International Journal of Pure and Applied Mathematics
Volume 12 No. 4 2004, 489-493

A CHARACTERIZATION FOR CURVES OF THE HEISENBERG GROUP

Mehmet Bektaş[1 §], Essin Turhan[2]
[1,2]Department of Mathematics
Firat University
23119 Elazığ, TURKEY
[1]e-mail: mbektas@firat.edu.tr
[2]e-mail: eturhan@firat.edu.tr

Abstract: T. Ikawa obtained in [4] the following characteristic ordinary differential equation:

$$\nabla_X \nabla_X \nabla_X X - K \nabla_X X = 0, \qquad K = k^2 - \tau^2 ,$$

for the circular helix which corresponds to the case that the curvatures k and τ of a time-like curve α on the Lorentzian manifold M are constant.

N. Ekmekçi and H. H. Hacisalihoglu generalized in [3] T. Ikawa's result, i.e. k and τ are variable, but $\frac{k}{\tau}$ is constant.

In [1] H.Balgetir, M.Bektas and M. Ergüt obtained a geometric characterization of Null Frenet curve with constant ratio of curvature and torsion (called null general helix).

In this paper, making use of method in [1], [3], [4], we obtained characterizations of a curve with respect to the Frenet frame of the three-dimensional Heisenberg group H_3.

AMS Subject Classification: 53B30
Key Words: Heisenberg group, helix

Received: February 20, 2004

§Correspondence author

1. Preliminaries

Let H_3 be the three-dimensional Heisenberg group:

$$H_3 = \left\{ w = w\left(x,y,z\right) = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x,y,z \in IR \right\},$$

and the Lie algebra

$$h_3 = \left\{ X = X\left(x,y,z\right) = \begin{bmatrix} 1 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : x,y,z \in IR \right\}.$$

The Heisenberg group H_3 can be seen as the Euclidean space IR^3 endowed with the multiplication (see [2])

$$\left(\widetilde{x},\widetilde{y},\widetilde{z}\right)\left(x,y,z\right) = \left(\widetilde{x}+x, \widetilde{y}+y, \widetilde{z}+z+\frac{1}{2}\widetilde{x}y - \frac{1}{2}\widetilde{y}x\right),$$

and with the Riemannian metric g given by

$$g = dx^2 + dy^2 + \left(dz + \frac{y}{2}dx - \frac{x}{2}dy\right)^2.$$

The metric g is invariant with respect to the left-translations corresponding to that multiplication. This metric is isometric to the other, also quite standard, which is left-invariant with respect to the composition arising from the multiplication of the 3×3 Heisenberg matrices.

At each point the metric g has an axial symmetry; the 4-dimensional group of its isometries contains the group of rotations around the z axis.

First of all we shall determine the Levi-Civita connection ∇ of the metric g with respect to the left-invariant orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},$$

which is dual to the coframe

$$\theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = dz + \frac{y}{2}dx - \frac{x}{2}dy.$$

We obtain

$$\begin{aligned} \nabla_{e_1}e_1 &= 0, \quad \nabla_{e_1}e_2 = \frac{1}{2}e_3, \quad \nabla_{e_1}e_3 = -\frac{1}{2}e_2, \\ \nabla_{e_2}e_1 &= -\frac{1}{2}e_3, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_3 = \frac{1}{2}e_1, \\ \nabla_{e_3}e_1 &= -\frac{1}{2}e_2, \quad \nabla_{e_3}e_2 = \frac{1}{2}e_1, \quad \nabla_{e_1}e_3 = 0. \end{aligned}$$

Also, we have the well-known Heisenberg bracket relations

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = [e_2, e_3] = 0.$$

Let $\alpha : I \longrightarrow H_3$ be a differentiable curve parametrized by arc length and let $\{T,\ N,\ B\}$ be orthonormal frame field tangent to H_3 along α and defined as follows: by T we denote the unit vector field α' tangent to α, by N the unit vector field in the direction of $\nabla_T T$ normal to α, and we chose B so that $\{T,\ N,\ B\}$ is a positive oriented orthonormal basis. Then we have the following Frenet equantions

$$\begin{aligned} &\nabla_T T = kN\,, \\ &\nabla_T N = -kT - \tau B\,, \\ &\nabla_T B = \tau N\,, \end{aligned} \tag{1.1}$$

where $k = ||\nabla_T T||$ is the geodesic curvature of α and τ geodesic torsion.

In this paper, we keep the name helix for a curve in a Riemanian manifold having constant both geodesic curvature and geodesic torsion.

2. A Characterization of Helix

Definition 2.1. Let α be a curve of a Heisenberg group H_3 and $F = \{T,\ N,\ B\}$ be the Frenet frame on H_3 along α. If both k and τ are positive constant along α, then α is called circular helix with respect to Frenet frame.

