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# ZERO-DIMENSIONAL SCHEMES WHICH ARE A SMALL MODIFICATION OF FAT POINTS AND POSTULATION

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**Abstract:** Here we give some very strong postulation result for general unions of zero-dimensional schemes very near to fat points.

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## 1. The Statements

For any $n$-dimensional integral variety $X$, any $P \in X_{reg}$ and any integer $k > 0$ let $kP$ denote the infinitesimal neighborhood of order $k-1$ of $P$ in $X$, i.e. the closed subscheme of $X$ with $(\mathcal{I}_{P,X})^k$ as its ideal sheaf. Thus $(kP)_{red} = \{P\}$ and $\mathrm{length}(kP) = \binom{n+k-1}{n-1}$. We will say that $kP$ is a $k$-point. Let $m_P$ be the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Since $X$ is smooth at $P$, $m_P^k/m_P^{k+1}$ is a vector space of dimension $\binom{n+k-1}{n-1}$ over the algebraically closed base field $\mathbb{K}$. For any linear subspace $E$ of $m_P^k/m_P^{k+1}$ there is a unique zero-dimensional subscheme, say $(kP, E)$ of $X$ such that $kP \subseteq (kP, E) \subseteq (k+1)P$ and $\mathrm{length}((kP,E)) = \mathrm{length}(kP) + \dim(E)$. We are only interested in the case $\dim(E) = 1$ and in this case we will denote any such scheme by the symbol $(k,1)P$ and call them $(k,1)$-points supported by $P$. Identify the completion of the local ring $\mathcal{O}_{X,P}$ with the formal power series ring $\mathbb{K}[[x_1, \dots, x_n]]$. We see

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that not all $(k,1)$-points supported by $P$ are equivalent under a formal change of coordinates. We will say that the $(k,1)$-point is special if $E$ can be represented by a $k$-power, i.e., if after a formal change of coordinates, we may take $E$ represented by $x_1^{k+1}$; any two such vector spaces can be made equivalent using a change of formal coordinates. Now assume $X = \mathbf{P}^n$. In this case any special $(k,1)$-point is uniquely determined by a line $D$ such that $P \in D$; the line $D$ is the unique line such that the intersection with this scheme has length $k+1$; all the other lines through $P$ have length $k$ intersection. Conversely, any pair $(P,D)$ with $P \in D \subseteq \mathbf{P}^n$ determines a special $(k,1)$-point with $P$ as support. Notice that the set of all $k$-points, the set of all $(k,1)$-points and the set of all special $(k,1)$-points is irreducible. Hence it make sense the following sentence: the general union of $x$ $k$-points, $y$ special $(k,1)$-points and $z$ $(k,1)$-point has a certain property. For any integral projective variety, any zero-dimensional scheme $Z \subset X$, any $L \in \mathrm{Pic}(X)$, any linear subspace $V \subseteq H^0(X,L)$ and any linear system $M$ of Cartier divisors on $X$, set $V(-Z) := V \cap H^0(X, \mathcal{I}_Z \otimes L)$ and $M(-Z) := \{D \in M : Z \subseteq D\}$. Thus $\dim(V) - \mathrm{length}(Z) \leq \dim(V(-Z)) \leq \dim(V)$. $Z$ is said to impose independent conditions to $V$ (or to the associated linear system) if $\dim(V(-Z)) = \dim(V) - \mathrm{length}(Z)$. We will say that $Z$ imposes the expected number of conditions to $V$ (or to the associated linear system) if $\dim(V(-Z)) = \max\{\dim(V) - \mathrm{length}(Z), 0\}$. Here we will prove the following results.

**Theorem 1.** *Fix integers $n \geq 2$, $x \geq 0$, $y \geq 0$ and $d > 0$. A general union $Z$ of $x$ 2-points and $y$ special $(2,1)$-points imposes the expected number of conditions to $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$ except at most in the following cases:*

*(a)* $d = 2$ *and* $x + y \geq 2$ *and* $x + 2y \leq n$*;*

*(b)* $n = 2$, $d = 4$, $x = 5$ *and* $y = 0$*;*

*(c)* $n = 3$, $d = 4$, $x = 9$ *and* $y = 0$*;*

*(d)* $n = 4$, $d = 3$, $x = 7$ *and* $y = 0$*;*

*(e)* $n = 4$, $d = 4$, $x = 14$ *and* $y = 0$.

