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COMPLETION OF THE SHAPLEY ENTROPY OF FUZZY MEASURES

Guo-Jun Wang[1 §], Ning He[2]

[1]Institute of Mathematics
Shaanxi Normal University
Xi'an 710062, P.R. CHINA
e-mail: gjwang@snnu.edu.cn

[2]Shaanxi University of Technology
Hanzhong 723001, P.R. CHINA
e-mail: he_ning0916@sina.com.cn

Abstract: A fuzzy measure with complete certainty has the minimal Shapley entropy and vice versa. This paper points out that a fuzzy measure with complete uncertainty has the maximal Shapley entropy but not vice versa. An example shows that fuzzy measures with maximal Shapley entropy may far from complete uncertainty. A complementary entropy of the Shapley entropy called the partitional entropy is proposed which behaves well if used together with the Shapley entropy. The sum of the two entropies is called the absolute entropy. It is proved that a fuzzy measure possesses complete certainty or complete uncertainty if and only if its absolute entropy attains the minimal value or the maximal value respectively. Moreover, extension of fuzzy measures is discussed, the regular extension of fuzzy measures is introduced which keeps certain basic properties of the original fuzzy measure unchanged.

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§Correspondence author

1. Introduction

In the literature, the terminology "fuzzy measure" has various meanings as listed in [3], among them the most general definition given in [12], [2] can be conveniently used to represent uncertainty, which is obtained from the one employed by Sugeno in [9], [10] by giving up the condition of monotone continuity. At first glance it seems that this definition is too vague and has nothing specific except the monotonity. But it is interesting that Yager introduced the concept of the Shapley entropy [12] associated with a fuzzy measure by making use of the Shapley index [8], and one can then distinguish uncertainty associated with different fuzzy measures by simply calculating the corresponding mathematical formulas. A series of examples such as possibility measure [13], [1], cardinality based measure, Dempster-Shafer (D-S) belief measure [6], probability measure, and the CBCD measure [12] introduced by Yager himself, etc. were investigated in detail, all examples show the accordance that fuzzy measures with complete uncertainty possess the maximal Shapley entropy and fuzzy measures with complete certainty possess the minimal Shapley entropy. For example, the probability measure which has the maximal Shapley entropy is evenly distributed. On the other hand, fuzzy measures with maximal Shapley entropy need not to possess this kind of homogeneity as shown by CBCD measures. The aim of this paper is to point out that fuzzy measures with the maximal Shapley entropy may far from homogeneity and a new entropy called the partitional entropy is proposed which can be seen as a complementary entropy of the Shapley entropy and can be used together with the Shapley entropy to characterize uncertainty of fuzzy measures more efficiently. In Section 1, it is proved that fuzzy measures with the maximal Shapley entropy can be quite un-even. First, a necessary and sufficient condition for a fuzzy measure on $X = \{x_1, x_2, x_3\}$ to have the maximal Shapley entropy is obtained, which convince the reader of the diversity of fuzzy measures having the maximal Shapley entropy. Then it is clarified that there exists a fuzzy measure μ on $X = \{x_1, \dots, x_n\}$ associated with the maximal Shapley entropy and meanwhile satisfies the condition that $\mu(x_1) = \alpha\mu(x_2)$, where α is any given non- negative real number which can be arbitrarily large. Marichal and Roubens characterized in [4] fuzzy measures with maximal Shapley entropy by using Bernoulli numbers which can reflect in certain extent that fuzzy measures with maximal Shapley entropy may be uneven. But the results in Section 1 of this paper reflect the unevenness more intuitively. In Section 2, the partitional entropy is proposed and the sum of the Shapley entropy and the partitional entropy is called the absolute entropy. It is proved that fuzzy measures possess complete certainty or possess complete uncertainty depends on the value of the absolute entropy of the measures being minimal or maximal respectively. In Section 3, a kind of extension of fuzzy measures called regular extension is introduced, which enable us start from a fuzzy measure μ on X to construct a fuzzy measure μ^* on $Y = X \cup \{x^*\}$, where $x^* \notin X$ and keep certain basic properties of μ unchanged, especially, if $H(\mu)$ is maximal, then so is $H(\mu^*)$. Lastly, Section 5 is a conclusion.

2. Diversity of Fuzzy Measures Having Maximal Shapley Entropy

Shannon, the founder of information theory, deemed that probabilistic certainty without randomness possesses minimal entropy and full random probabilistic uncertainty possesses maximal entropy, and vice versa [7]. Unfortunately, the maximality of the Shapley entropy of a fuzzy measure can in no way guarantee the uncertainty of the fuzzy measureas will be clarified in this section.

