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AN EQUILIBRIUM MODEL FOR THE DAY-AHEAD TRADE IN ELECTRICITY MARKETS

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Abstract: We present a model for an electricity market consisting of market participants that are as well producers as retailers of electrical power.

The participants are modelled by their production capacities, production costs, risk aversion and the random demand on electrical power of the costumers for the next day. This demand has to be supplied by a combination of own generation and the day–ahead trade. If the demand exceeds own generation plus the amount purchased in the day–ahead trade of the forgoing day, the missing amount must be purchased on the real–time marked to a high price. This price enters the calculation of the optimal position in the day–ahead trade as a random variable.

We give conditions for the existence of an equilibrium forward price in the day–ahead trade, present some examples, and calculate the optimal forward price for some examples.

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1. Introduction

We introduce an electricity market model using some settings from Bessembinder [1]. Let $I \in \mathbb{N}$ be the number of market participants (agents). Each agent acts as a retailer and a producer of electrical power at the same time.

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Assume that no outside speculators are active in the market, meaning that the agents purchase and sell their energy to each other due to different power demand, production technologies, costs, available capacities, liabilities, risk aversion, etc. We suppose that power may be transmitted without costs, and no restrictions on transmission capacities exist.

In Hinz, Weber and Weber [2] a one–step model for such an electricity market has been presented. One of the simplifications in this model criticized by practitioners was the assumption that there is an initially known fixed price, called upper penalty price, for electrical power purchased on the real–time marked by an agent to meet his demand in the case that this demand is bigger than his production capacity plus the energy purchased by forward contracts.

A first attempt to avoid this assumption has been made in Standfuss [6]. There the upper penalty price has been modelled as a discrete random variable with a finite number of values.

In this paper the existence of an equilibrium forward price is proved in the case that the upper penalty price has a continuous distribution on a bounded interval. The equilibrium forward price and the optimal traded volumes are characterized and some examples are presented.

2. The Electricity Market Model

Let us consider the following rules which reflect simplified electricity trading norms currently used in Germany. Each agent $i \in \{1, \dots, I\}$ retails the power to his final consumers and sells it at a fixed price p_i^R (USD/MWh). The final consumer's one–day–demand is modeled by a random variable $Q_i \geq 0$ exogenously given on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. The i–th agent is obligated to supply the need of his consumers. To do that, he may use each combination of the following methods:

1. The generation of power in his own production unit with the upper bound $c_i \geq 0$ (MWh within one day).

2. The forward trade one day before delivery at the forward price $p > 0$. In what follows, we denote by $q_i \in \mathbb{R}$ (MWh) the forward traded volume of the i–th agent. A negative q_i is interpreted as forward sale of $|q_i|$ MWh.

3. In the case of power shortage (that is, if $Q_i > q_i + c_i$) the agent may complete his supply buying the missing amount $Q_i - (q_i + c_i) > 0$ at an upper penalty price P^U (USD/MWh). In the case of power excess (that

is, if $Q_i > q_i$) the agent may sell the extra amount $Q_i - q_i > 0$ of power at the lower penalty price p^L. Whereas p^L is assumed to be fixed, the upper penalty price P^U is a random variable on $(\Omega, \mathcal{F}, \mathrm{P})$ taking values in an interval $]p^L, p^U]$.

Denote by p_0^U the lower bound of the distribution of P^U in the sense that

$$\begin{aligned} &\mathrm{P}(P^U < p) \cdot \mathrm{P}(P^U > p) > 0, \quad p \in]p_0^U, p^U[\ , \\ &\mathrm{P}(P^U < p_0^U) = \mathrm{P}(P^U > p^U) = 0. \end{aligned} \tag{1}$$

The case of a non–random upper penalty price has been considered in [2].

We suppose the production technology used by i–th agent yields electricity to a price p_i^C (USD/MWh), and that

$$p^L < p_i^C < p^U \quad \text{for all} \quad i = 1, \ldots, I. \tag{2}$$

We assume for all $i = 1, \ldots, I$, that the i–th agent knows the joint distribution of Q_i and P^U when he trades the forwards. After forward trading is finished, but before he has to allocate his power plants, he gets knowledge of the upper penalty price $P^U(\omega)$. We mention that he will not produce energy if $P^U(\omega) \leq p_i^C$. Then he will purchase all energy he needs to the price $P^U(\omega)$.

As an outcome of his strategy, the agent obtains a random earning G_i^p, which depends on the forward position q_i, on the forward price p, on the demand Q_i, and the upper penalty price P^U, and that is given by

$$\begin{aligned} G_i^p(q_i, Q_i, P^U) \ &= \ p_i^R Q_i - p q_i - P^U (Q_i - (c_i + q_i))^+ 1_{\{P^U > p_i^C\}} \\ &- \ P^U (Q_i - q_i)^+ 1_{\{P^U \leq p_i^C\}} + p^L (q_i - Q_i)^+ \\ &- \ f_i - K_i(q_i, Q_i, P^U). \end{aligned}$$

Here, we use the notation $(x)^+ = \max\{x, 0\}$ for all $x \in \mathbb{R}$, and denote by $f_i > 0$ the fixed costs (USD within one day) of the i–th production unit and by $K_i(q_i, Q_i, P^U)$ the production costs of power depending on the forward position q_i, agent's demand Q_i and the upper penalty price P^U.

Let us explain how the production costs $K_i(q_i, Q_i, P^U)$ are calculated (see Figure 1).

If $P^U(\omega) \leq p_i^C$ then there is no production, and $K_i(q_i, Q_i(\omega), P^U)(\omega) = 0$.

