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NEW SOLITON-LIKE SOLUTION OF A (3+1)-DIMENSIONAL NON-INTEGRABLE EQUATION

Jian-qin Mei[1 §], Hong-qing Zhang[2]
[1,2]Department of Applied Mathematics
Dalian University of Technology
Dalian 116024, P.R. CHINA
[1]e-mail: meijqin408@yahoo.com.cn
[2]e-mail: zhanghq@dlut.edu.cn

Abstract: In this article, a new generally projective Riccati equation method is presented and applied to explore exact solutions of a (3+1)-dimensional not completely integrable equation. As a result, many exact solutions, including new solitary wave solutions and periodic solutions, are obtained.

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1. Introduction

In the study of nonlinear science, soliton theory plays a very important roles and has been applied in almost all natural sciences, such as physics, mechanics, biology, chemistry, etc. Many scientists have paid attention to seeking for soliton solutions of nonlinear evolutionary equations (NLEEs) and obtained lots of significant results by using all kinds of methods [1], [5], [6], [3], [7]. A lot of NLEEs in (1+1)-dimensions and (2+1)-dimensions have been broadly researched and it is much more difficult to find some exact physical significant soliton solutions in (3+1)-dimensions than in (1+1)-dimensions.

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§Correspondence author

In this paper, we consider the following (3+1)-dimensional equation:

$$u_{yt} - u_{xz} + u_{xxxy} - 3u_x u_{xy} - 3u_y u_{xx} = 0\,, \tag{1}$$

which was introduced by Jimbo and Miwa [4] as the second equation in so-called Kadomtsev-Petviashvili hierarchy of equations and equation (1) is not completely integrable in the usual sense (see [2]). In this paper, we introduce a new method (called new generally projective Riccati equation expansion method) and use our method to explore some new travelling-like solutions for equation (1).

2. A New Generally Projective Riccati Equation Expansion Method

Our method is summed up as follows:

For a given NLEE with independent variables, $x = (t, x_1, x_2, ..., x_m)$, and dependent variable u, We look for its solution in form:

$$u = \sum_{i=1}^{n} \sigma^{i-1}(\omega(x))[a_i(x)\sigma(\omega(x)) + b_i(x)\tau(\omega(x))] + a_0(x)\,, \tag{2}$$

which satisfies:

$$\sigma^{'} = k\tau^2, \quad \tau^{'} = ek\sigma\tau - \frac{l}{k}\tau(1 - l\sigma)\,, \tag{3}$$

where $' = \frac{d}{d\omega}$, $e = \pm 1$, r, j are constants and $r \neq 0$. And it is easy to see that (5) admits the first integral with $r \neq 0$:

$$\tau^2 = \frac{1}{k^2}(1 - l\sigma)^2 + e\sigma^2\,. \tag{4}$$

To determine u explicitly, we take the following four steps:

Step 1. Determining the parameter n above by balancing the highest-order linear term with nonlinear term in the given NLEE.

Step 2. Substituting system (4) along with conditions (5), (6) into the given NLEE and collecting the coefficients of $\sigma^i\tau^j (j = 0, 1, i = 0, 1, 2, 3, ...)$, then setting their coefficients to zero to derive a set of partial differential equations on the unknown functions a_i, $b_i (i = 1, 2, ..., n)$, a_0 and unknown constants r, j.

Step 3. Solving the system partial differential equations obtained in Step 2.

Step 4. We know that (5) yields to the following solutions:

Case 1. When $e = -1$,

$$\sigma_1 = \frac{\tanh(\omega)}{k + l\tanh(\omega)}, \quad \tau_1 = \frac{sech(\omega)}{k + l\tanh(\omega)}. \tag{5}$$

Case 2. When $e = 1$,

$$\sigma_1 = \frac{\tan\omega)}{k + l\tan(\omega)}, \quad \tau_1 = \frac{\sec(\omega)}{k + l\tan(\omega)}. \tag{6}$$

Substituting a_i, b_i, r, j obtained in Step 3 along with (5)(6) into (2) to obtain the soliton-like, periodic-like solutions for the NLEE in concern.