First, we list the basic concepts below to make our discussion clear.

Definition 1. A mapping $\mu : 2^X \to [0,1]$ is called a fuzzy measure on X if $X \neq \emptyset$ and

(i) $\mu(\emptyset) = 0, \mu(X) = 1$;

(ii) If $A \subset B$, then $\mu(A) \leq \mu(B)$.

Definition 2. Suppose that μ is a fuzzy measure on $X = \{x_1, \ldots, x_n\}$, then for any x_i in X, its Shapley index [8] v_i is defined by

$$v_i = \sum_{k=0}^{n-1} \gamma_k \quad d_k^{(i)}\ , \tag{1}$$

where

$$\gamma_k = \frac{(n-1-k)!k!}{n!}, \quad d_k^{(i)} = \sum_{E \subset X - \{x_i\}, |E| = k} [\mu(E \cup \{x_i\}) - \mu(E)]. \tag{2}$$

Definition 3. (see [12]) Suppose that μ is a fuzzy measure on $X = \{x_1, \ldots, x_n\}$, then the Shapley entropy associated with μ is defined by

$$H(\mu) = -\sum_{i=1}^{n} v_i \ln(v_i). \tag{3}$$

The Shapley index was proposed early is the 50's of the 20-th century, which can be used to help indicate the amount of information in a fuzzy measure when it is being used to represent our knowledge about an uncertain variable. And so it is reasonable to use this Shapley index to compare the amount of uncertainty contained in fuzzy measures. This idea naturally leads to Definition 3. The reasonability of the Shapley entropy has been explained in [12] by a series of examples, and it is interesting that a fuzzy measure has complete certainty if and only if it has the minimal Shapley entropy. On the other hand, the relation between complete uncertainty and the maximal Shapley entropy is complicated, it is an "one way" relation, i.e., fuzzy measures with complete uncertainty has the maximal Shapley entropy but not vice versa. In fact, in case a fuzzy measure μ on X does not have complete certainty it will then have some randomness. The whole space X may have different features in accordance with different aspects of X, and properties associated with a variable is only related to one aspect. Therefore fuzzy measures with the maximal Shapley entropy may have diverse appearances even if X contains only a few elements. In fact, we have the following theorem.

Theorem 1. *Suppose that μ is a fuzzy measure on $X = \{x_1, x_2, x_3\}$, and*

$$\mu(x_1) = a, \quad \mu(x_2) = b, \quad \mu(x_3) = c, \quad 0 \leq c \leq b \leq a \leq 1\,. \tag{4}$$

Then the nesessary and sufficient condition for μ having the maximal Shapley entropy is $2a - c \leq 1$.

The proof of this theorem can be found in Appendix.

Note that there are many triples (a, b, c) which satisfy condition (4) and $2a - c \leq 1$. Moreover, the values of μ on two-elements sets can still be chosen freely only if the monotone conditions are fulfilled, i.e.,

$$\mu(x_1, x_2) = a + \epsilon, \quad \mu(x_1, x_3) = a + \delta, \quad \mu(x_2, x_3) = b + \eta, \tag{5}$$

$$0 \leq \epsilon \leq 1 - a, \quad 0 \leq \delta \leq 1 - a, \quad 0 \leq \eta \leq 1 - b. \tag{6}$$

Since $0 \leq 2a-c-b \leq 1-b$ whenever $2a-c \leq 1$, it follows that $[2a-c-b, 1-b] \subset [0, 1]$. Choose η in $[2a - c - b, 1 - b]$ and let

$$\varepsilon = \eta - (2a - c - b), \qquad \delta = \eta - (2a - 2b),$$

then (6) holds. Hence there are many different sets of numbers $(a, b, c, \varepsilon, \delta, \eta)$ of which the corresponding fuzzy measures have maximal Shapley entropy.