Definition 2.2. Let α be a curve of a Heisenberg group H_3 and $F = \{T,\ N,\ B\}$ be the Frenet frame on H_3 along α. A curve α such that

$$\frac{k}{\tau} = \text{const.}$$

is called a general helix with respect to Frenet frame.

Theorem 2.1. *Let α be a curve of a Heisenberg group H_3. α is a general helix with respect to Frenet frame $F = \{T,\ N,\ B\}$ if and only if*

$$\nabla_T \nabla_T \nabla_T T - \kappa \nabla_T B = 3k' \nabla_T N\,, \tag{2.1}$$

where $\kappa = \left(\frac{k''}{\tau} - \frac{k^3}{\tau} - k\tau\right)$.

Proof. Soppose that α is a general helix with respect to the Frenet frame $F = \{T,\ N,\ B\}$. Then, from (1.1), we have

$$\nabla_T \nabla_T \nabla_T T = -3kk'T + \left(k'' - k^3 - k\tau^2\right) N - \left(2k'\tau + k\tau'\right) B\,. \tag{2.2}$$

Now, since α is general helix with respect to Frenet frame, then by

$$\frac{k}{\tau} = \text{const.}$$

and this opon the derivation gives rise to

$$k'\tau = k\tau'. \tag{2.3}$$

If we substitue the equation

$$N = \frac{1}{\tau}\nabla_T B \tag{2.4}$$

and (2.3) in (2.2) we obtain (2.1).

Conversely let us assume that the equation

$$\nabla_T\nabla_T\nabla_T T - \kappa\nabla_T B = 3k'\nabla_T N$$

holds. We show that the curve α is a general helix. Differentiating covariantly of

$$N = \frac{1}{k}\nabla_T T, \tag{2.5}$$

we obtain

$$\nabla_T N = -\frac{k'}{k^2}\nabla_T T + \frac{1}{k}\nabla_T\nabla_T T\,, \tag{2.6}$$

and so,

$$\nabla_T\nabla_T N = \left(-\frac{k'}{k^2}\right)'\nabla_T T - 2\frac{k'}{k^2}\nabla_T\nabla_T T + \frac{1}{k}\nabla_T\nabla_T\nabla_T T. \tag{2.7}$$

If we use (2.1) in (2.7) and some calculations, we have

$$\nabla_T\nabla_T N = -k'T + \left[\left(-\frac{k'}{k^2}\right)' k - 2\frac{k'}{k^2} + \frac{\kappa}{k}\tau\right] N - \frac{k'\tau}{k}B. \tag{2.8}$$

Also we obtain

$$\nabla_T\nabla_T N = -k'T - \left(k^2 + \tau^2\right) N - \tau' B. \tag{2.9}$$

Since (2.8) and (2.9) are equal, routine calculations show that α is a general helix. □

Corollary 2.1. *Let α be curve of a Heisenberg group H_3. α is a circular helix with respect to the Frenet frame $F = \{T,\ N,\ B\}$ if and only if*

$$\nabla_T\nabla_T\nabla_T T + \left(\frac{k^3}{\tau} + k\tau\right)\nabla_T B = 0\,. \tag{2.10}$$

Proof. From the hypotesis of Corollary 2.1. and since α is a circular helix, we can show easily (2.10). □

Corollary 2.2. *Let α be a curve of a Heisenberg group H_3. α is a circular helix with respect to the Frenet frame $F = \{T,\ N,\ B\}$ if and only if*

$$\nabla_T \nabla_T \nabla_T T + k\left(k^2 + \tau^2\right) N = 0. \tag{2.11}$$

Proof. From the hypotesis of Corollary 2.2. and the equation (1.1) and since α is a circular helix,we can show easily (2.11). □

Corollary 2.3. *Let α be a curve of a Heisenberg group H_3. α is a circular helix with respect to the Frenet frame $F = \{T,\ N,\ B\}$ if and only if*

$$\nabla_T \nabla_T \nabla_T T + \left(k^2 + \tau^2\right) \nabla_T T = 0. \tag{2.12}$$

Proof. From the hypotesis of Corollary 2.3. and the equation (2.1) and since α is a circular helix, we can show easily (2.12). □

References

[1] H. Balgetir, M. Bektas, M. Ergüt, On a characterization of null helix, *Bull. Ins. Math. Aca. Sin.*, **29**, No. 1 (2001), 71-78.

[2] R. Caddeo, C. Onıciuc, P. Piu, Explicit formulas for non-geodesic biharmonic curves of the heisenberg group, *Arxiv Math.*, DG/0311221 V1 (2003), 1-10.

[3] N. Ekmekçi, H.H. Hacisalihoglu, On Helices of a Lorentzian manifold, *Commun. Fac. Sci.*, Üniv. Ank. Series **A1** (1996), 45-50.

[4] T. Ikawa, On curves and submanifolds in an indefinite Riemannian manifold, *Tsukuba J. Math.*, **9** (1985), 353-371.