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EXTREME VALUE TYPE DISTRIBUTIONS WITH BOUNDED SUPPORT

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Abstract: Underlying the derivation of the classical extreme value distributions is the group $\mathcal{A}^+$ of orientation preserving affine transformations of the real line. The group $\mathcal{F}^+$ of orientation preserving fractional linear transformations is the only Lie group of analytic transformations of the (extended) real line that contains $\mathcal{A}^+$. This singular place occupied by $\mathcal{F}^+$ is one of the reasons to consider the consequences of using $\mathcal{F}^+$ in place of $\mathcal{A}^+$.

It is shown that there exists a unique compactly supported four parameter distribution G, which is a weak limit of rescaled distributions of sample maxima as the sample size tends to infinity. In contrast to the case of classical max-value distributions, the "rescalings" for G come from $\mathcal{F}^+$, not $\mathcal{A}^+$. The same holds for minima.

As in the classical case, the distribution function G is alternatively characterized as "stable", i.e., for $n = 1, 2, ...$, one has $G^n = G \circ q_n$ for some $q_n \in \mathcal{F}^+$ (on the support of G). It is shown that the sequence q_n can be extended to a continuous, hence smooth, one parameter subgroup $q_{\exp(\sigma)}$ of $\mathcal{F}^+$. This allows to obtain G from a simple differential equation.

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1. Introduction

A practical question that has led to the classical theory of maximal (minimal) value distributions is what kind of distribution can be proposed for the maximal (minimal) value of a random variable ξ in a long series of independent trials, *if little is known about the distribution of ξ itself.* The theory implies that the choice can be limited to three families of distributions for maxima and to three for minima, because only those can appear in the limit (appropriately understood), as the number of trials tends to infinity. These families of distributions are referred to as "maximal (minimal) value distributions" or, collectively, as "extreme value distributions", see Galambos [6], Gumbel [7], Leadbetter et al [11], Resnick [12]. The distributions from all three families have unbounded supports.

Specifically, let $\xi_1, \xi_2, ...$ be independent identically distributed random variables whose common distribution function (d.f.) is $F(x)$. Put $\eta_n = \max(\xi_1, ..., \xi_n)$. Since $F_{\eta_n}(x) = [F(x)]^n$, then, as $n \to \infty$, only degenerate d.f.'s may appear as the limits. This leads to the classical reformulation of the original question. One introduces n-dependent affine transformations $x' = Ax + B$ of the x-axis to scale back the misbehavior of $F_{\eta_n} = F^n$ for large n and asks which non-degenerate d.f.'s $G(x)$ can be obtained as $\lim_{n\to\infty}[F(A_n x + B_n)]^n$ for some d.f. F and some sequences A_n, B_n, $n = 1, 2, ...$. The limiting d.f.'s are referred to as d.f.'s of *maximal value type.* In the approach that originates with the heuristic argument of Fisher and Tippett, Fisher et al [4], one proves that $G(x)$ is of max-value type if and only if it is *max-stable*, i.e., for each $n = 1, 2, ...$, there are constants $a_n > 0$, b_n such that $[G(x)]^n = G(a_n x + b_n)$; max-stable d.f.'s are then classified. Domains of attraction of max-value type d.f.'s are then described.

Evidently, the notions of max-value-type and max-stable d.f.'s use the group $\mathcal{A}^+$ of all orientation preserving affine transformations of the real line. More generally, if $\mathcal{G} \supset \mathcal{A}^+$ is a Lie group of transformations of the real line, then one may ask (i) which d.f.'s $G(x)$ can be obtained as $\lim_{n\to\infty}[F(q_n(x))]^n$ for some d.f. F and some $q_n \in \mathcal{G}$, $n = 1, 2, ...$ ("maximal value type relative to $\mathcal{G}$" d.f.'s), (ii) which d.f.'s $G(x)$ have the property that there exists a sequence $p_n \in \mathcal{G}$, such that $[G(x)]^n = G(p_n(x))$, for $n = 1, 2, ...$ ("max-stable relative to $\mathcal{G}$" d.f.'s), and (iii) whether the two classes of d.f.'s are related. The scope of these questions narrows dramatically if only Lie groups of *analytic* transformations are considered. Indeed, there is only one connected Lie group of analytic transformations of the (extended) real line, which strictly contains $\mathcal{A}^+$, namely, the group $\mathcal{F}^+$ of all orientation preserving fractional linear transformations. It is the uniqueness of the ambient group that, perhaps, makes the above questions worth considering.