Otherwise the following holds:

If $Q_i(\omega) < q_i$ then the demand is supplied from the forward purchase, that is, no own production is needed and $K_i(q_i, Q_i(\omega), P^U(\omega)) = 0$.

If $Q_i(\omega) \geq q_i$ and $Q_i(\omega) < q_i + c_i$, then the agent produces the quantity $Q_i(\omega) - q_i \geq 0$, and so $K_i(q_i, Q_i(\omega), P^U(\omega)) = p_i^C(Q_i(\omega) - q_i)$.

If $Q_i(\omega) \geq q_i + c_i$ the production runs at the upper limit of his unit, hence $K_i(q_i, Q_i(\omega), P^U(\omega)) = p_i^C c_i$.

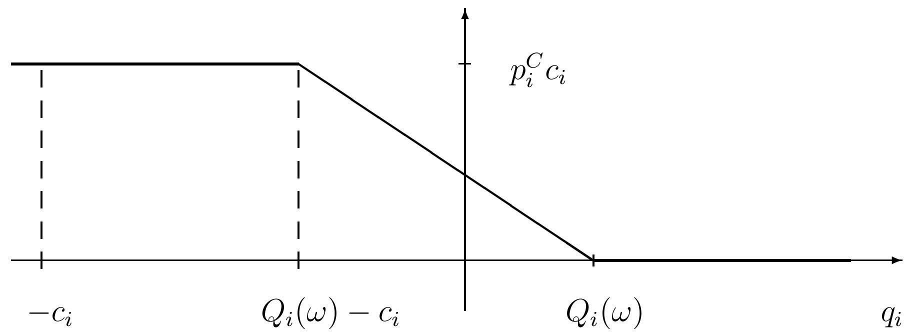

Figure 1: The production costs $K_i(\,\cdot\,, Q_i(\omega),\, P^U(\omega))$ if $P^U(\omega) > p_i^C$.

An important feature of our market's description is the setting of participant's risk aversion by appropriate utility functionals. Let us define the utility functional of the i–th agent by

$$\mathcal{U}_i^p(q) = E(U_i(G_i^p(q, Q_i, P^U))) \qquad \text{for all } q \in \mathbb{R}\,,\ \ p \in [p^L, p^U], \tag{3}$$

where the utility function $U_i : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with a positive and decreasing derivative $\dot{U}_i$ for all $i = 1, \ldots, I$, and $E(\,\cdot\,)$ denotes the expectation with respect to the measure P. Note that from now we restrict ourselves to consider the forward prices p within the interval $[p^L, p^U]$ since in the real market the forward price will never exceed p^U (no purchasers) or fall below p^L (no sellers).

The basic idea of the equilibrium market modeling is that the trades and prices in a real market are described by an equilibrium state of its model. The model equilibrium is characterized by the assumption that each agent acts optimally, and at the same time some boundary conditions are satisfied. In our model, an equilibrium is given by the agent's forward trades $(q_1^*, \ldots, q_I^*) \in \mathbb{R}^I$ and by the forward electricity price $p^* \in [p^L, p^U]$ such that for all $i = 1, \ldots, I$ the utility functional $\mathcal{U}_i^{p^*}(\cdot)$ is maximized at q_i^*, and the forward contracts are in zero net supply: $\sum_{i=1}^I q_i^* = 0$. Our aim is to give conditions ensuring the existence of an equilibrium and to calculate the equilibrium forward price p^* and the optimal traded volumes q_i^*.

3. The Equilibrium

To ensure that (3) is well–defined and for technical reasons we suppose the following integrability conditions are satisfied:

$$\begin{aligned} &E(U_i(aQ_i+b))<\infty,\\ &E(\dot{U}_i(aQ_i+b))<\infty\,,\ \text{for all } a,b\in\mathbb{R},\ i=1,\dots,I. \end{aligned} \tag{4}$$

Further, we make the following assumptions to the distributions of P^U and Q_i, $i=1,\dots,I$:

(A1) $p_i^R \in]p_0^U, p^U[$ for all $i=1,\dots,I$.

(A2) The functions $q \mapsto \mathrm{P}(Q_i < q \mid P^U < p)$ and $q \mapsto \mathrm{P}(Q_i < q \mid P^U > p)$ are continuous and strictly increasing on $\mathbb{R}^+$ for all $i=1,\dots,I$ and $p \in]p_0^U, p^U[$.

(A3) $\mathrm{P}(P^U > p_i^C \mid Q_i = q) > 0$ for all $q \geq 0$, $i=1,\dots,I$.

(A4) The functions $p \mapsto \mathrm{P}(\ P^U < p | \ Q_i < q)$ are continuous and strictly increasing on $]\,p_0^U,\, p^U\,[$ for all $i=1,\dots,I$, and $q>0$.

Theorem 1. *Let the conditions (A1) through (A4) be satisfied. Then there exists an equilibrium.*

In preparation for the proof of the theorem we study the properties of the functions

$$h_i^p : \mathbb{R} \to \mathbb{R},\ q \mapsto E(U_i(G_i^p(q,Q_i,P^U))),$$

especially the properties of their derivatives $h_i^{p\,\prime}$, $i=1,\dots,I$.

