QUASILINEAR EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Abstract: In the paper the author, using Aleksandrov-Pucci maximum principle, proves an $L^\infty$ a priori estimate and also uniqueness for weak solution $u$ of a Dirichlet problem associated to quasilinear strictly elliptic equations with Charatheodory coefficients. The results obtained are a first step in the study of weak solvability of boundary value problems for quasilinear elliptic equations.

AMS Subject Classification: 31B10, 43A15, 35K20, 32A37, 46E35
Key Words: quasilinear operator, divergence form equations, uniqueness of solution of Dirichlet problem

1. Introduction and Assumptions

The main goal of the present paper is to establish an a priori bound of $\|u\|_{L^\infty(\Omega)}$ for the weak solutions $u \in W^{1,q}(\Omega), q > n$, of the equation:

$$Mu = 0 \quad \text{almost everywhere in } \Omega,$$

where

$$M \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x,u) \frac{\partial}{\partial x_j} u) + b(x,u). \quad (1.1)$$

The coefficients $a_{ij}(x,z), b(x,z) \in \Omega \times \mathbb{R}$ are Charatheodory functions and
also \( a_{ij} \) are uniformly elliptic and such that is true the local uniform continuity with respect to the second variable \( u \), uniformly in \( x \).

On the coefficients \( b(x,u) \) we point out the characteristic hypotheses that it is less or equal to a function of the Lebesgue space \( L^q(\Omega) \), \( q > n \).

The second result is uniqueness of the solution of the Dirichlet problem associated to the above equation, then we have extended the uniqueness result obtained in [8] from the linear case to the quasilinear one.

Divergence form quasilinear elliptic equations have been studied by Ladyzenskaya and Ural’ceva in [5], where they consider the terms \( a_{ij} \) to be sufficiently smooth. Also in the book [3] the authors have supposed regularity assumptions on the coefficients.

We wish to mention the study made by Simon in [10], where he suppose that the terms having first partial derivatives are local Lipschitz continuous and also the recent paper [6], where are considered Hölder continuous coefficients for a divergence form elliptic equation to obtain \( L^\infty \) estimates for the gradient of the solution.

We point out that our study generalizes those results because in this note the coefficients \( a_{ij} \) are discontinuous, precisely belong to the class VMO, and it is possible to prove that \( C^0 \subset VMO \).

The class VMO, first considered by Sarason in the note [9], has been used by many authors, we want to recall the two papers [1] and [2], where has been very well studied the linear elliptic equations in nondivergence form, and the study made in [7] for nondivergence form equations.

The vanishing mean oscillation functions made a subset of the set of bounded mean oscillation functions, known as BMO class and defined by John and Nirenberg in the note [4].

Let us now establish the assumptions we will need in the following.

Let us consider \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n > 2 \), \( a_{ij}(x,z) \) and \( b(x,z) \) be Charatheodory functions or equivalently measurable on \( x \in \Omega, \forall z \in \mathbb{R} \), and continuous respect to \( z \) for all almost \( x \in \Omega \). Moreover we assume that:

\( \exists \lambda \) positive constant such that

\[
a_{ij}(x,z)\xi_i\xi_j \geq \lambda |\xi|^2 \text{ a.a. } x \in \Omega, \forall z \in \mathbb{R}, \quad (1.2)
\]

and

\[
a_{ij}(x,z) = a_{ji}(x,z) \text{ a.a. } x \in \Omega, \forall z \in \mathbb{R}. \quad (1.3)
\]

Let \( a_{ij} \) be local uniform continuous in \( z \), uniformly in \( x \):

\[
|a_{ij}(x,z) - a_{ij}(x,z')| \leq a(x)\mu_M(|z-z'|) \text{ a.e. } \Omega, \forall z, z' \in [-M,M],
\]
where \( a(x) \in L^\infty(\Omega) \), \( \mu_M(t) \) is a non-decreasing function, and \( \lim_{t \to 0} \mu_M(t) = 0 \), \( a_{ij}(x,0) \in L^\infty(\Omega) \).