In this paper, the questions (i)-(iii) are considered for the group $\mathcal{F}^+$ and d.f.'s G with *bounded* support (it can be shown that "max-value relative to $\mathcal{F}^+$" d.f.'s with unbounded support amount to the three classical families.) Namely, we show (Theorem 1) that there exist non-degenerate d.f.'s G with bounded support that are of max-value type relative to $\mathcal{F}^+$ (called "M-value type d.f.'s" below). To this end, we define, in Section 3.2, a notion of M-stable d.f., show that a d.f. G is of M-value type if and only if G is M-stable (Lemma 7), and classify M-stable d.f.'s. The proofs are essentially based on an analog of a theorem of Khintchine (Lemma 4). The structure of the proofs can be partially traced to Haan [8]. The main departure occurs in the proof of Theorem 1, where a significant shortcut results from the use of a well-known fact that continuity of a one parameter subgroup of a Lie group implies its differentiability (see the proof of Lemma 8). This allows to obtain an M-stable G from a simple differential equation.

Section 2 contains the needed terminology and statement of the result.

Section 3 contains proofs.

In Section 4, we discuss among other things the $\mathcal{F}^+$-types of d.f.'s and limiting relations between the new and the old families of d.f.'s. We also reference an applied context in which an extreme value type distribution with bounded support appears to be useful.

2. Main Result

Let G be a distribution function (d.f.). Denote $x_G = \inf\{x : G(x) > 0\}$, $X_G = \sup\{x : G(x) < 1\}$. Evidently, $x_G \leq X_G$ and also $x_G < X_G$, if and only if, G is non-degenerate. If $x_G, X_G \in \mathbb{R}$, then G is said to have *bounded support* $[x_G, X_G]$.

Let us call G *strongly non-degenerate* if G has bounded support and is continuous at both x_G and X_G. If G is strongly non-degenerate, then G is non-degenerate.

Let G_n, $n = 1, 2, ...,$ be functions defined at least on $[x_G, X_G]$. We write $G_n \xrightarrow{W} G$ if $G_n(x) \to G(x)$ for all $x \in [x_G, X_G]$ that are continuity points of G.

Denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ equipped with the usual one point compactification topology: open subsets of $\mathbb{R}$ together with complements in $\overline{\mathbb{R}}$ of compact subsets of $\mathbb{R}$; $\overline{\mathbb{R}}$ is homeomorphic to a circle, and the order on $\mathbb{R}$ induces an orientation on $\overline{\mathbb{R}}$. We extend any d.f. F to $\overline{\mathbb{R}}$ by defining $F(\infty) = 0$. Thus extended function (still denoted F) remains right continuous.

Denote by $\mathcal{F}$ the group of fractional linear transformations of $\overline{\mathbb{R}}$, $\mathcal{F} = \{q(x) = \frac{ax+b}{cx+d} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0\}$. A transformation $q \in \mathcal{F}$ is *orientation preserving* if $ad - bc > 0$ (the inequality does not depend on the above representation of q as a ratio). Orientation preserving fractional linear transformations form an index 2 subgroup $\mathcal{F}^+$ of $\mathcal{F}$.

Throughout the paper, $\circ$ denotes composition of mappings. Exponents refer to either pointwise powers or iterated compositions. The usage should be clear from the context, since d.f.'s will never be composed with d.f.'s and transformations will never be multiplied pointwise.

Definition 1. A d.f. G is of *M-value type*, if G is strongly non–degenerate and there exist a d.f. F and $q_n \in \mathcal{F}^+$, $n = 1, 2, \ldots$, such that $F^n \circ q_n \xrightarrow{W} G$.

Theorem 1. *A d.f. G is of M-value type, if and only if,*

$$G(x) = \begin{cases} 0, & x \leq m\,, \\ \exp\left[-\beta\left(\frac{M-x}{x-m}\right)^{\alpha}\right], & m < x < M\,, \\ 1, & M \leq x\,, \end{cases} \tag{2.1}$$

for some real numbers $M > m$, $\alpha > 0$, $\beta > 0$.

We will refer to the family (2.1) as *maximal value distribution of Type IV* (see Section 4.1 for a discussion of "types").

3. Proofs

3.1. Preliminaries

Denote by $\mathcal{A}$ the group of affine transformations of $\mathbb{R}$, $\mathcal{A} = \{q(x) = ax + b : a, b \in \mathbb{R},\ a \neq 0\}$. Each $q \in \mathcal{A}$ can be extended to a transformation of $\overline{\mathbb{R}}$ by defining $q(\infty) = \infty$. This allows one to identify $\mathcal{A}$ with a subgroup of $\mathcal{F}$, and we will do so. Let $\mathcal{A}^+ = \mathcal{A} \cap \mathcal{F}^+ = \{q(x) = ax + b : a > 0\}$.