Lemma 1. *For all $i=1,\dots,I$ we have:*

(i) *The function h_i^p is differentiable for each $p\in[p^L,p^U]$, and*

$$\begin{aligned} h_i^{p\,\prime}(q) &= E(H_i^p(q))\,,\\ H_i^p(q) &= \dot{U}_i(G_i^p(q,Q_i,P^U))\Big(p^L 1_{\{q\geq Q_i\}} - p\\ &\quad + P^U 1_{\{q<Q_i\}} 1_{\{P^U\leq p_i^C\}}\\ &\quad + P^U 1_{\{q<Q_i-c_i\}} 1_{\{P^U>p_i^C\}}\\ &\quad + p_i^C 1_{\{Q_i-c_i\leq q<Q_i\}} 1_{\{P^U>p_i^C\}}\Big),\\ &\quad \textit{for all } q\in\mathbb{R},\quad p\in[p^L,p^U]\,. \end{aligned} \tag{5}$$

(ii) *The function* $H_i^p(\,.\,)$ *is decreasing on* $\mathbb{R}$ *for all* $\omega \in \Omega$, $p \in [p^L, p^U]$.
If $p \in]p^L, p^U[$ *then* $h_i^{p\,\prime}$ *is decreasing and continuous on* $\mathbb{R}$, *and strictly decreasing on* $]-c_i, \infty[$.

Proof. (i) Let $\omega \in \Omega$, $q, \tilde{q} \in \mathbb{R}$ and $p \in [p^L, p^U]$, then

$$\begin{aligned}
&|G_i^p(q, Q_i, P^U)(\omega) - G_i^p(\tilde{q}, Q_i, P^U)(\omega)| \\
&\leq \; p|q - \tilde{q}| \\
&\quad +P^U(\omega)|(Q_i(\omega) - (c_i + q))^+ - (Q_i(\omega) - (c_i + \tilde{q}))^+|1_{\{P^U(\omega) > p_i^C\}} \\
&\quad +P^U(\omega)|(Q_i(\omega) - q)^+ - (Q_i(\omega) - \tilde{q})^+|1_{\{P^U(\omega) \leq p_i^C\}} \\
&\quad +p^L|(q - Q_i(\omega))^+ - (\tilde{q} - Q_i(\omega))^+| \\
&\quad +|K_i(q, Q_i(\omega), P^U(\omega)) - K_i(\tilde{q}, Q_i(\omega), P^U(\omega))| \;\leq\; 4p^U|q - \tilde{q}|\,.
\end{aligned}$$

Hence, we obtain for all $q, \tilde{q} \in \mathbb{R}$

$$\begin{aligned}
&|U_i(G_i^p(q, Q_i(\omega), P^U(\omega))) - U_i(G_i^p(\tilde{q}, Q_i(\omega), P^U(\omega)))| \\
\leq\;& |\dot{U}_i(\xi)| 4p^U |q - \tilde{q}|\,,
\end{aligned}$$

where ξ is between $G_i^p(q, Q_i(\omega), P^U(\omega))$ and $G_i^p(\tilde{q}, Q_i(\omega), P^U(\omega))$. Thus,

$$\xi \geq -p^U Q_i - f_i - p(q \vee \tilde{q}) - p_i^C c_i,$$

and (4) gives a dominating integrable function

$$\dot{U}_i(-p^U Q_i - f_i - p(|q| + \varepsilon) - p_i^C c_i) 4p^U\,,$$

for

$$\frac{U_i(G_i^p(q, Q_i, P^U)) - U_i(G_i^p(\tilde{q}, Q_i, P^U))}{q - \tilde{q}}, \text{ for all } \; \tilde{q} \text{ with } |q - \tilde{q}| < \varepsilon.$$

Passing through the limit $\tilde{q} \to q$, we find with (A2) the almost sure convergence of the above quotient to $H_i^p(q)$ for all $q \in \mathbb{R}$, $p \in [p^L, p^U]$.

Hence, the derivative $h_i^{p\,\prime}$ of h_i^p exists, and $h_i^{p\,\prime}(q) = E(H_i^p(q))$ for all $q \in \mathbb{R}$ and $p \in [p^L, p^U]$.

(ii) First, we show that $H_i^p(\cdot)(\omega)$ is decreasing on $\mathbb{R}$. Let $\omega \in \Omega$.

The function $G_i^p(\,.\,, Q_i(\omega), P^U(\omega))$ is differentiable on each of the intervals $]-\infty, Q_i(\omega) - c_i[$, $]Q_i(\omega) - c_i, Q_i(\omega)[$ and $]Q_i(\omega), \infty[$, the derivative

$$\begin{aligned}
q \mapsto \quad & (p^L 1_{\{q \geq Q_i\}} - p + P^U 1_{\{q < Q_i\}} 1_{\{P^U \leq p_i^C\}} \\
& + P^U 1_{\{q < Q_i - c_i\}} 1_{\{P^U > p_i^C\}} + p_i^C 1_{\{Q_i - c_i \leq q < Q_i\}} 1_{\{P^U > p_i^C\}})
\end{aligned} \tag{6}$$

is the second factor of (5), and it is constant on each of these intervals.

At points q of such an interval where (6) is positive the function $G_i^p(\,.\,, Q_i(\omega), P^U(\omega))$ is increasing. As $\dot{U}$ is decreasing, this implies that H_i^p is decreasing at q.

At points q of such an interval where (6) is negative the function $G_i^p(\,.\,, Q_i(\omega), P^U(\omega))$ is decreasing. This implies that $\dot{U}(G_i^p(\,.\,, Q_i(\omega), P^U(\omega))$ is increasing at q, and the product H_i^p is decreasing at q.

If the function defined by (6) is equal to 0 at a point of such an interval, it is equal to 0 on the whole interval as well as H_i^p.