We set that \( a_{ij}(x,z) \in VMO \) with respect to \( x \) and are loc. unif. in \( z \):

\[
\sup_{\rho \leq r} \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} \left| a_{ij}(x,z) - \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} a_{ij}(y,z) dy \right| \, dx = \eta_M(r),
\]

for all \( z \in [-M,M] \) and satisfy the following condition

\[
\lim_{r \to 0} \eta_M(r) = 0,
\]

where \( \Omega_\rho = \Omega \cap B_\rho \) and \( B_\rho \) is in class of the balls with radius \( \rho \) centered at the points of \( \Omega \).

As it concerns \( b(x,z) \) we suppose that:

\[
|b(x,z)| \leq b_1(x) \quad \text{a.e. } \Omega \quad \forall \, z \in \mathbb{R},
\]

where \( b_1(x) \in L^q(\Omega) \), \( q > n \).

### 2. A Priori Estimates

**Theorem 2.1.** Let \( u \in W^{1,q}(\Omega), \, q > n \) be a solution of the quasilinear equation (1.1) and (1.2) and (1.6) be true. Then there exists a constant \( C \) dependent on \( n \) and \( \text{diam} \Omega \) such that

\[
\|u\|_{L^\infty(\Omega)} \leq C(n,|\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}).
\]

**Proof.** Let us set \( \Omega^+ = \{x \in \Omega : u(x) > 0\} \). Then using Lemma 10.8 in [1] and taking into account that \( u \in W^{1,q}(\Omega) \) we have

\[
\sup_{\Omega^+} u \leq C(n,|\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}) \quad (u^+ = \max(u(x),0)).
\]

If we consider the function \( v = -u \) we have

\[
\sup_{\Omega} v \leq C(n,|\Omega|) \cdot (\|u^+\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}),
\]

and then

\[
-\inf_{\Omega} u \leq C(n,|\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)})
\leq C(n,|\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}),
\]
and
\[ u(x) \geq \inf_{\Omega} u \geq -C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}). \]

The last inequality combined with (2.2) gives
\[ \sup_{\Omega} |u| \leq C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}). \]

3. Uniqueness Result

**Theorem 3.1.** Let \( a_{ij} \) be bounded and measurable functions independent of \( z \), let (1.2) condition be true, \( b(x, z) \) non increasing in \( z \), a.e. in \( \Omega \). If \( u, v \in W^{1,q}(\Omega) \) are solutions of the following Dirichlet problem
\[
\begin{cases}
\mathcal{M} \tau = 0 & \text{a.e. in } \Omega, \\
\tau = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(3.1)

then \( u = v \) in \( \overline{\Omega} \).

**Proof.** The difference \( w = u - v \in W^{1,q}(\Omega) \) solves
\[ D_i(a_{ij}(x)D_jw) + b(x, u) - b(x, v) = 0 \quad \text{a.e. in } \Omega. \]
(3.2)

Let us set \( \Omega^+ = \{ x \in \Omega : w(x) > 0 \} \), we have \( u(x) > v(x) \) a.e. in \( \Omega^+ \), and from the hypothesis \( b(x, u(x)) \leq b(x, v(x)) \) a.e. in \( \Omega^+ \), then
\[ b(x, u(x)) - b(x, v(x)) \leq 0. \]

Thus (3.2) conducts to the following inequality
\[ D_i(a_{ij}(x)D_jw) \geq 0 \quad \text{almost everywhere in } \Omega^+. \]

Applying the maximum principle for divergence form operators
\[ w(x) \leq \sup_{\Omega^+} w \leq \sup_{\partial \Omega^+} w^+ = 0 \quad (w = 0 \text{ on } \partial \Omega), \]
then \( w \leq 0 \) in \( \overline{\Omega} \).

If we substitute \( w \) by \(-w\) we have \( w \geq 0 \), that is equivalent to \( u = v \) in \( \Omega \). \( \square \)
References