Example 1. Let $a = 0.5, b = 0.3, c = 0.2$, then $2a - c \leq 1$ is satistied. Choose η in the interval $[2a - c - b, 1 - b] = [0.5, 0.7]$ say, $\eta = 0.6$, then

$$\epsilon = \eta - (2a - c - b) = 0.1, \quad \delta = \eta - (2a - 2b) = 0.2\,,$$

and so,

$$\mu(x_1, x_2) = a + \epsilon = 0.6, \quad \mu(x_1, x_3) = a + \delta = 0.7,$$
$$\mu(x_2, x_3) = b + \eta = 0.9.$$

Therefore

$$\begin{aligned} v_1 &= \gamma_0 \mu(x_1) \\ &\quad + \gamma_1[(\mu(x_1, x_2) - \mu(x_2)) + (\mu(x_1, x_3) - \mu(x_3))] + \gamma_2[1 - \mu(x_2, x_3)] \\ &= \frac{1}{3} \times 0.5 + \frac{1}{6}[(0.6 - 0.3) + (0.7 - 0.2)] + \frac{1}{3}(1 - 0.9) = \frac{1}{3}. \end{aligned}$$

Similarly it can verify that $v_2 = v_3 = \frac{1}{3}$, and we see that μ is indeed associated with the maximal Shapley entropy $\ln 3$, while its distribution is not even.

Example 2. In the above example we have $a = \frac{5}{3}b$, i.e., $\mu(x_1) = \frac{5}{3}\mu(x_2)$. If we assume that $a = 10^6 b$ and $c = 0$, then we can still construct a fuzzy measure with the maximal Shapley entropy by the same procedure. Generally speaking, we can get a fuzzy measure with the maximal Shapley entropy and satisfying the condition that $\mu(x_1) = \alpha\mu(x_2)$, where α can be any larger number. Moreover, α can also be taken to be zero because $\mu(x_3) = c = 0 = 0\mu(x_2)$ and we can change the subscript. In fact, we have the following more general result.

Theorem 2. *Suppose that $X = \{x_1, \ldots, x_n\}$, then there exists a fuzzy measure μ on X which has the maximal Shapley entropy and satisfies the condition that $\mu(x_1) = \alpha\mu(x_2)$, where α is any given real number in the half line* $[0, +\infty)$.

Theorem 2 is a corollary of Theorem 5 in Section 3.

3. The Partitional Entropy and the Absolute Entropy of Fuzzy Measures

First let us consider the cardinality based measures [12]. Let

$$0 = D_0 \leq D_1 \leq \cdots \leq D_n = 1 \tag{7}$$

be a set of monotonically non-decreasing values. A fuzzy measure $\mu : 2^X \to [0, 1]$ is said to be cardinality based if

$$\forall E \subset X, \quad \mu(E) = D_{|E|}\,, \tag{8}$$

where $|E|$ is the cardinal of E. This measure has the property that every single element subset of X has one and the same value of measure no matter what single point it is, and every two element subset of X has another one and the same value of measure no matter where the subset located, and so on. This is a very strong homogeneity and the Shapley entropy associated with it is certainly maximal as shown in [12], [2]. But the reverse is not true as shown by the examples in Section 1 of this paper. Therefore, one may reasonably expect certain kind of entropy of which the maximal value of the entropy leads to the homogeneous property mentioned above. The Shapley entropy is obviously weaker than what we need.

In this section we first propose a complementary entropy to the Shapley entropy which is the partitional entropy and prove that the partitional entropy can well reflect the property of complete uncertainty but it is not adequate for reflecting complete certainty. It is very interesting that the situation will turn to be satisfactory if we add this new entropy to the Shapley entropy. This is why we call the partitional entropy complementary entropy of the Shapley entropy. The resulted entropy will then be called the absolute entropy, it behaves very well as shown by the theorem at the end of this section.

Definition 4. Suppose that $X = \{x_1, \ldots, x_n\}$. An m-partition of X is a class

$$\mathcal{P} = \{E_1, \ldots, E_m\}$$

of subsets of X such that

$$\bigcup_{i=1}^{m} E_i = X,\ \ E_i \neq \emptyset,\ \ E_i \cap E_j = \emptyset,\quad i,j \leq m,\ \ i \neq j,\ \ 1 < m \leq n. \tag{9}$$

The group $\{e_1, \ldots, e_m\}$ of positive integers is called the type of $\mathcal{P}$ and denoted $T(\mathcal{P})$, where $e_i = |E_i| \geq 1, i = 1, \ldots, m$. The set which consist of all partitions of X will be denoted by $\prod(X)$.

It is obvious that $e_1 + \cdots + e_m = n$. The order of e_i's is of little importance, for example, $\{1, 2, 3, 3\}$ and $\{3, 2, 1, 3\}$ are one and the same type ($n = 9$).