The function H_i^p has at $q = Q_i(\omega)$ a jump of the size

$$\dot{U}_i(G_i^p(Q_i(\omega), Q_i(\omega), P^U(\omega)))(P^U 1_{\{P^U \leq p_i^C\}} + p_i^C 1_{\{P^U > p_i^C\}} - p^L) \geq 0$$

(left limit minus right limit).

If $P^U(\omega) > p_i^C$ then H_i^p has at $q = Q_i(\omega) - c_i$ a jump of the size

$$\dot{U}_i(G_i^p(Q_i(\omega) - c_i, Q_i(\omega), P^U(\omega)))(P^U - p_i^C) \geq 0,$$

(left limit minus right limit) otherwise it is continuous there. Combined with the monotonic behavior this yields

$$\begin{aligned} & H_i^p(q)(\omega) - H_i^p(\tilde{q})(\omega) \\ \geq\ & \dot{U}_i(G_i^p(Q_i(\omega) - c_i, Q_i(\omega), P^U(\omega)))(P^U - p_i^C) 1_{\{P^U(\omega) > p_i^c\}} \end{aligned}$$

for all $q, \tilde{q} \in \mathbb{R}$ and $\omega \in \Omega$ satisfying $q < Q_i(\omega) - c_i < \tilde{q}$. Hence

$$\begin{aligned} & H_i^p(q) - H_i^p(\tilde{q}) \\ \geq\ & \dot{U}_i(G_i^p(Q_i - c_i, Q_i, P^U(\omega)))(P^U - p_i^C) 1_{\{Q_i(\omega) - c_i \in]q, \tilde{q}[\}} 1_{\{P^U(\omega) > p_i^c\}}. \end{aligned}$$

Note that there is a $C > 0$ such that $G_i^p(Q_i - c_i, Q_i, P^U(\omega))) > C$ on $\{Q_i(\omega) - c_i \in\,]q, \tilde{q}[\ \}$. Taking the expectation on both sides we get with (A2) for $q < \tilde{q}$, $\tilde{q} > c_i$

$$\begin{aligned} & {h_i^p}'(q) - {h_i^p}'(\tilde{q}) \\ =\ & E(H_i^p(q)) - E(H_i^p(\tilde{q})) \\ \geq\ & C \cdot E((P^U - p_i^C) 1_{\{Q_i - c_i \in]q, \tilde{q}[\}} 1_{\{P^U > p_i^c\}}) \\ \geq\ & C \cdot E((P^U - p_i^C) 1_{\{P^U > p_i^c\}} | Q_i - c_i \in]q, \tilde{q}[) \cdot P(Q_i - c_i \in]q, \tilde{q}[) \\ >\ & 0\,, \end{aligned}$$

since $P(Q_i - c_i \in]q, \tilde{q}[\,) > 0$. If $\tilde{q} < c_i$ we get only the estimate

$${h_i^p}'(q) - {h_i^p}'(\tilde{q}) \ \geq\ 0.$$

The continuity of $h_i^{p\,\prime}$ follows from the almost sure convergence

$$\lim_{q_n \to q} H_i^p(q_n) = H_i^p(q)$$

dominated by the integrable function $\dot{U}_i(-p^U Q_i - f_i - \sup_{n\in\mathbb{N}} pq_n - p_i^C c_i)p^U$. □

It is clear that a maximizer q_i^* of h_i^p, $h_i^p(q_i^*) \geq h_i^p(q)$, $q \in \mathbb{R}$, is an optimal forward position of the i–th agent if the forward price is p. Therefore we next prove a lemma that allows to study the existence of zeros of the function $h_i^{p\,\prime}$.

Lemma 2. *For all $i = 1, \dots, I$ we have:*

(i) If $q \leq -c_i$, then the expression $h_i^{p\,\prime}(q)$ is strictly increasing in p.

(ii) We have $\lim_{q_n \uparrow \infty} h_i^{p\,\prime}(q_n) < 0$, $p \in [p^L, p^U]$.

(iii) If $\dot{U}_i$ is not a constant function on $\mathbb{R}$, then $h_i^{p\,\prime}$ is strictly decreasing for $q \in]-\infty, -c_i]$, and, for $p \leq E(P^U)$, there exists $q_0(p)$ such that $h_i^{p\,\prime}(q) > 0$ for all $q < q_0(p)$. If $\dot{U}_i$ is a constant function on $\mathbb{R}$, then $h_i^{E(P^U)\,\prime}(q) = 0$ for all $q \in]-\infty, -c_i]$, and $h_i^{p\,\prime}(-c_i) > 0$ for $p^L < p < E(P^U)$.

(iv) We have $h_i^{p^U\,\prime}(q) < 0$ and $h_i^{p^L\,\prime}(q) > 0$ for all $q \in \mathbb{R}$.

(v) The function $[p^L, p^U] \to \mathbb{R}$, $p \mapsto h_i^{p\,\prime}(q)$ is continuous for all $q \in \mathbb{R}$.

Proof. (i) For any $\omega \in \Omega$ and $q \leq -c_i$ we have $G_i^p(q, Q_i(\omega), P^U(\omega))$ is strictly increasing in p. Further,

$$H_i^p(q)(\omega) = \dot{U}_i(G_i^p(q, Q_i(\omega), P^U(\omega)))(P^U(\omega) - p).$$

As $\dot{U}$ is increasing and positive $H_i^p(q)(\omega)$ is strictly increasing. Taking expectations we get the statement.