Any $q \in \mathcal{F}$ is a homeomorphism of $\overline{\mathbb{R}}$ onto itself, and if $q \neq \mathrm{id}_{\overline{\mathbb{R}}}$, then q has at most two fixed points.

For $p \in \mathcal{F}$, $x \in \overline{\mathbb{R}}$, the set $\mathcal{O}_p(x) = \{p^k(x) : k \in \mathbb{Z}\}$ will be called the p-orbit of x. Notice that p and p^{-1} have the same orbit.

Proposition 1. *Let $p \in \mathcal{F}^+$. Suppose p has exactly two fixed points $m, M \in \overline{\mathbb{R}}$, $m \neq M$. Then, for any $x \in \overline{\mathbb{R}}$ such that $x \neq m$ and $x \neq M$, both m and M are accumulation points of the p-orbit of x. In particular, if $p \in \mathcal{A}^+$,* *then p is a dilation and both the center of the dilation and ∞ are accumulation points.*

Proof. Without loss of generality, $m \neq \infty$. Take any $q \in \mathcal{F}$ such that $q(m) = m$ and $q(M) = \infty$. Then $q \circ p \circ q^{-1} \in \mathcal{F}^+$ has m and ∞ as its only fixed points. Therefore $p_1 = q \circ p \circ q^{-1}|_{\mathbb{R}}$ is an orientation preserving dilation with its center at m and $p_1 \neq \mathrm{id}_{\mathbb{R}}$. So, evidently, p_1-orbit of any $y \in \mathbb{R}$, $y \neq m$, has m and ∞ as its accumulation points. In particular, m and ∞ are accumulation points for $\mathcal{O}_{p_1}(q(x))$. But $\mathcal{O}_p(x) = q^{-1}(\mathcal{O}_{p_1}(q(x)))$ and the rest follows from $M = q^{-1}(\infty)$, $m = q^{-1}(m)$, and q being a homeomorphism. □

Proposition 2. *Let $p \in \mathcal{F}^+$. Suppose p has exactly two fixed points $m, M \in \mathbb{R}$, $m < M$. If $x \in (m, M)$, then*
(i) $\mathcal{O}_p(x) \subseteq (m, M)$;
(ii) $\inf \mathcal{O}_p(x) = m$, $\sup \mathcal{O}_p(x) = M$.

Proof. Let us show that $p(x) \in (m, M)$ for any $x \in (m, M)$. Put $\alpha = p^{-1}(\infty)$. One can write p as $p(x) = \frac{(M+m-\alpha)x - Mm}{x-\alpha}$. Since $p \in \mathcal{F}^+$, then the "determinant" $\Delta \equiv (M + m - \alpha)(-\alpha) + Mm = (M - \alpha)(m - \alpha) > 0$.

Take $x \in (m, M)$, i.e. $(x - m)(x - M) < 0$. From a straightforward calculation, $(p(x) - m)(p(x) - M) = \frac{(x-m)(x-M)\Delta}{(x-\alpha)^2} < 0$. Hence $p(x) \in (m, M)$. Notice that $p^{-1} \in \mathcal{F}^+$ and has m and M as its only fixed points, so $p^{-1}(x) \in (m, M)$ as well. By induction, $\mathcal{O}_p(x) \subseteq (m, M)$, which proves (i).

By Proposition 1, m and M are accumulation points of $\mathcal{O}_p(x)$. This and (i) imply (ii). □

Proposition 3. *Let G, G^* be strongly non-degenerate d.f.'s and $q, Q \in \mathcal{F}^+$. Suppose that*

$$G(q(x)) = G^*(x) = G(Q(x)), \qquad x \in [x_{G^*}, X_{G^*}]. \tag{3.1}$$

Then $q = Q$.

Proof. Notice that

$$q([x_{G^*}, X_{G^*}]) = [x_G, X_G]. \tag{3.2}$$

Indeed, $x \in (x_{G^*}, X_{G^*})$ implies $0 < G^*(x) < 1$, i.e. $0 < G(q(x)) < 1$, which implies $q(x) \in (x_G, X_G)$. Together with the continuity of q this implies

$$q([x_{G^*}, X_{G^*}]) \subseteq [x_G, X_G]. \tag{3.3}$$

But $G(q(X_{G^*})) = G^*(X_{G^*}) = 1$ and, since G^* is strongly non-degenerate, $G(q(x_{G^*})) = G^*(x_{G^*}) = 0$. In view of (3.3), this is only possible if $q(x_{G^*}) = x_G$ and $q(X_{G^*}) = X_G$. Equation (3.2) now follows, since $q([x_{G^*}, X_{G^*}])$ is connected.