Definition 5. Suppose that μ is a fuzzy measure on $X = \{x_1, \ldots, x_n\}$, the function $\eta : 2^X \to [0, +\infty)$ defined by

$$\forall E \subset X, \quad \eta(E) = \frac{\mu(E)}{a_{|E|}}\,, \tag{10}$$

is called the ratio function of μ, where a_k is the average μ-measure of all subsets of X with cardinal k, and $\frac{0}{0}$ is assumed to be 1.

Remark 1. It is obvious that $\eta \geq 0$ and $\eta(\emptyset) = \eta(X) = 1$. Note that if $a_k = 0$, then every subset with cardinal k has measure 0 and hence η is well defined by (10). Moreover, it may happen that $\eta(E) > 1$. For example, suppose that $\mu(x_1) = \cdots = \mu(x_{n-1}) = 0$ and $\mu(x_n) = 1$, then $a_1 = \frac{1}{n}$ and $\eta(x_n) = n$. It can be proved that

$$\eta(E) \leq \binom{n}{[\frac{n}{2}]+1},$$

but this goes beyond our discussion and is hence omitted.

Proposition 1. *Suppose that μ is a cardinality based measure on $X = \{x_1, \ldots, x_n\}$, then the ratio function η of μ is a constant* 1. *Conversely, if η is a constant c, then μ is a cardinality based measure and $c = 1$.*

Proof. let μ be a cardinality based measure and $\mu(E) = e$ where $|E| = k$, then every subset of X with cardinal k has the same measure e, hence $a_k = e$ and it follows from (10) that $\eta(E) = 1$. Conversely, assume that η is a constant c, i.e., $\eta(E) = c$. This means $\mu(E) = c \cdot a_{|E|}$, i.e., the measure of E is decided by its cardinal, therefore μ is a cardinality based measure and $c = 1$. □

Definition 6. Suppose that μ is a fuzzy measure on $X = \{x_1, \ldots, x_n\}$ and η its ratio function. Let $\mathcal{P} = \{E_1, \ldots, E_m\}$ be a m-partition of X, then the local partitional entropy of μ at $\mathcal{P}$ is defined by

$$H'(\mu, \mathcal{P}) = -\sum_{i=1}^{m} \xi(\mathcal{P}, i) \ln(\xi(\mathcal{P}, i)), \quad (11)$$

where

$$\xi(\mathcal{P}, i) = \eta(E_i) / \sum_{j=1}^{m} \eta(E_j), \quad i = 1, \ldots, m. \quad (12)$$

And the partitional entropy of μ is defined by

$$H'(\mu) = \sum_{\mathcal{P} \in \prod(X)} H'(\mu, \mathcal{P}). \quad (13)$$

Example 3. Consider the fuzzy measure μ given in Example 1. It is easy to verify that $a_1 = 0.33, a_2 = 0.73, a_3 = 1.00$, and the ratio function η is as follows

$$\begin{array}{llll} \eta(x_1) = 1.50, & \eta(x_2) = 0.90, & \eta(x_3) = 0.60, & \\ \eta(x_1, x_2) = 0.82, & \eta(x_2, x_3) = 0.95, & \eta(x_2, x_3) = 1.23, & \eta(X) = 1.00. \end{array}$$

There are 3 partitions of type $\{1, 2\}$ and one partition of type $\{1, 1, 1\}$. They are

$$\begin{array}{ll} \mathcal{P}_0 = \{\{x_1\}, \{x_2\}, \{x_3\}\}, & \mathcal{P}_1 = \{\{x_1\}, \{x_2, x_3\}\}, \\ \mathcal{P}_2 = \{\{x_2\}, \{x_1, x_3\}\}, & \mathcal{P}_3 = \{\{x_3\}, \{x_1, x_2\}\}. \end{array}$$

Then

$$\xi(\mathcal{P}_0,1)=\frac{\eta(x_1)}{\sum_{i=1}^{3}\eta(x_i)}=0.50,\quad \xi(\mathcal{P}_0,2)=0.30,\quad \xi(\mathcal{P}_0,3)=0.20$$

and the local partitional entropy of μ at $\mathcal{P}_0$ is

$$H'(\mu,\mathcal{P}_0)=-(0.50\ln 0.50+0.30\ln 0.30+0.20\ln 0.20)=1.03.$$

The local partitional entropy of μ at $\mathcal{P}_1$ can be calculated as follows

$$\xi(\mathcal{P}_1,1)=\frac{\eta(x_1)}{\eta(x_1)+\eta(x_2,x_3)}=0.55,\quad \eta(\mathcal{P}_1,2)=0.45,$$