(ii) For $p \in]p^L, p^U[$, we have

$$h_i^{p\,\prime}(-c_i) = E(H_i^p(-c_i)) = E(\dot{U}_i(G_i^p(-c_i, Q_i, P^U)))(p^U - p) > 0.$$

Now, let $(q_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be increasing with $\lim_{n\to\infty} q_n = \infty$. Then, by (ii) of Lemma 1, $(H_i^p(q_n))_{n\geq 1}$ is decreasing. Moreover, if $q_n \geq Q_i(\omega)$, then

$$H_i^p(q_n)(\omega) \leq \dot{U}_i(G_i^p(Q_i(\omega), Q_i(\omega), P^U(\omega)))(p^L - p) < 0.$$

Hence

$$\begin{aligned} \lim_{q_n \uparrow \infty} h_i^{p\,\prime}(q_n) &\leq \lim_{q_n \uparrow \infty} E(H_i^p(q_n) \vee \dot{U}_i(G_i^p(Q_i, Q_i, P^U)))(p^L - p) \\ &= E(\dot{U}_i(G_i^p(Q_i, Q_i, P^U)))(p^L - p) < 0. \end{aligned}$$

(iii) Let $q \le -c_i$. Then

$$\begin{aligned} & G_i^p(q, Q_i, P^U)(\omega) \\ = \;& Q_i(\omega)(p_i^R - P^U(\omega)) \\ - \;& f_i + c_i(P^U(\omega) - p_i^C)1_{\{P^U(\omega) > p_i^C\}} \;+\; q(P^U(\omega) - p)\,. \end{aligned} \tag{7}$$

Thus, the random variables

$$(\dot{U}_i(G_i^p(q, Q_i, P^U)(\omega) \wedge (\dot{U}_i(\infty) + 1))1_{\{P^U(\omega) > p\}}$$

and

$$(\dot{U}_i(G_i^p(q, Q_i, P^U)(\omega) \vee \dot{U}_i(\infty))1_{\{P^U(\omega) < p\}}$$

tend as $q \to -\infty$ for any $\omega \in \Omega$ to $(\dot{U}_i(-\infty) \wedge (\dot{U}_i(\infty) + 1))1_{\{P^U(\omega) > p\}}$, and to $\dot{U}_i(\infty)1_{\{P^U(\omega) < p\}}$, respectively. Here we used the conventions $\dot{U}_i(\infty) := \lim_{x\to\infty} \dot{U}_i(x)$, $\dot{U}_i(-\infty) := \lim_{x\to-\infty} \dot{U}_i(x) \in]0, \infty]$, and $\min\{\infty, x\} = x$, $x \in \mathbb{R}$. We get

$$\begin{aligned} & E\big(\dot{U}_i(G_i^p(q, Q_i, P^U))(P^U - p)\big) \\ \ge \;& E\big(\big(\dot{U}_i(G_i^p(q, Q_i, P^U)) \wedge (\dot{U}_i(\infty) + 1)\big)(P^U - p)1_{\{P^U(\omega) > p\}}\big) \\ + \;& E\big(\big(\dot{U}_i(G_i^p(q, Q_i, P^U)) \vee \dot{U}_i(\infty)\big)(P^U - p)1_{\{P^U(\omega) < p\}}\big)\,. \end{aligned}$$

The right hand side of this estimate converges as $q \to -\infty$ dominated by $\max\{(\dot{U}_i(\infty)+1)|p^U - p|,\ p^U\big(\dot{U}_i(\infty) \vee E(Q_i(p_i^R - p^U) - f_i - c_i(p^U - p_i^C))\big)\}$ (see (4)) to

$$\begin{aligned} & \big(\dot{U}_i(-\infty) \wedge (\dot{U}_i(\infty) + 1)\big)E\big((P^U - p)1_{\{P^U(\omega) > p\}}\big) \\ + \;& \dot{U}_i(\infty)E\big((P^U - p)1_{\{P^U(\omega) < p\}}\big) \;\ge\; \dot{U}_i(\infty)(E(P^U) - p). \end{aligned}$$

If $\mathrm{P}(P^U > p) > 0$, then equality in the above estimates holds exactly if $\dot{U}$ =constant. If $\dot{U}$ =constant> 0, then the above limit is equal to 0 exactly if $p = E(P^U)$, and it is easy to check that also $h_i^{E(P^U)}(-c_i) = 0$, and thus, for such an agent any position $q \in]-\infty, -c_i]$ is optimal, if the forward price is $p = E(P^U)$. Further, we get with (i) that $h_i^{p\,\prime}(-c_i) > 0$, for all $p^L < p < E(P^U)$.

For any other utility function the limit is positive if $p \le E(P^U)$, and thus, there exists $q_0(p)$ such that with (ii) of Lemma 1 $h_i^{p\,\prime}(q) > 0$ for $q < q_0(p)$.

If $\dot{U}$ is not constant everywhere, then there exists an interval $]a, b[\subset \mathbb{R}$, $a < b$, where $\dot{U}$ is strictly decreasing. Let $q < \tilde{q} < -c_i$, and $p \in [p^L, p^U[$. By the assumptions (A1) through (A4) we get with (7) that the set

$$D_{[a,b]}^{p,q} = \{\, P^U > p \,\} \;\cap\; \{\, G_i^p(q, Q_i, P^U) \in]a, b[\,\}$$

has positive probability, and for any $\omega \in D^{p,q}_{[a,b]}$ we have

$$H_i^p(q)(\omega) > H_i^p(\tilde{q})(\omega).$$

With (i) we have

$$H_i^p(q) - H_i^p(\tilde{q}) \;\geq\; (H_i^p(q) - H_i^p(\tilde{q}))1_{D^{p,q}_{[a,b]}}.$$

Taking expectations we get the result.