Remark. In order to explore the explicit solutions of the partial differential equations derived in Step 2, we may choose special forms of a_i, b_i and ω (as we do in Section 3).

3. Applications: New Explicit Solutions for Equation (1)

In order to seek solutions for equation (1), according to the step 1 above, first we can get $n = 1$, then we can suppose that:

$$u = f(t,y,z) + g(t,y,z)\sigma(\alpha x + q(t,y,z)) + h(t,y,z)\tau(\alpha x + q(t,y,z)), \tag{7}$$

where σ and τ satisfy (5), (6).

With the aid of maple, substituting (7) along with (3), (4) into the equation (1) and collecting coefficients of polynomial of $\sigma^i\tau^j$ ($j = 0,1, i = 0,1,2,4,5$), and setting each coefficient to zero, we can derive a set of partial differential equations on unknown functions f, g, h, q and unknown constants k, l. Using the powerful package of *Maple*, solving this system, we can derive the following nontrivial solutions whether $e = -1$ or $e = 1$:

Case 1. $l = l$, $q(y,t,z) = F_2(t) + F_1(z)$, $f(y,t,z) = -\frac{\left(\frac{d}{dz}F_1(z)\right)y}{3\alpha} + F_3(t,z)$, $g(y,t,z) = \frac{2\alpha\left(l^2-k^2\right)}{3k}$, $h(y,t,z) = 0$.

Case 2. $l = l$, $q(y,t,z) = F_2(t) + F_1(z)$, $f(y,t,z) = -\frac{\left(\frac{d}{dz}F_1(z)\right)y}{3\alpha} + F_3(t,z)$, $g(y,t,z) = \frac{\alpha\left(l^2-k^2\right)}{18k}$, $h(y,t,z) = \frac{\alpha}{18}\sqrt{22\,l^2 - 22\,k^2}$.

Case 3. $g(y,t,z) = 0$, $q(y,t,z) = q(y,t,z)$, $l = \pm k$, $f(y,t,z) = F_3(t,z) \pm \frac{\left(\frac{d}{dz}F_2(z)\right)y}{3F_2(z)\alpha}$, $h(y,t,z) = F_1(t)\,F_2(z)\,e^{-q(y,t,z)}$.

Case 4. $q(y,t,z) = q(y,t,z)$, $l = \pm k$,

$$g(y,t,z) = F_1(t)\,F_2(z)\,e^{-2\,q(y,t,z)},$$

$$f(y,t,z) = F_3(t,z) \mp \left(\frac{F_1(t)\,F_2(z)\,e^{-2\,q(y,t,z)}}{2k} - \frac{\left(\frac{d}{dz}F_2(z)\right)y}{6F_2(z)\,\alpha}\right),$$

$h(y,t,z) = F_4(t)\sqrt{F_2(z)}e^{-q(y,t,z)}$.

According to these results along with (5), we can obtain the following soliton-like solutions for equation (1):

$$u_1 = \frac{2\alpha\left(l^2-k^2\right)\tanh(\omega)}{3k(k+l\tanh(\omega))} - \frac{\left(\frac{d}{dz}F_1(z)\right)y}{3\alpha} + F_3(t,z)\ , \tag{8}$$

where $\omega = \alpha\,x + F_2(t) + F_1(z)$.

$$u_2 = \frac{\alpha[\left(l^2-k^2\right)\tanh(\omega) + k\sqrt{22\,l^2-22\,k^2}sech(\omega)]}{18k(k+l\tanh(\omega))} - \frac{\left(\frac{d}{dz}F_1(z)\right)y}{3\alpha} + F_3(t,z)\ , \tag{9}$$

where $\omega = \alpha\,x + F_2(t) + F_1(z)$.