$$H'(\mu,\mathcal{P}_1)=-(0.55\ln 0.55+0.45\ln 0.45)=0.68.$$

Similarly we have

$$H'(\mu,\mathcal{P}_2)=0.69,\quad H'(\mu,\mathcal{P}_3)=0.68,$$

hence

$$H'(\mu)=\sum_{i=0}^{3}H'(\mu,\mathcal{P}_i)=3.08.$$

Proposition 2. *Suppose that μ is a fuzzy measure on $X=\{x_1,\dots,x_n\}$, then the partitional entropy $H'(\mu)$ is located in the closed interval $[0,M_n]$, where*

$$M_n=\sum_{m=2}^{n}\pi_m\ln m\,, \tag{14}$$

and π_m is the number of different m-partitions of X $(m=2,\dots,n)$.

Proof. It is well known that if $\sum_{i=1}^{m}b_i=1$ and $b_i\geq 0$ $(i=1,\dots,m)$, then

$$0\leq -\sum_{i=1}^{m}b_i\ln b_i\leq \ln m.$$

Therefore it follows from (11) and (13) that

$$0\leq H'(\mu)\leq \sum_{m=2}^{n}\pi_m\ln m. \qquad \square$$

In Example 3, $n=3,\pi_2=3,\pi_3=1$ and $M_3=3\ln 2+\ln 3=3.18$, we see that $H'(\mu)=3.08$ is indeed located in $[0,3.18]$.

Theorem 3. *Suppose that μ is a fuzzy measure on $X = \{x_1, \ldots, x_n\}$ and $H'(\mu)$ attains the maximal value M_n given in (14), then μ has the complete uncertainty, i.e., μ is a cardinality based measure. Conversely, if μ is a cardinality based measure, then $H'(\mu) = M_n$.*

The proof of this theorem can be found in Appendix.

Example 4. Consider the fuzzy measure μ on $X = \{x_1, \ldots, x_n\}$ with complete certainty, i.e., there is an x_i say, x_1, such that $\mu(E) = 1$ if $x_1 \in E$ and $\mu(E) = 0$ if $x_1 \notin E$. Let $\mathcal{P} = \{E_1, \ldots, E_m\}$ be any m-partition of X, then there is one and only one E_i say, E_1, contains x_1. Now it follows from (12) that $\xi(\mathcal{P}, 1) = 1$ and $\xi(\mathcal{P}, i) = 0 (i > 1)$. On account of (11) we see that the local partitional entropy of μ at $\mathcal{P}$ equals to 0. Hence it follows from (13) that $H'(\mu) = 0$.

The reverse of Example 4 is only partially true, i.e., the following proposition holds.

Proposition 3. *Suppose that μ is a fuzzy measure on $X = \{x_1, \ldots, x_n\}$ and $H'(\mu) = 0$, then there exists a variable x_i of X such that $\mu(x_i) \neq 0$ and $\mu(E) = 0$ if and only if $x_i \notin E (E \subset X)$.*

The proof can be found in Appendix.

Remark 2. Under the condition $H'(\mu) = 0$, Proposition 3 only asserts that there exists x_i such that $\mu(E) = 0$ if and only if $x_i \notin E$. This shows that μ has a portion of certainty if its partitional entropy attains the minimal value 0, but what about $x_i \in E$? It does not answer this question. In fact, if $x_i \in E$, then $\mu(E)$ can be assigned to be any number in the interval $[\mu(x_i), 1]$ only if the monotonicity is taking into account. Fuzzy measures like this may apparently deviate from the fuzzy measure with complete certainty given in Example 4. This is a short coming of the partitional entropy in reflecting complete certainty. This situation will be much better if we combine the partitional entropy with the Shapley entropy.

Definition 7. Suppose that μ is a fuzzy measure on X, then the absolute entropy $\overline{H}(\mu)$ of μ is the sum of its Shapley entropy and its partitional entropy, i.e.,

$$\overline{H}(\mu) = H(\mu) + H'(\mu). \tag{15}$$

It is evident that the value of $\overline{H}(\mu)$ is contained in the interval $[0, H_n]$ where

$$H_n = \ln n + M_n = \ln n + \sum_{m=2}^{n} \pi_m \ln m. \tag{16}$$

Suppose that $\overline{H}(\mu)$ attains its maximal value H_n, then $H'(\mu)$ attains its maximal value M_n and it follows from Theorem 3 that μ is the cardinality based measure. And suppose that $\overline{H}(\mu)$ attains its minimal value 0, then the Shapley entropy $H(\mu)$ also attains its minimal value 0 and hence μ possesses the complete certainty as shown in [12]. Conversely, the Shapley entropy and the partitional entropy of the fuzzy measure μ with complete certainty are both equal to 0 as shown in [12] and by Example 4 above respectively and hence $\overline{H}(\mu) = 0$. Moreover, the Shapley entropy and the partitional entropy of the fuzzy measure with complete uncertainty are both equal to their maximal value as shown in [12] and by Theorem 3 above respectively, hence the following theorem is obtained.