*In all these cases $Z$ fails to give the expected number of conditions to $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$.*

**Theorem 2.** *Fix integers $x \geq 0$ and $y \geq 0$. Let $X$ be an integral projective variety and $M$ a linear system (even a non-complete one) such that a general union of $x + y$ $k$-points of $X$ imposes independent conditions to $M$. Then a* *general union of $x$ $k$-points and $y$ special $(k,1)$-points of $X$ imposes the expected number of conditions to $M$.*

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

## 2. The Proofs

**Remark 1.** Let $X$ be an integral projective variety and $M$ a linear system on $X$. It is easy to check using [3], Proposition 2.3, that if $M(-kP) \neq \emptyset$ for a general $P \in X$, then $\dim(M(-A)) = \dim(M(-B)) - 1$, where $A$ is a general $(k,1)$-point of $X$ and $B$ is a genearl $k$-point of $X$. Since for any $Q \in X_{reg}$ the ideal $m_Q^{k+1}$ is spanned as a $\mathbb{K}$-vector space by $m_Q^k$ and the union of all special $(k,1)$-points of $X$ supported by $Q$, we see that $\dim(M(-A)) = \dim(M(-B)) - 1$ even if we take as $A$ a general element of the variety of all special $(k,1)$-points of $X$. In particular to prove Theorem 2 it is sufficient to prove the same assertion for the union of $x$ general $k$-points and $y$ general $(k,1)$-points.

**Remark 2.** Let $X$ be an integral projective variety and $M$ a linear system on $X$. Fix a general $P \in X$ and call $A$ a general special $(k,1)$-point supported by $P$. Assume that a general $k$-point of $X$ gives the expected number of conditions to $M$. If this is the case because $V(-kP) = \emptyset$, then obviously $V(-A) = \emptyset$ and hence a general special $(k,1)$-point of $X$ gives the expected number of conditions to $M$. Now assume $M(-kP) \neq \emptyset$. By Remark 1 we have $\dim(M(-A)) = \dim(M(-B)) - 1$, where $A$ is a general element of the set of all special $(k,1)$-points. Hence $A$ gives the expected number of conditions to $M$. More is true. Assume now that $kP$ does not give the expected number of conditions to $M$. Thus $M \neq \emptyset$. If $M$ consists just of a divisor, then $M(-A) = \emptyset$ by Remark 1. If $\dim(M) > 0$, then Remark 1 implies that $\dim(M(-A)) = \dim(M(-kP)) - 1$ and hence the amounts of failure with respect to $M$ of $kP$ and of $A$ are the same.

*Proof of Theorem 2.* Assume that the result fails. Among the failing data with respect to $M$ with $x' + y' \leq x + y$, choose the one with minimal $y'$ and among the ones with this $y'$ choose the one with a minimal $x'$. Call $B$ a general union of $x' + 1$ $k$-points of $X$ and $y' - 1$ special $((k,1)$-points of $X$. By the inductive assumption $B$ imposes the expected number of conditions to $V$. By Remark 2 the same is true for a general union of $x'$ $k$-points of $X$ and $y'$ special $((k,1)$-points of $X$, contradiction. □

*Proof of Theorem 1.* Use the list the list of all triples $(n, d, x)$ such that the general union of $x$ double points of $\mathbf{P}^n$ does not give the expected number of conditions to $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$ (see e.g. [2], Theorem at p. 982; the result is due to J. Alexander and A. Hirschowitz (see [1])). We obtain exactly all cases with $y = 0$. Furthermore, in all cases with $d \geq 3$ the amount of failure for the pair $(x, 0)$ is exactly one and the corresponding linear system $M(-A)$ is zero-dimensional. Hence by Remark 2 none of these cases gives an exceptionasl case for some $y > 0$. We leave the details of the case by case checking for $d = 2$ to the interested reader. Apply Theorem 2 to conclude in the non exceptional cases. □

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