As q and Q are on equal footing, one has that (3.2) holds for Q as well. Put $p = q \circ Q^{-1}$. From (3.1) and (3.2), one has

$$G(p(X)) = G(X), \qquad X \in [x_G, X_G]. \tag{3.4}$$

Assume that $p \neq \mathrm{id}_{\overline{\mathbb{R}}}$. Since x_G and X_G are fixed points of p and $x_G \neq X_G$, then p has exactly two fixed points. Take $x \in (x_G, X_G)$ and consider its p-orbit $\mathcal{O}_p(x)$. By Proposition 2 (i), applied to p, $m \equiv x_G$, $M \equiv X_G$, and x, one has $\mathcal{O}_p(x) \subseteq (x_G, X_G)$. This and (3.4) imply that G takes the same value at all $X \in \mathcal{O}_p(x)$. Furthermore, Proposition 2 (ii), implies that $\inf \mathcal{O}_p(x) = x_G$ and $\sup \mathcal{O}_p(x) = X_G$. Since G is non-decreasing and right continuous, then G is constant on the interval $[\inf \mathcal{O}_p(x), \sup \mathcal{O}_p(x)) = [x_G, X_G)$, which contradicts strong non-degeneracy of G. Thus $p = \mathrm{id}_{\overline{\mathbb{R}}}$. □

The following lemma is an analog, for fractional linear transformations, of a theorem of Khintchine, see Balkema [1] and Leadbetter et al [11]. The proof is tedious, yet straightforward.

Lemma 4. *Let G, G^* be strongly non-degenerate d.f.'s. For $n = 1, 2, \dots$, let $q_n, q_n^* \in \mathcal{F}^+$ and F_n be d.f.'s. Suppose that $F_n \circ q_n \xrightarrow{W} G$.*

(i) If $F_n \circ q_n^* \xrightarrow{W} G^*$, then there is $q \in \mathcal{F}^+$ such that $q_n^{-1} \circ q_n^* \to q$ pointwise and $G^*(x) = G(q(x))$ for all $x \in [x_{G^*}, X_{G^*}]$.

(ii) If there is $q \in \mathcal{F}^+$ such that $q_n^{-1} \circ q_n^* \to q$ pointwise and $G^*(x) = G(q(x))$ for all $x \in [x_{G^*}, X_{G^*}]$, then $F_n(q_n^*(x)) \longrightarrow G^*(x)$ for all $x \in (x_{G^*}, X_{G^*})$ that are continuity points of G^*.

Note. In part (ii), it is possible to have no convergence at x_{G^*} or X_{G^*} (even though these, by assumption, are continuity points of G^*).

3.2. M-Stable Distributions

Definition 2. A d.f. G is *M-stable* if G is strongly non–degenerate and, for each $k = 1, 2, \dots$, there is $p_k \in \mathcal{F}^+$such that $G^k(p_k(x)) = G(x)$ for all $x \in [x_G, X_G]$.

Note. The transformations p_k in the definition of M-stability are uniquely defined, as follows from Proposition 3.

Proposition 5. *A strongly non-degenerate d.f. G is M-stable if and only if there exist d.f.'s F_n and transformations $q_n \in \mathcal{F}^+$, $n = 1, 2, \dots$, such that, for each $k = 1, 2, \dots$,*

$$F_n \circ q_{nk} \xrightarrow{W} G^{1/k} \quad \text{as} \quad n \to \infty. \tag{3.5}$$

Proof. Suppose (3.5) holds for each $k = 1, 2, \ldots$. Fix k. Notice that $G^{1/k}$ is strongly non-degenerate, since G is. Lemma 4 applied to G, F_n, q_n, $q_n^* \equiv q_{nk}$ ($n = 1, 2, \ldots$), and $G^* \equiv G^{1/k}$ implies that there exist $p_k \in \mathcal{F}^+$, $k = 1, 2, \ldots$, such that $G^{1/k}(x) = G(p_k(x))$ for any $x \in [x_{G^{1/k}}, X_{G^{1/k}}] = [x_G, X_G]$, which shows that G is M-stable.

Suppose G is M-stable. Put $F_n = G^n$. Then, for any $x \in [x_G, X_G]$, $F_n(p_{nk}(x)) = [G^{nk}(p_{nk}(x))]^{1/k} = [G(x)]^{1/k}$, and (3.5) holds trivially for $q_m \equiv p_m$, $m = 1, 2, \ldots$. □

Proposition 6. *Let G be a strongly non-degenerate d.f.. Then:*

(i) G is M-stable if and only if there exists a family $q_s \in \mathcal{F}^+$, $s > 0$, such that

$$G^s(x) = G(q_s(x)), \qquad x \in [x_G, X_G], \quad s > 0; \tag{3.6}$$

(ii) the family q_s, $s > 0$, in part (i) is unique and $q_1 = \mathrm{id}_{\overline{\mathbb{R}}}$;

(iii) the above $q_s(x)$ is a continuous function of s (with values in $\overline{\mathbb{R}}$) for any $x \in \overline{\mathbb{R}}$.

