-x < t < x$:

$$K_+(x,t) = \frac{\alpha^+}{2}B\Omega\left(\frac{x+t}{2}\right) + \int\limits_{(x+t)/2}^{x} B\Omega(\xi)K_-(\xi, t+x-\xi)d\xi,$$

$$K_-(x,t) = \frac{\alpha^-}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B\Omega\left(\frac{t-x+2a}{2}\right)$$
$$+\int\limits_{(x+t)/2}^{x} B\Omega(\xi)K_+(\xi,t+x-\xi)d\xi + \frac{\alpha^-}{2}B\Omega\left(\frac{x-t+2a}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

When we write successive approximations for each case:
I. For $t < x < a$

$$K_+^{(0)}(x,t) = \frac{1}{2}B\Omega\left(\frac{x+t}{2}\right), \quad K_-^{(0)}(x,t) = 0,$$
$$K_+^{(n)}(x,t) = \int\limits_{(x+t)/2}^{x} B\Omega(\xi)K_-^{(n-1)}(\xi,t+x-\xi)d\xi, \quad n = 1,2,\ldots,$$
$$K_-^{(n)}(x,t) = \int\limits_{(x+t)/2}^{x} B\Omega(\xi)K_+^{(n-1)}(\xi,t+x-\xi)d\xi, \quad n = 1,2,\ldots.$$

II. For $x > a, \ -x < t < x-2a$:

$$K_+^{(0)}(x,t) = \frac{\alpha^+}{2}B\Omega\left(\frac{x+t}{2}\right), \quad K_-^{(0)}(x,t) = 0\,,$$
$$K_+^{(n)}(x,t) = \alpha^+\int\limits_{(x+t)/2}^{a} B\Omega(\xi)K_-^{(n-1)}(\xi,t-\xi+x)d\xi$$
$$+\alpha^-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\int\limits_{(2a-x-t)/2}^{a} B\Omega(\xi)K_+^{(n-1)}(\xi,t+x+\xi-2a)d\xi$$
$$+\int\limits_{a}^{x} B\Omega(\xi)K_-^{(n-1)}(\xi,t+x-\xi)d\xi,$$
$$K_-^{(n)}(x,t) = \alpha^+\int\limits_{(x+t)/2}^{a} B\Omega(\xi)K_+^{(n-1)}(\xi,t-\xi+x)d\xi$$
$$+\alpha^-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\int\limits_{(2a-x-t)/2}^{a} B\Omega(\xi)K_-^{(n-1)}(\xi,t+x+\xi-2a)d\xi$$
$$+\int\limits_{a}^{x} B\Omega(\xi)K_+^{(n-1)}(\xi,t+x-\xi)d\xi.$$

III. For $x > a, \; x-2a < t < 2a-x$:

$$K_+^{(0)}(x,t) = \frac{\alpha^+}{2} B\Omega\left(\frac{x+t}{2}\right),$$
$$K_-^{(0)}(x,t) = \frac{\alpha^-}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B\Omega\left(\frac{t-x+2a}{2}\right),$$

$$K_+^{(n)}(x,t) = \alpha^+ \int_{(x+t)/2}^{a} B\Omega(\xi) K_-^{(n-1)}(\xi, t-\xi+x)d\xi$$
$$+\alpha^- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int_{(2a-x-t)/2}^{a} B\Omega(\xi) K_+^{(n-1)}(\xi, t+x+\xi-2a)d\xi$$
$$+\int_a^x B\Omega(\xi) K_-^{(n-1)}(\xi, t+x-\xi)d\xi,$$
$$K_-^{(n)}(x,t) = \alpha^+ \int_{(x+t)/2}^{a} B\Omega(\xi) K_+^{(n-1)}(\xi, t-\xi+x)d\xi$$
$$+\alpha^- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int_{(2a-x-t)/2}^{a} B\Omega(\xi) K_-^{(n-1)}(\xi, t+x+\xi-2a)d\xi$$
$$+\int_a^x B\Omega(\xi) K_+^{(n-1)}(\xi, t+x-\xi)d\xi.$$