(iv) From (5) it is easy to see that $H_i^{p^L}(q)(\omega) \geq 0$ for all ω with $Q_i(\omega) \leq q$, and thus

$$H_i^{p^L}(q) \;\geq\; \dot{U}(p_i^R Q_i)(-p^L + (p_i^C \wedge P^U)1_{\{q<Q_i\}}).$$

We get with (1), (2), (A2), and an arbitrary $a > 0$

$$\begin{aligned}
& h_i^{p^L}{}'(q) \\
\geq\; & E\big(\dot{U}(p_i^R Q_i)((p_i^C \wedge P^U) - p^L)1_{\{q<Q_i\}}\big) \\
\geq\; & \dot{U}(p_i^R(q^+ + a))E\big((p_i^C \wedge P^U) - p^L)1_{\{q<Q_i<q^++a\}}1_{\{P^U>(p_0^U+p^U)/2\}}\big) \\
\geq\; & \dot{U}(p_i^R(q^+ + a))((p_i^C \wedge \frac{p_0^U + p^U}{2}) - p^L) \\
& \quad \times \mathrm{P}(q < Q_i < q^+ + a \,|P^U > \frac{p_0^U + p^U}{2})\mathrm{P}(P^U > \frac{p_0^U + p^U}{2}) \\
>\; & 0.
\end{aligned}$$

From (5) we get with (2)

$$\begin{aligned}
H_i^{p^U}(q) \;&\leq\; \dot{U}(G_i^{p^U}(q, Q_i, P^U))((P^U \vee p_i^C) - p^U) \\
&\leq\; \dot{U}(p_i^R Q_i)((P^U \vee p_i^C) - p^U).
\end{aligned}$$

Taking expectation we get for $a > 0$ with (1), (2), and (A2)

$$\begin{aligned}
& h_i^{p^U}{}'(q) \\
\leq\; & E\big(\dot{U}(p_i^R Q_i)((p_i^C \vee P^U) - p^U)1_{\{Q_i<a\}}1_{\{p_i^C<P^U<(p_i^C+p^U)/2\}}\big) \\
\leq\; & -\frac{1}{2}\dot{U}(p_i^R a)\;(p^U - p_i^C)\;\mathrm{P}(Q_i < a \,|p_i^C < P^U < \frac{p_i^C + p^U}{2}) \\
& \times \mathrm{P}(p_i^C < P^U < \frac{p_i^C + p^U}{2}) \;<\; 0.
\end{aligned}$$

(v) Let $p \in [p^L, p^U]$ and $q \in \mathbb{R}$. If $(p_n)_{n\in\mathbb{N}} \subset [p^L, p^U]$ converges to p, then $(H_i^{p_n}(q))_{n\in\mathbb{N}}$ converges almost sure to $H_i^p(q)$ dominated by $\dot{U}_i(-p^U Q_i - f_i - \sup_{n\in\mathbb{N}} p_n q - p_i^C c_i)p^U$, implying that $h_i^{p_n}{}'(q) = E(H_i^{p_n}(q)) \to E(H_i^p(q)) = h_i^p{}'(q)$ for $n \to \infty$. □

Now we are able to prove the Theorem 1.

Proof. i) By (*i*) and (*iv*) of Lemma 2 we have that for any $i \in \{1, \dots, I\}$ there is a unique $\hat{p}_i^0 \in]p^L, p^U[$ such that $h_i^{\hat{p}_i^0}{}'(-c_i) = 0$, and $h_i^{\hat{p}_i^0}{}'(p) > 0$ for all $p \in [p^L, \hat{p}_i^0[$. Denote $p_i^0 = \max\{E(P^U), \hat{p}_i^0\}$.

For $i \in \{1, \dots, I\}$ and $p \in]p^L, p_i^0[$ we use the intermediate value theorem to conclude from (ii) of Lemma 1 and (ii) and (iii) of Lemma 2 that there exists a unique solution $q_i^*(p) \in \mathbb{R}$ of $h_i^p{}' = 0$.

In view of (i) from Lemma 1, the maximum of h_i^p is reached at that point $q_i^*(p)$. Obviously, $q_i^*(p)$ is an optimal forward position of the i–th agent, given the forward price p.

ii) Let us show that q_i^* depends continuously on $p \in]p^L, p_i^0[$.

Given a sequence $(p_n)_{n \in \mathbb{N}} \subset]p^L, E(P^U)[$ converging to $p \in]p^L, E(P^U)[$ and $\varepsilon > 0$, we conclude from the strict monotonic decrease of $h_i^p{}'$ on $\mathbb{R}$ that $h_i^p{}'(q_i^*(p) - \varepsilon) > 0 > h_i^p{}'(q_i^*(p) + \varepsilon)$. From (v) of Lemma 2 we see that there exists $K \in \mathbb{N}$ such that $h_i^{p_n}{}'(q_i^*(p) - \varepsilon) > 0$ and $h_i^{p_n}{}'(q_i^*(p) + \varepsilon) < 0$ for all $n \geq K$. The continuity of $h_i^{p_n}{}'$ implies that $q_i^*(p_n) \in]q_i^*(p) - \varepsilon, q_i^*(p) + \varepsilon[$ for all $n \geq K$. Hence $\lim_{n \to \infty} q_i^*(p_n) = q_i^*(p)$. This implies the continuity of q_i^* on $]p^L, E(P^U)[$.