Proof. (i) If G is M-stable, then, by definition, there exist $p_n \in \mathcal{F}^+$, $n = 1, 2, \ldots$, such that, for any $x \in [x_G, X_G]$,

$$G^n(p_n(x)) = G(x). \tag{3.7}$$

Hence, for any $t > 0$ and any $x \in [x_G, X_G]$, $G^{[nt]}(p_{[nt]}(x)) = G(x)$ (here [ ] denotes integer part), and so $G^n(p_{[nt]}(x)) = G^{n/[nt]}(x) \to G^{1/t}(x)$, as $n \to \infty$. Consequently, since $[x_G, X_G] = [x_{G^{1/t}}, X_{G^{1/t}}]$, one has

$$G^n \circ p_{[nt]} \xrightarrow{W} G^{1/t}. \tag{3.8}$$

Due to (3.7), (3.8), and strong non-degeneracy of $G^{1/t}$, Lemma 4 applies to G, $F_n \equiv G^n$, $q_n \equiv p_n$, $G^* \equiv G^{1/t}$, $q_n^* \equiv p_{[nt]}$ and implies that, for any $t > 0$, there exists $p_t \in \mathcal{F}^+$ such that $G^{1/t}(x) = G(p_t(x))$ for all $x \in [x_{G^{1/t}}, X_{G^{1/t}}] = [x_G, X_G]$. Now (3.6) follows, if one puts $t = 1/s$ and $q_s = p_{1/s}$.

Conversely, if (3.6) holds, then G is clearly M-stable.

(ii) The uniqueness follows from Proposition 3, and $q_1 = \mathrm{id}_{\overline{\mathbb{R}}}$ follows from the uniqueness.

(iii) Let us verify first the continuity of $q_s(x)$ as a function of s at $s = 1$.

Since $G^s \to G$ pointwise as $s \to 1$, then, by (3.6), $G(q_s(x)) \to G(x)$ as $s \to 1$ for all $x \in [x_G, X_G]$. Let $s_n \to 1$. Lemma 4 applied to G, $G^* \equiv G$, $q_n \equiv \mathrm{id}_{\overline{\mathbb{R}}}$, $q_n^* \equiv q_{s_n}$, $F_n \equiv G$ implies that there exists $q \in \mathcal{F}^+$ such that $\mathrm{id}_{\overline{\mathbb{R}}}^{-1} \circ q_{s_n} \to q$ pointwise and $G(x) = G(q(x))$ for all $x \in [x_G, X_G]$. But q is unique (Proposition 3), hence $q = \mathrm{id}_{\overline{\mathbb{R}}}$. Thus $q_{s_n} \to \mathrm{id}_{\overline{\mathbb{R}}} = q_1$ pointwise. Since $s_n \to 1$ was arbitrary, then q_s is continuous in s at $s = 1$.

Notice that

$$q_{st} = q_s \circ q_t, \quad s, t > 0. \tag{3.9}$$

Indeed, for $x \in [x_G, X_G]$, $G(q_{st}(x)) = G^{st}(x) = [G^t(x)]^s = [G(q_t(x))]^s = G(q_s(q_t(x)))$, $s, t > 0$, where all three equalities follow from (3.6), the last one because $q_t(x) \in [x_G, X_G]$ (from (3.6), $q_t((x_G, X_G)) \subseteq (x_G, X_G)$, so the same inclusion holds for the closures). Equation (3.9) follows now from Proposition 3 applied to G, $G^* \equiv G^{st}$, $q \equiv q_{st}$, $Q \equiv q_s \circ q_t$.

Take $x \in [x_G, X_G]$. For $s, t > 0$, $q_t(x) = q_{s(t/s)}(x) = q_s(q_{t/s}(x)) \to q_s(x)$ as $t \to s$ (q_r is continuous in r at $r = 1$ and $q_s(y)$ is continuous in y). Thus $q_s(x)$ is a continuous function of s. □

3.3. Classification of *W*-Limits of Distributions of Sample Maxima

As in the classical extreme value theory, we will recognize M-value distributions to be M-stable and proceed to classify the latter. However, the proof of the classification contains a twist which both simplifies it and makes it different from commonly found expositions of the classical result (Proposition 6 (iii) and Lemma 8).

Lemma 7. *A d.f. is of M-value type if and only if it is M-stable.*

Proof. If G is M-stable, then G is the (trivial) W-limit of $G^n \circ p_n$ for those $p_n \in \mathcal{F}^+$, which appear in Definition 2, i.e. G is of M-value type.