Theorem 4. *Suppose that μ is a fuzzy measure on $X = \{x_1, \ldots, x_n\}$ and $\overline{H}(\mu)$ is the absolute entropy of μ, then:*

(i) $0 \leq \overline{H}(\mu) \leq H_n$.

(ii) $\overline{H}(\mu) = 0$ if and only if μ possesses the complete certainty.

(iii) $\overline{H}(\mu) = H_n$ if and only if μ possesses the complete uncertainty.

4. Regular Extension of Fuzzy Measures

Let μ be a fuzzy measure on $X = \{x_1, \ldots, x_n\}$ and $Y = X \cup \{x^*\}$ has one element more than X, i.e., $x^* \notin X$. In this section we construct a fuzzy measure μ_c^* on Y which is in a sense a good extension of μ.

Definition 8. Suppose that μ is a fuzzy measure on $X = \{x_1, \ldots, x_n\}$, $x^* \notin X, Y = X \cup \{x^*\}$. Assume that $c \in [0, 1], \alpha = 1 - c$. Define a fuzzy measure μ_c^* on Y as follows:

(i) $\forall E \subset X, \mu_c^*(E) = \alpha\mu(E)$.

(ii) $\forall E \subset X, \mu_c^*(E \cup \{x^*\}) = c + \alpha\mu(E)$.

μ_c^* is called the regular c-extension of μ.

Theorem 5. *The regular c-extension μ_c^* of μ has the following properties in common with μ:*

(i) Let v_i^* be the Shapley index of x_i with respect to μ_c^*, then $v_i^* = \alpha v_i$, where v_i is the Shapley index of x_i with respect to $\mu (i = 1, \ldots, n)$.

(ii) If the Shapley entropy of μ attains the maximal value, then so is the Shapley entropy of μ_c^*, where $c = \frac{1}{n+1}$.

(iii) If μ is the Dempster-Shafer belief measure, then so is μ_c^*.

(iv) If μ is the Dempster-Shafer plausibility measure, then so is μ_c^*.

Proof. It is clear that μ_c^* is a fuzzy measure on Y. First note that if $\gamma_k^* = \frac{(n-k)!k!}{(n+1)!}$, then it is easy to verify that

$$\gamma_k^* + \gamma_{k+1}^* = \gamma_k, \tag{17}$$

where γ_k is defined by (2). Now the Shapley index v_i^* can be calculated as follows

$$\begin{aligned} v_i^* &= \sum_{k=0}^{n} \gamma_k^* \sum_{F\subset Y-\{x_i\},|F|=k} [\mu_c^*(F\cup\{x_i\})-\mu_c^*(F)] \\ &= \gamma_n^*(1-\mu_c^*(Y-\{x_i\})) + \sum_{k=0}^{n-1}\gamma_k^* \sum_{E\subset X-\{x_i\},|E|=k} [\mu_c^*(E\cup\{x_i\})-\mu_c^*(E)] \\ &\quad + \sum_{k=1}^{n-1}\gamma_k^* \sum_{E\subset X-\{x_i\},|E|=k-1} [\mu_c^*(E\cup\{x^*,x_i\})-\mu_c^*(E\cup\{x^*\})]\,. \end{aligned}$$

Assume that $\mu(X-\{x_i\})=\theta_i$, then it follows from Definition 8 that $\mu_c^*(Y-\{x_i\})=c+\alpha\theta_i$, hence it follows from (ii) of Definition 8 that

$$\begin{aligned} v_i^* &= \gamma_n^*(1-c-\alpha\theta_i) + \textstyle\sum_{k=0}^{n-1}\gamma_k^*\alpha d_k^{(i)} + \sum_{k=1}^{n-1}\gamma_k^*\alpha d_{k-1}^{(i)} \\ &= \alpha\gamma_n^*(1-\theta_i) + \textstyle\sum_{k=0}^{n-1}\gamma_k^*\alpha d_k^{(i)} + \sum_{k=0}^{n-2}\gamma_{k+1}^*\alpha d_k^{(i)} \\ &= \alpha\gamma_n^*(1-\theta_i) + \alpha\textstyle\sum_{k=0}^{n-1}(\gamma_k^*+\gamma_{k+1}^*)d_k^{(i)} - \gamma_n^*\alpha d_{n-1}^{(i)}. \end{aligned}$$