If $p_i^0 > E(P^U)$ then $q_i^*(p) > -c_i$ for $p \in]p^L, p_i^0[$, and the same arguments as above show the continuity on $]p^L, p_i^0[$.

iii) If $\dot{U}$ is not a constant function then, by (ii) of Lemma 1 and (ii) and (iii) of Lemma 2, there exists a unique zero $q_i^*(E(P^U)) \in \mathbb{R}$ of $h_i^{E(P^U)}{}'$, and by the same arguments used in (ii) it can be shown that

$$\lim_{p \uparrow E(P^U)} q_i^*(p) \ = \ q_i^*(E(P^U)).$$

If $\dot{U}$ is a constant function then we know from (ii) of Lemma 1 and (i) and (iii) of Lemma 2 that

$$\begin{aligned} h_i^{E(P^U)}{}'(q) = 0 \quad &\text{if } q \leq -c_i, \quad h_i^{E(P^U)}{}'(q) < 0 \quad \text{if } \ q > -c_i \\ h_i^p{}'(-c_i) > 0 \quad &\text{if } p < E(P^U), \quad h_i^p{}'(-c_i) < 0 \quad \text{if } \ p > E(P^U). \end{aligned}$$

Again, it follows with (v) of Lemma 2 that $q_i^*(p) \downarrow -c_i$ if $p \uparrow E(P^U)$.

(iv) From (iv) and *(v)* of Lemma 2 it follows that $\lim_{p \downarrow p^L} h_i^p{}'(q) > 0$ for all $q \in \mathbb{R}$, and, thus, $q_i^*(p) > 0$ for small enough $p > p^L$. This implies that $\sum_{i=1}^I q_i^*(p) > 0$ for $p > p^L$ small enough.

By (ii) and (iii) we get that $p \mapsto \sum_{i=1}^I q_i^*(p)$ is a continuous function on $]p^L, E(P^U)]$ having either a zero $p^* \in]p^L, E(P^U)]$ or, if not, then a finite limit $\hat{q} > 0$ exists as $p \uparrow E(P^U)$.

In the first case p^* is an equilibrium forward price to the optimal forward positions $q_i^*(p^*)$ of the market's participants $i = 1, \dots, I$.

In the second case $p^* = E(P^U)$ is an equilibrium forward price, if there is an agent with linear utility function. Then the missing amount $\hat{q} > 0$ of forwards to get the trade in zero net supply will be sold by participants with linear utility function. If there are more than one such participants the extra amount sold by each of them is not unique. However, by (iii) of Lemma 2 any solution leads to an optimal forward position for all participants.

(v) It remains to consider the case that there are no agents with linear utility function, and there is no $p \in]p^L, E(P^U)]$ with $\sum_{i=1}^I q_i^*(p) = 0$. Let us assume this for the rest of the proof. In this case $\sum_{i=1}^I q_i^*(E(P^U)) > 0$. As above, one can show that

$$q_i^*(p) \downarrow q_i^*(\hat{p}_i^0) = -c_i \text{ as } p \uparrow \hat{p}_i^0\,, \text{ and } q_i^*(p) \uparrow q_i^*(\hat{p}_i^0) = -c_i \text{ as } p \downarrow \hat{p}_i^0.$$

By (ii) of Lemma 1 and (v) of Lemma 2 the function

$$(p, q) \mapsto h_i^{p\,\prime}(q)$$

is continuous on $]p^L, p^U[\, \times \mathbb{R}$. For $q \leq -c_i$ it is strictly increasing in p (see (i) of Lemma 2), and thus there exists a unique $p_i^*(q) \in\,]\hat{p}_i^0, p^U[$ with $h_i^{p_i^*(q)\,\prime}(q) = 0$. Evidently $q_i^*(p_i^*(q)) = q$. By (i) and (iii) of Lemma 2 the function $(p, q) \mapsto h_i^{p\,\prime}(q)$ is strictly decreasing in p and in q on $]\hat{p}_i^0, p^U[\, \times\,]-\infty, -c_i[$. Thus, $p_i^*(q)$ is continuous and strictly decreasing. Let

$$p_i^\infty = \lim_{q \to -\infty} p_i^*(q), \quad p^\infty = \min\{p_i^\infty,\ i = 1, \dots, I\}.$$

We get that the function $p \mapsto \sum_{i=1}^I q_i^*(p)$ is continuous on $]p^L, p^\infty[$ and

$$\lim_{p \uparrow p^\infty} \sum_{i=1}^I q_i^*(p) \;=\; -\infty.$$

Thus, there exists $p^* \in]E(P^U), p^\infty[$ with $\sum_{i=1}^I q_i^*(p^*) = 0$. □

Corollary 1. *From the proof of the Theorem 1 it follows that* p^* *and* $(q_1^*(p^*), \dots, q_I^*(p^*))$ *are obtained as a solution of*

$$h_i^{p\,\prime}(q_i) = 0\,,$$

for all $i = 1, \dots, I$ *subject to* $\sum_{i=1}^I q_i = 0$, *where*

$$h_i^{p\,\prime}(q_i) \;=\; E(\dot{U}_i(G_i^p(q_i, Q_i, P^U))).$$

If there is an agent with linear utility function, and there exists no solution with $p^* \in]p^L, E(P^U)[$*, then an equilibrium price is* $p^* = E(P^U)$*. In this case the* $q_i^*(p^*)$ *are to obtain as follows: If* U_i *is not linear, then* $q_i^*(p^*)$ *is the unique solution of* $h_i^{E(P^U)}(q_i) = 0$*. If* U_i *is linear, then* $q_i^*(p^*) \leq -c_i$ *such that* $\sum_{i=1}^{I} q_i^*(p^*) = 0$*.*

Remark 1. The problem of uniqueness of the equilibrium can not be answered in the general setting considered here. It will depend on the choice of the utility functions. For example we can give the following corollary.