Conversely, suppose G is of M-value type, say, $F^n \circ q_n \xrightarrow{W} G$, where F is a d.f. and $q_n \in \mathcal{F}^+$. Then, for any k, $F^{nk} \circ q_{nk} \xrightarrow{W} G$, whence $F^n \circ q_{nk} \xrightarrow{W} G^{1/k}$ (G and $G^{1/k}$ have the same continuity points and supports), i.e. (3.5) holds for $F_n = F^n$. Therefore, by Proposition 5, G is M-stable. □

Lemma 8. *If a d.f. G is M-stable, then there exists a family $q_s \in \mathcal{F}^+$, $s > 0$, such that $q_1 = \mathrm{id}_{\overline{\mathbb{R}}}$,*

$$G^s(x) = G(q_s(x)), \qquad x \in [x_G, X_G], \quad s > 0\,, \tag{3.10}$$

and:

(i) for every $x \in [x_G, X_G]$, $q_s(x) \in \mathbb{R}$ and, as a function of s, $q_s(x)$ is infinitely differentiable for all $s > 0$;

(ii) $q_s(x)$ is infinitely differentiable in s at $s = 1$ for every $x \in \mathbb{R}$ in the sense that, for every $x \in \mathbb{R}$, there exists $\varepsilon > 0$ such that $q_s(x) \in \mathbb{R}$ for $1-\varepsilon < s < 1+\varepsilon$ and, as a function of s, $q_s(x)$ is infinitely differentiable at $s = 1$.

Proof. If G is a M-stable d.f., then Proposition 6 implies that there exists a family $q_s \in \mathcal{F}^+$, $s > 0$, such that $q_1 = \mathrm{id}_{\overline{\mathbb{R}}}$, (3.10) holds, and $q_s(x)$ is a continuous function of s for any $x \in \overline{\mathbb{R}}$. In addition, (3.9) holds.

For $-\infty < \sigma < +\infty$, put

$$Q_\sigma = q_s, \; s = e^\sigma. \tag{3.11}$$

Then

$$Q_0 = \mathrm{id}_{\overline{\mathbb{R}}} \,, \tag{3.12}$$

and (3.9) gives

$$Q_{\sigma+\tau} = Q_\sigma \circ Q_\tau \,, \tag{3.13}$$

i.e. Q_σ is a continuous one-parameter subgroup of $\mathcal{F}^+$. It is known that continuity of a one-parameter subgroup of a Lie group implies its infinite differentiability, see Freudenthal et al [5] and Varadarajan [13] ($\mathcal{F}^+$ is a Lie group isomorphic to the group $SL_2(\mathbb{R})$ of 2×2 real matrices with determinant 1). Therefore

$$Q_\sigma(x) = \frac{a_\sigma x + b_\sigma}{c_\sigma x + d_\sigma}, \tag{3.14}$$

where $a_\sigma, ..., d_\sigma$ are infinitely differentiable functions of σ. As remarked immediately before (3.9), for every $x \in [x_G, X_G]$, one has $Q_\sigma(x) \in [x_G, X_G] \subset \mathbb{R}$, and so $c_\sigma x + d_\sigma \neq 0$. It follows that, for all $x \in [x_G, X_G]$ and all $n \in \mathbb{N}$, the derivative $d^n Q_\sigma(x)/d\sigma^n$, which is a rational function in x with a power of $c_\sigma x + d_\sigma$ in the denominator, exists. Thus $Q_\sigma(x)$ and, consequently, $q_s(x) = Q_{\ln s}(x)$ are infinitely differentiable in σ and in s, respectively, for all $x \in [x_G, X_G]$, and (i) is proved.

Since $Q_\sigma(x)|_{\sigma=0} = q_s(x)|_{s=1} = x$, Proposition 6 (ii), then, for any $x \in \mathbb{R}$, the denominator $c_\sigma x + d_\sigma|_{\sigma=0} = d_0 \neq 0$, hence $d^n Q_\sigma(x)/d\sigma^n|_{\sigma=0}$ exists for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, which proves (ii).[1] □

Remark 1. If Q_σ $(-\infty < \sigma < +\infty)$ is a differentiable one-parameter subgroup of $\mathcal{F}^+$, i.e. Q_σ satisfies (3.12) - (3.14), where $a_\sigma, ..., d_\sigma$ are differentiable functions of σ, then the *velocity field* of Q_σ is defined by (cf. proof of Lemma 8(ii))

$$v(x) = \left.\frac{dQ_\sigma(x)}{d\sigma}\right|_{\sigma=0}, \quad x \in \mathbb{R}$$

[1]If $\overline{\mathbb{R}}$ is identified with the projective line equipped with the natural differential structure, then the conclusion of Lemma 8 would hold for all $x \in \overline{\mathbb{R}}$ (no need for a separate part (ii) then). Similarly, the concept of the velocity field of Q_σ below is more natural for the projective line where such a field generates back the one-parameter group of diffeomorphisms Q_σ. However, in the present context, we do not need to use this language.