Since

$$d_{n-1}^{(i)} = \sum_{E\subset X-\{x_i\},|E|=n-1} [\mu(E\cup\{x_i\})-\mu(E)] = 1-\mu(X-\{x_i\}) = 1-\theta_i\,,$$

we have from (17) and (1) that

$$v_i^* = \alpha\gamma_n^*(1-\theta_i) + \alpha\sum_{k=0}^{n-1}\gamma_k d_k^{(i)} - \alpha\gamma_n^*(1-\theta_i) = \alpha v_i.$$

This proves (i).

If the Shapley entropy of μ attains its maximum $\ln n$, then $v_i=\frac{1}{n}(i=1,\ldots,n)$. Let $c=\frac{1}{n+1}$, then $\alpha=\frac{n}{n+1}$. It follows from (i) that $v_i^*=\frac{n}{n+1}\cdot\frac{1}{n}=\frac{1}{n+1}$, and the Shapley index of x^* is $1-\sum_{i=1}^{n}v_i^*=\frac{1}{n+1}$ as well, hence $H(\mu_c^*)$ attains its maximum $\ln(n+1)$.

Suppose that μ is a Dempster-Shafer belief measure, then there is a (D-S) structure m such that

$$\forall E\subset X, \mu(E) = \sum_{G\subset E} m(G).$$

Define m^* as follows

$$m^*(G) = \begin{cases} \alpha m(G), & x^* \notin G, \\ c, & G = \{x^*\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{G \subset Y} m^*(G) = \alpha \sum_{G \subset X} m(G) + c = \alpha + c = 1,$$

hence m^* is a (D-S) structure on Y. Now we have

$$\forall E \subset X, \mu_c^*(E) = \alpha\mu(E) = \alpha \sum_{G \subset E} m(G) = \sum_{G \subset E} m^*(G).$$

$$\mu_c^*(E \cup \{x^*\}) = c + \alpha\mu(E) = m^*(x^*) + \sum_{G \subset E} m^*(G) = \sum_{G \subset E \cup \{x^*\}} m^*(G).$$

This proves (iii).

Property (iv) can be proved similarly and the proof is hence omitted. □

Corollary. *Theorem 2 is true.*

Proof. In Example 2 we see that there exists a fuzzy measure μ on $\{x_1, x_2, x_3\}$ of which the Shapley entropy is maximal and $\mu(x_1) = \alpha\mu(x_2)$ holds where $\alpha \in [0, \infty)$. Because regular extension keeps the ratio of $\mu(x_1)$ over $\mu(x_2)$, use the regular extension $(n-3)$-times and we obtain Theorem 2. □

5. Conclusion

We have pointed out that the Shapley entropy of a fuzzy measure should be strengthened by combining with the partitional entropy, and the resulted entropy, the absolute entropy, reflects both complete certainty and complete uncertainty in a harmonious manner. We have also introduced the regular extension of fuzzy measures which keeps certain properties of the original fuzzy measure unchanged. As an application, the fact that the Shapley entropy is not satifactory for characterizing complete uncertainty is proved.

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Appendix

1. Proof of Theorem 1. The Shapley index of x_1 can be calculated from (1), (2), (4) and (5) as follows

$$\begin{aligned} v_1 &= \gamma_0\mu(x_1) + \gamma_1[(\mu(x_1,x_2) - \mu(x_2)) + (\mu(x_1,x_3) - \mu(x_3))] \\ &\quad + \gamma_2[1 - \mu(x_2,x_3)] \\ &= \frac{1}{3}a + \frac{1}{6}[(a+\epsilon-b) + (a+\delta-c)] + \frac{1}{3}(1-b-\eta) \\ &= \frac{1}{6}(2+4a-3b-c+\epsilon+\delta-2\eta). \end{aligned}$$