Corollary 2. *If all* $U_1, \ldots, U_I$ *are linear functions, then the equilibrium is unique, and* $p^* < E(P^U)$*.*

Proof. Let $\dot{U}_i$ =constant. Then it is easy to see from (5) that for any $q \in]-c_i, \infty[$ and $\omega \in \Omega$ the expression $H_i^p(q)(\omega)$ is strictly increasing in p and thus, the same holds for $h_i^{p\,\prime}(q)$. From the proof of Theorem 1 it follows that for any $q \in]-c_i, \infty[$ there exists a unique $p \in]p^L, E(P^U)[$ with $h_i^{p\,\prime}(q) = 0$. This implies that the function $p \mapsto q^*(p)$ is strictly decreasing for $p \in]p^L, E(P^U)]$ with $q_i^*(p^L) = \infty$ and $q_i^*(E(P^U)) = -c_i < 0$. This holds for all $i = 1, \ldots, I$ and consequently for $\sum_{i=1} q_i^*(p)$. This implies the statement of the corollary. See also Example 2 and Example 3 below. □

4. Examples and Discussion

From Corollary 1 and the proof of Theorem 1 it is easy to see that the solutions $q_i^*(p)$of the equations $h_i^{p\,\prime}(q_i) = 0$ are the optimal forward traded volumes of the agents if the forward price is $p \in]p^L, E(P^U)[$. Then, p^*, the forward price for the market in equilibrium, solves $\sum_{i=1}^{I} q_i^*(p^*) = 0$.

Example 1. Suppose the i–th agent is small, meaning that the influence of his forward trade to the forward price $p^* \in]p^L, E(P^U)[$ is negligible. Then his optimal forward position q_i^* is approximately given by the unique solution of $h_i^{p^*\,\prime}(q_i) = 0$.

Example 2. If the utility function U_i is given by $U_i(x) = a_i x + b_i$, $a_i, b_i \in \mathbb{R}$, $a_i > 0$, then $h_i^{p\,\prime}(q_i) = 0$ is equivalent to

$$\begin{aligned} p &= p^L \mathrm{P}(q \geq Q_i) \\ &\quad + E\big[\, P^U (1_{\{q<Q_i\}} 1_{\{P^U \leq p_i^C\}} + 1_{\{q<Q_i - c_i\}} 1_{\{P^U > p_i^C\}})\,\big] \\ &\quad + p_i^C \mathrm{P}(Q_i - c_i \leq q < Q_i,\, P^U > p_i^C) \\ &= p_i^C \mathrm{P}(Q_i > q_i) \;+\; p^L \mathrm{P}(Q_i \leq q_i) \\ &\quad +\; E((P^U - p_i^C)^+ \,|\, q_i + c_i < Q_i) \mathrm{P}(q_i + c_i < Q_i) \\ &\quad -\; E((p_i^C - P^U)^+ \,|\, q_i < Q_i) \mathrm{P}(q_i < Q_i). \end{aligned}$$

Thus, for such a market's participant the forward price p appears as a sum of four terms depending on his optimal forward position at this price, meaning that p is equal to his variable production costs p_i^C times the probability that own production will happen, plus a possible gain from selling to the lower penalty price an extra amount of power exceeding his demand, plus a surcharge for the risk that he must purchase power at an upper penalty price that is bigger then p_i^C reduced by possible gain from purchasing power at an upper penalty price that is less then p_i^C.

If all utility functions U_i, $i = 1, \ldots, I$ are linear, then the equilibrium is unique (Corollary 2), and it is the solution of the system

$$\begin{aligned} p &= p_i^C \;-\; (p_i^C - p^L) \mathrm{P}(Q_i \leq q_i) \\ &\quad -\; E((p_i^C - P^U)^+ \,|\, q_i < Q_i) \mathrm{P}(q_i < Q_i) \\ &\quad +\; E((P^U - p_i^C)^+ \,|\, q_i + c_i < Q_i) \mathrm{P}(q_i + c_i < Q_i), \\ 0 &= q_1 + \;\ldots\; + q_I\,, \quad i = 1, \ldots, I, . \end{aligned}$$

Example 3. Consider a market consisting of I identical agents:

$p_i^C = p_j^C$, $f_i = f_j$, $c_i = c_j$, $U_i = U_j$, (P^U, Q_i) and (P^U, Q_j) have the same distribution for all $i, j = 1, \ldots, I$.

Clearly, $q_i^*(p^*) = 0$ for all $i = 1, \ldots, I$, and the equilibrium price p^* is calculated from $h_i^{p^*\,\prime}(0) = 0$. For example, if utility functions satisfy $U_i(x) = a_i x + b_i$, $a_i, b_i \in \mathbb{R}$, $a_i > 0$, for all $x \in \mathbb{R}$, $i = 1, \ldots, I$, then (compare with Example 2):

$$\begin{aligned} p^* &= p_1^C \;-\; E((p_1^C - P^U)^+) \\ &\quad + E((P^U - p_1^C)^+ \,|\, c_1 < Q_1) \mathrm{P}(c_1 < Q_1)\,. \end{aligned}$$

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