IV. For $x > a$, $2a-x < t < x$:

$$K_+^{(0)}(x,t) = \frac{\alpha^+}{2} B\Omega\left(\frac{x+t}{2}\right),$$

$$K_-^{(0)}(x,t) = \frac{\alpha^-}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B\Omega\left(\frac{t-x+2a}{2}\right) + \frac{\alpha^-}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B\Omega\left(\frac{x-t+2a}{2}\right),$$

$$K_+^{(n)}(x,t) = \int_{(x+t)/2}^{a} B\Omega(\xi) K_-^{(n-1)}(\xi, t-\xi+x)d\xi,$$
$$K_-^{(n)}(x,t) = \int_{(x+t)/2}^{a} B\Omega(\xi) K_+^{(n-1)}(\xi, t-\xi+x)d\xi.$$

Now, let us obtain that the written successive approximations are convergent for the situation I. and also get that the evaluation for the vector function $K(x,t)$, for $t<x<a$:

$$K_{+}^{(0)}(x,t)=\frac{1}{2}B\Omega\left(\frac{x+t}{2}\right),\quad K_{-}^{(0)}(x,t)=0\,,$$

$$\int\limits_{-x}^{x}\left\|K_{+}^{(0)}(x,t)\right\|dt=\frac{1}{2}\int\limits_{-x}^{x}\|B\|\left\|\Omega\left(\frac{x+t}{2}\right)\right\|dt=\int\limits_{0}^{x}\|\Omega\left(t\right)\|\,dt\leq\sigma(x)$$

$$K_{-}^{(1)}(x,t)=\int\limits_{(x+t)/2}^{x}B\Omega\left(\xi\right)K_{+}^{(0)}(x,t)d\xi=\frac{1}{2}\int\limits_{(x+t)/2}^{x}B\Omega\left(\xi\right)B\Omega\left(\frac{x+t}{2}\right)d\xi,$$

$$\int\limits_{-x}^{x}\left\|K_{-}^{(1)}(x,t)\right\|dt\leq\frac{\sigma^{2}(x)}{2!}.$$

Analogously, for $n=k$, the truth of the following inequalities

$$\int\limits_{-x}^{x}\left\|K_{+}^{(2k)}(x,t)\right\|dt\leq\frac{\sigma^{(2k+1)}(x)}{(2k+1)!},$$

$$\int\limits_{-x}^{x}\left\|K_{-}^{(2k+1)}(x,t)\right\|dt\leq\frac{\sigma^{(2k+2)}(x)}{(2k+2)!},$$

is obtained. Let us show that similar inequalities are satisfied for also $n=k+1$:

Since

$$K_{+}^{(2k+2)}(x,t)=\int\limits_{(x+t)/2}^{x}B\Omega\left(\xi\right)K_{-}^{(2k+1)}(\xi,t+x-\xi)d\xi,$$

$$\int\limits_{-x}^{x}\left\|K_{+}^{(2k+2)}(x,t)\right\|dt\leq\int\limits_{-x}^{x}\int\limits_{(x+t)/2}^{x}\|\Omega\left(\xi\right)\|\left\|K_{-}^{(2k+1)}(\xi,t+x-\xi)\right\|d\xi dt$$

$$=\int\limits_{-x}^{x}\|\Omega\left(\xi\right)\|\int\limits_{\xi}^{2\xi-x}\left\|K_{-}^{(2k+1)}(\xi,\tau)\right\|d\tau d\xi$$

$$= \frac{1}{(2k+2)!} \int_0^x \|\Omega(\xi)\| \, \sigma^{(2k+2)}(\xi) d\xi = \frac{\sigma^{(2k+3)}(x)}{(2k+3)!}.$$

Similarly,

$$\int_{-x}^{x} \left\| K_-^{(2k+3)}(x,t) \right\| dt \le \frac{\sigma^{(2k+4)}(x)}{(2k+4)!}.$$

We get the following evaluations that series $\sum_{n=0}^{\infty} \int_{-x}^{x} \left\| K_\pm^{(n)}(x,t) \right\| dt$ are absolutely and uniformly convergent with respect to x over $[0,\pi]$. Then if we consider that $K(x,t) = K_+(x,t) + K_-(x,t)$ then we get $\int_{-x}^{x} \|K(x,t)\| \, dt \le e^{\sigma(x)} - 1.$