$$u_3 = \frac{F_1(t)\,F_2(z)\,e^{-q(y,t,z)}sech(\alpha\,x+q(t,y,z))}{k \pm k\tanh(\alpha\,x+q(t,y,z))} + F_3(t,z) \pm \frac{\left(\frac{d}{dz}F_2(z)\right)y}{3F_2(z)\,\alpha} \tag{10}$$

$$u_4 = \frac{F_1(t)\,F_2(z)\,e^{-2\,q(y,t,z)}\tanh(\alpha\,x+q(t,y,z))}{k \pm k\tanh(\alpha x+q(t,y,z))} + \frac{F_4(t)\,F_2(z)\,e^{-q(y,t,z)}sech(\alpha\,x+q(t,y,z))}{k \pm k\tanh(\alpha\,x+q(t,y,z))} + F_3(t,z) \mp \left(\frac{F_1(t)\,F_2(z)\,e^{-2\,q(y,t,z)}}{2k} - \frac{\left(\frac{d}{dz}F_2(z)\right)y}{6F_2(z)\,\alpha}\right). \tag{11}$$

And according to (6), we can derive some periodic-like solutions for equation (1):

$$u_5 = \frac{2\alpha\left(l^2-k^2\right)\tan(\omega)}{3k(k+l\tan(\omega))} - \frac{\left(\frac{d}{dz}F_1(z)\right)y}{3\alpha} + F_3(t,z)\ , \tag{12}$$

$$u_6 = \frac{\alpha[\left(l^2-k^2\right)\tan(\omega) + k\sqrt{22\,l^2-22\,k^2}\sec(\omega)]}{18k(k+l\tan(\omega))} - \frac{\left(\frac{d}{dz}F_1(z)\right)y}{3\alpha} + F_3(t,z)\ , \tag{13}$$

where $\omega = \alpha\, x + F_2\,(t) + F_1\,(z)$.

$$u_7 = \frac{F_1\,(t)\,F_2\,(z)\,e^{-q(y,t,z)}\sec(\alpha\,x + q\,(t,y,z))}{k \pm k\tan(\alpha\,x + q\,(t,y,z))} + F_3\,(t,z) \pm \frac{\left(\frac{d}{dz}F_2\,(z)\right)y}{3F_2\,(z)\,\alpha}, \quad (14)$$

where $k \neq 0$ and $q\,(t,y,z)$ is an arbitrary analytic function.

$$u_8 = \frac{F_1\,(t)\,F_2\,(z)\,e^{-2\,q(y,t,z)}\tan(\alpha\,x + q\,(t,y,z))}{k \pm k\tan(\alpha x + q\,(t,y,z))} + \frac{F_4\,(t)\,F_2\,(z)\,e^{-q(y,t,z)}\sec(\alpha\,x + q\,(t,y,z))}{k \pm k\tan(\alpha\,x + q\,(t,y,z))} + F_3\,(t,z) \mp (\frac{F_1\,(t)\,F_2\,(z)\,e^{-2\,q(y,t,z)}}{2k} - \frac{\left(\frac{d}{dz}F_2\,(z)\right)y}{6F_2\,(z)\,\alpha}), \quad (15)$$

where $k \neq 0$ and $q\,(t,y,z)$ is an arbitrary analytic function.

The above constants α, k, l and the analytic functions F_1, F_2, F_3, F_4 not to be restricted are all arbitrary.

It is obvious to see that almost all these soliton-like solutions for equation (1)((8)-(15)) are quite new and have not appeared in those known literatures.

4. Conclusion

In summary, by using our newly generally projective Riccati equation expansion method, we have considered a (3+1)-dimensional KdV-type equation and derived many types of soliton-like solutions for it, most of which have not appeared in those known literatures. And with the aid of the powerful package of *Maple*, the course of solving PDE can be carried out in computer.

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