(here a tangent vector field on $\mathbb{R}$ is identified with a function). Moreover, $v(x)$ is a polynomial of degree ≤ 2. Indeed,

$$v(x) = \left. \frac{dQ_\sigma(x)}{d\sigma} \right|_{\sigma=0} = \left. \frac{P_\sigma(x)}{(c_\sigma x + d_\sigma)^2} \right|_{\sigma=0},$$

where $P_\sigma(x)$ is a polynomial of degree ≤ 2 in x. But, $Q_0 = \mathrm{id}_{\overline{\mathbb{R}}}$. Therefore, for any $x \in \mathbb{R}$, $c_\sigma x + d_\sigma|_{\sigma=0} = d_0 \neq 0$, hence $v(x)$ is a polynomial of degree ≤ 2. In fact, any polynomial vector field on $\mathbb{R}$ of degree ≤ 2 is a velocity field of a one-parameter group of orientation preserving fractional linear transformations.

Proof of Theorem 1. If G is given by (2.1), then it is strongly non-degenerate, and direct computation shows that $G^k(x) = G(q_k(x))$, $x \in [m, M]$, $k = 1, 2, ...,$ where

$$q_k(x) = \frac{(M - k^{1/\alpha} m)x - Mm(1 - k^{1/\alpha})}{(1 - k^{1/\alpha})x + Mk^{1/\alpha} - m}, \tag{3.15}$$

i.e. G is M-stable, hence of M-value type by Lemma 7.

Conversely, suppose that G is of M-value type. By Lemma 7, G is M-stable. By Lemma 8, there exists a family $q_s \in \mathcal{F}^+$, $s > 0$, such that $q_1 = \mathrm{id}_{\overline{\mathbb{R}}}$, (3.10) holds and $q_s(x)$ is differentiable in s at $s = 1$ for all $x \in \mathbb{R}$. Let Q_σ be the one-parameter subgroup introduced in the proof of Lemma 8, i.e. $Q_\sigma = q_s$, $s = e^\sigma$ (cf. (3.11)-(3.13)). Let $v(x)$ denote its velocity field (see Remark 1).

Let us show that twice iterated logarithm of G is differentiable on (x_G, X_G). Denote

$$g(x) = \ln(-\ln G(x)), \quad x \in (x_G, X_G). \tag{3.16}$$

Then (3.10) can be rewritten, after taking logarithms twice, as

$$g(Q_\sigma(x)) - g(x) = \sigma, \quad x \in (x_G, X_G). \tag{3.17}$$

Fix $x \in (x_G, X_G)$ such that $v(x) \neq 0$. By implicit function theorem, the equation $Q_\sigma(x) = X$ defines a function $\sigma = \sigma(X)$ defined and differentiable in a neighborhood of x and such that

$$\sigma(x) = 0 \quad \text{and} \quad \sigma'(x) = \frac{1}{v(x)} \ . \tag{3.18}$$

Thus, (3.17) and (3.18) imply that, for any X close enough to x, $X \neq x$,

$$\frac{g(X) - g(x)}{X - x} = \frac{\sigma(X)}{X - x} \ \to \ \frac{1}{v(x)} \quad \text{as} \quad X \to x. \tag{3.19}$$

From (3.16) and (3.19), one has $\frac{d}{dx}\ln(-\ln G(x)) = \frac{1}{v(x)}$, $x \in (x_G, X_G)$, $v(x) \neq 0$, whence

$$G(x) = \exp\left[-\exp\int\frac{dx}{v(x)}\right], \quad x \in (x_G, X_G), \quad v(x) \neq 0. \tag{3.20}$$

By Remark 1, $v(x)$ is a polynomial of degree ≤ 2. Degrees 0 and 1 are inconsistent with the assumption that $x_G, X_G \in \mathbb{R}$ (without this assumption, polynomials $v(x)$ of degree 0 and 1 in (3.20) yield the three classical families of maximal value distributions). If $v(x)$ has degree 2 and less than two real roots, then (3.20) yields no d.f.'s (even with unbounded support). Suppose that $v(x)$ has degree 2 and has two distinct real roots, say, $v(x) = \lambda(x-M)(x-m), \quad \lambda \neq 0, \quad m < M$, then the general solution of (3.20) on (x_G, X_G) is (put $\alpha = \frac{1}{\lambda(M-m)}$)