Let us assume that $\Omega(x)$ function is differentiable. In this case when the expression of the function $Y(x,\lambda)$ substitute in the equation (1.1); $B\frac{dy}{dx} + y = \lambda y$,

$$B\frac{d}{dx}\left(Y_0(x,\lambda) + \int_{-x}^{x} K(x,t) e^{-\lambda B t} dt \right) + \Omega(x) \left(Y_0(x,\lambda) + \int_{-x}^{x} K(x,t) e^{-\lambda B t} dt \right) = \lambda \left(Y_0(x,\lambda) + \int_{-x}^{x} K(x,t) e^{-\lambda B t} dt \right),$$

$$BY_0'(x,\lambda) + B\frac{d}{dx}\left(\int_{-x}^{x} K(x,t) e^{-\lambda B t} dt \right) + \Omega(x) Y_0(x,\lambda)$$
$$+\Omega(x) \int_{-x}^{x} K(x,t) e^{-\lambda B t} dt = \lambda Y_0(x,\lambda) + \lambda \int_{-x}^{x} K(x,t) e^{-\lambda B t} dt,$$

here since $BY_0'(x,\lambda) = \lambda Y_0(x,\lambda)$,

$$B\frac{d}{dx}\left(\int_{-x}^{2a-x} K(x,t) e^{-\lambda B t} dt + \int_{2a-x}^{x} K(x,t) e^{-\lambda B t} dt \right)$$

$$+\Omega(x)Y_0(x,\lambda)+\Omega(x)\int_{-x}^{x}K(x,t)e^{-\lambda Bt}dt=\lambda\int_{-x}^{x}K(x,t)e^{-\lambda Bt}dt\,,$$

$$B\Big[-K(x,2a-x-0)e^{-\lambda B(2a-x)}-K(x,x)e^{-\lambda Bx}$$
$$+K(x,2a-x+0)e^{-\lambda B(2a-x)}\Big]+B\int_{-x}^{x}\frac{\partial K(x,t)}{\partial x}e^{-\lambda Bt}dt+\Omega(x)Y_0(x,\lambda)$$
$$+\Omega(x)\int_{-x}^{x}K(x,t)e^{-\lambda Bt}dt=\lambda\int_{-x}^{x}K(x,t)e^{-\lambda Bt}dt\,,$$

is obtained.

Using integration in parts:

$$B\left[K(x,2a-x+0)-K(x,2a-x-0)\right]e^{-\lambda B(2a-x)}-BK(x,x)e^{-\lambda Bx}$$
$$-\Omega(x)Y_0(x,\lambda)+\int_{-x}^{x}\left[B\frac{\partial K(x,t)}{\partial x}+\Omega(x)K(x,t)\right]e^{-\lambda Bt}dt$$
$$=-\left[K(x,2a-x+0)-K(x,2a-x-0)\right]Be^{-\lambda B(2a-x)}-K(x,x)Be^{-\lambda Bx}$$
$$-\int_{-x}^{x}\frac{\partial K(x,t)}{\partial t}Be^{-\lambda Bt}dt,$$

From this last equality it follows that:

1) $B\dfrac{\partial K(x,t)}{\partial x}+\Omega(x)K(x,t)=-\dfrac{\partial K(x,t)}{\partial t}B;$

2) $K(x,0)B\begin{pmatrix}0\\-1\end{pmatrix}=0;$

3) for $0<x<a$ $BK(x,x)+\Omega(x)K(x,t)=K(x,x)B;$

4) for $x<a$ $B\left[K(x,2a-x+0)-K(x,2a-x-0)\right]+\Omega(x)==-\left[K(x,2a-x+0)-K(\right.$

5) for $x>a$, $BK(x,x)+\alpha^{+}\Omega(x)=K(x,x)B,$

$$B\left[K(x,2a-x+0)-K(x,2a-x-0)\right]+\alpha^{-}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\Omega(x)$$
$$=-\left[K(x,2a-x+0)-K(x,2a-x-0)\right]B,\quad(2.5)$$

Thus we proved following theorem:

Theorem. *Let us say that* $\int\limits_0^\pi \|\Omega(x)\|\,dx < +\infty$. *Then each solution* $Y(x,\lambda)$ *satisfying conditions (1.2)-(1.3) of the equation (1.1) has the expression (2.1) and moreover* $\int\limits_{-x}^{x} \|K(x,t)\|\,dt < e^{\sigma(x)} - 1$, *where* $\sigma(x) = \int\limits_0^x \|\Omega(t)\|\,dt$.

If the function $\sigma(x)$ is differentiable then the function $K(x,t)$ satisfies the conditions of (2.5).

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