Similarly we have

$$v_2 = \frac{1}{6}(2-2a+3b-c+\epsilon-2\delta+\eta),$$

$$v_3 = \frac{1}{6}(2-2a+2c-2\epsilon+\delta+\eta).$$

The necessary and sufficient condition for Shapley entropy $H(\mu)$ of μ being maximal is that $v_1 = v_2 = v_3$. It is easy to prove that this is equivalent to the following condition:

$$b - c = \delta - \epsilon, \quad 2a - b - c = \eta - \epsilon. \tag{18}$$

It follows from (6) that $b + \eta \leq 1$. Since $\eta = 2a - b - c + \epsilon$ we have

$$b + \eta = 2a - c + \epsilon \leq 1\,,$$

and therefore the condition (i) that $2a - c \leq 1$ is necessary. Conversely, assume that a, b, c are given and the conditions $0 \leq c \leq b \leq a \leq 1$ and $2a - c \leq 1$ are fulfilled, then $2a - b - c \leq 1 - b$. Choose any number in $[2a-b-c, 1-b]$ to be η, then $\eta \geq 2a - b - c \geq 2a - 2b$. Let $\epsilon = \eta - (2a - b - c), \delta = \eta - (2a - 2b)$, then both (6) and (18) are fulfilled and Theorem 1 is proved. □

2. Proof of Theorem 3. $H'(\mu)$ attains the maximum if and only if every local partitional entropy attains its maximum. First consider the unique n-partition $\mathcal{P} = \{\{x_1\}, \ldots, \{x_n\}\}$. The local partitional entropy of μ at $\mathcal{P}$, $H'(\mu, \mathcal{P})$, attains the maximal value $\ln n$. This happens only if

$$\xi(\mathcal{P}, 1) = \cdots = \xi(\mathcal{P}, n) = \frac{1}{n}.$$

Hence we have from (12) that

$$\eta(x_1) = \cdots = \eta(x_n) = c. \tag{19}$$

Now assume that E is any non-empty subset of X with $|E| = k \le n-1$. Without any loss of generality (by virtue of (19)) we can assume that $E = X - \{x_1, \ldots, x_{n-k}\}$. Consider the $(n-k+1)$-partition $\mathcal{P}_E = \{\{x_1\}, \ldots, \{x_{n-k}\}, E\}$. Because the local partitional entropy of μ at $\mathcal{P}_E$ attains its maximum $\ln(n-k+1)$ we have

$$\xi(\mathcal{P}_E, 1) = \cdots = \xi(\mathcal{P}_E, n-k) = \xi(\mathcal{P}_E, n-k+1).$$

This implies that

$$\eta(x_1) = \cdots = \eta(x_{n-k}) = \eta(E) = c.$$

Hence η is a constant c and the first half of the proof is completed by virtue of Proposition 1.

Conversely, let μ be a cardinality based measure, then it follows from Proposition 1 that the ratio function η of μ is constant 1. Let $\mathcal{P} = \{E_1, \ldots, E_m\}$ be any m-partition of X, then it follows from (12) that $\xi(\mathcal{P}, i) = \frac{1}{m}(i = 1, \ldots, m)$, hence it follows from (11) that $H'(\mu, \mathcal{P})$ attains its maximum $\ln m$. Since $\mathcal{P}$ is arbitrary, hence $H'(\mu)$ attains its maximum M_n. This completes the proof. □

3. Proof of Proposition 3. Consider the n-partition $\mathcal{P} = \{\{x_1\}, \ldots, \{x_n\}\}$. Assume that $H'(\mu) = 0$, then because every local partition entropy is non-negative we have from (13) that $H'(\mu, \mathcal{P}) = 0$. On account of (11) we see that there exists a unique $\xi(\mathcal{P}, i)$ say, $\xi(\mathcal{P}, 1)$, equals 1 and all others equal 0. Hence it follows from (12) that $\eta(x_1) \ne 0$ and $\eta(x_2) = \cdots = \eta(x_n) = 0$. Now assume that $x_1 \notin E \subset X$ and $|E| = k(1 \le k \le n-1)$. Consider the 3-partition $\mathcal{P}^* = \{\{x_1\}, E, X - (E \cup \{x_1\})\}$ (when $k = n-1$, the empty set $X - (E \cup \{x_1\})$ should be cancelled from $\mathcal{P}^*$ and $\mathcal{P}^*$ becomes a 2-partition). Since $H'(\mu, \mathcal{P}^*) = 0$ it can be proved similarly that there exists only one non-zero number among $\eta(x_1), \eta(E)$, and $\eta(X - (E \cup \{x_1\}))$, and it is $\eta(x_1)$ as we have seen. Hence $\eta(E) = 0$ and it follows from (10) that $\mu(E) = 0$. Since it is obvious that $\mu(E) \ne 0$ whenever $x_1 \in E$, the proof of Proposition 3 is completed. □