$$G(x) = \begin{cases} \exp\left[-\beta_1\left(\frac{M-x}{m-x}\right)^\alpha\right], & x < m, \quad x \in (x_G, X_G), \\ \exp\left[-\beta_2\left(\frac{M-x}{x-m}\right)^\alpha\right], & m < x < M, \quad x \in (x_G, X_G), \\ \exp\left[-\beta_3\left(\frac{x-M}{x-m}\right)^\alpha\right], & M < x, \quad x \in (x_G, X_G), \end{cases} \tag{3.21}$$

where arbitrary $\beta_1, \beta_2, \beta_3 > 0$ come from integration constants. One can verify that, for $G(x)$ given by (3.21) to satisfy $G(x) \to 0$ as $x \downarrow x_G$ and $G(x) \to 1$ as $x \uparrow X_G$, one must have $x_G = m$, $X_G = M$, and $\alpha > 0$, i.e. $G(x)$ must be given by (2.1). □

4. Concluding Remarks

4.1. The infinitesimal (i.e. velocity field) point of view on the families q_s makes it easy to arrive at the following limiting relations between d.f.'s of Types I - IV (the stratification of the four-dimensional space of the extended collection of maximal value distributions). By a continuous change of parameters of any d.f. of either Type II or Type III, one can obtain any d.f. of Type I as a pointwise limit. By a continuous change of parameters of any d.f. Type IV, one can obtain, as a pointwise limit, any d.f. of Type II, Type III and, therefore, of Type I.

4.2. The 'Type IV *minimal* value distribution' can be formally obtained as $1 - G(-x)$, where $G(x)$ is given by (2.1).

4.3. D.f.'s G_1, G_2, are said to have 'the same type' if $G_1 = G_2 \circ q$ for some $q \in \mathcal{A}^+$, Haan [8], Leadbetter et al [11]. One may also say that such

G_1, G_2 have the same $\mathcal{A}^+$-type. This is an equivalence relation, since $\mathcal{A}^+$ is a group. The affine version of Lemma 4 (Khintchine's Lemma/Theorem in Haan [8]/Leadbetter et al [11]) implies that the set of maximal value distributions contains with every G all d.f.'s of the same $\mathcal{A}^+$-type. The classical maximal value distributions have the following representatives, one per $\mathcal{A}^+$-type [*ibid.*]:

$$\text{Type I: } F(x) = \exp\left[-\exp\left(-x\right)\right], \quad -\infty < x < +\infty\,, \tag{4.1}$$

$$\text{Type II}_\alpha\text{:} \qquad F(x) = \begin{cases} 0 \quad , & x \le 0\,, \\ \exp\left(-x^{-\alpha}\right) \quad , & x > 0\,, \end{cases} \tag{4.2}$$

$$\text{Type III}_\alpha\text{:} \quad F(x) = \begin{cases} \exp\left(-(-x)^{\alpha}\right) \quad , & x < 0\,, \\ 1 \quad , & x \ge 0\,, \end{cases} \tag{4.3}$$

where $\alpha > 0$ is an arbitrary constant. Similarly, one can say that d.f.'s G_1, G_2 are of the same $\mathcal{F}^+$-type if $G_1 = G_2 \circ q$ for some $q \in \mathcal{F}^+$ (an equivalence relation since $\mathcal{F}^+$ is a group). Then maximal value distributions from the extended list I - IV (equations (4.1), (4.2), (4.3), (2.1)) have the following representatives, one per $\mathcal{F}^+$-type:

$$\text{Type I}'\text{: } F(x) = \exp\left[-\exp\left(-x\right)\right], \quad -\infty < x < +\infty\,, \tag{4.4}$$

$$\text{Type II}'_\alpha\text{:} \qquad F(x) = \begin{cases} 0 \quad , & x \le 0\,, \\ \exp\left(-x^{-\alpha}\right) \quad , & x > 0\,, \end{cases} \tag{4.5}$$

where $\alpha > 0$ is an arbitrary constant.

4.4. One application of the classical extreme value theory can be found in the Crack Diffusion Theory, Chudnovsky et al [2], [3] and Kunin [9]. A central character there, is the 'specific fracture energy' γ, which is argued to represent certain minima. So its distribution is chosen from among the three families of minimal value distributions (that $\gamma \ge 0$ eliminates two of the three). However, besides being non-negative, γ has well known theoretical upper bounds. This is one of the instances when one wishes that there were distributions which could be justifiably called 'extreme value distributions' and yet would have bounded supports.

4.5. In Kunin [10] one finds comparisons of approximations by minimal value distributions of Type III and Type IV. Two of examples there are numerical, and the third one is based on experimental data related to brittle fracture.

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