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COMPARISON BETWEEN DISTRIBUTIONS OF SUMS OF RANDOM VARIABLES AND ROBUSTNESS OF SOME APPLIED STOCHASTIC MODELS

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Abstract: The aim of the paper is to show that the probability metric approach to the comparison between distributions of two sums of random variables can be successfully applied in the analysis of stability (robustness) of many stochastic models with optimization applications. For this purpose we extend some known estimates of the rate of convergence in the central limit theorem to compare sums of random variables not necessarily Gaussian. Then we use the inequalities obtained to get new quantitative stability estimates in the following models: the S. Andersen risk process, renewal functions, optimization of the replacement period in the block-replacement model, optimization of the initial capital securing a prescribed risk in the classical risk model and a certain ingredient of a dam model with compound Poisson inputs.

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1. Introduction

The problem of robustness (stability) is of practical importance in applied models because of an almost inevitable uncertainty of the input data. As a consequence of such uncertainty one usually deals with some "approximating" model in place of a "real" one. Quantitative estimation of discrepancy in outputs of these two models in terms of given upper bounds for errors in input data is a matter under discussion in this paper. The detailed inquiry into these topics can be found, for instance, in the references: Beirlant et al [3], Borovkov [4], Gertsbakh [8], Gordienko et al [11], Kalashnikov [16], Kalashnikov [17], Prokhorov et al [22], Rachev [23] and Zolotarev [30] (uncontrolled stochastic models), and in Van Dijk et al [5], Van Dijk et al [6], Gordienko [10], Gordienko et al [12], Hilgert et al [14], Montes de Oca et al [19], Müller [20] and Whitt [29] (controlled stochastic models).

There are several stochastic models (both controlled or not) involving as an important integrant sums $Z = \sum_{k=1}^{\nu} X_k$ of independent and identically distributed (i.i.d.) random variables. The number of summands ν can be fixed or random (independent of $X_1, X_2, \dots$) in certain circumstances (see: Abdel-Hammed [1], Bae et al [2], Beirlant et al [3], Borokov [4], Feller [7], Gordienko et al [11], Grandell [13], Kass et al [15], Kalashnikov [16], Kalashnikov [17], Prokhorov et al [22], Rachev [23], Shorgin [27] and Tijms [28]). In such models "the input information" takes the form of a common distribution function F of random variables X_k, $k \geq 1$, where F could be known only approximately. For this reason a researcher has to relay on some known approximation G of the distribution function F. For instance G, could be obtained from statistical procedures or (and) theoretical simplifications. Thus, if we introduce a sequence of i.i.d. random variables $Y_1, Y_2, \dots$ with common distribution function G, then the available sum $\tilde{Z} = \sum_{k=1}^{\nu} Y_k$ is used instead of Z as the corresponding ingredient in the model.

With a view to the robustness problem inequalities of the type

$$\rho(Z, \tilde{Z}) \leq c[\mu(X_1, Y_1)]^{\gamma}, \qquad \gamma > 0\,, \tag{1.1}$$

are of interest. In (1.1) (and in what follows)

$$\rho(Z, \tilde{Z}) \equiv \rho(F_Z, F_{\tilde{Z}}) := \sup_{x \in \mathbb{R}} |F_Z(x) - F_{\tilde{Z}}(x)| \tag{1.2}$$

is the *uniform metric*, and $\mu(X_1, Y_1) \equiv \mu(F, G)$ is a suitable probability metric (see Kalashnikov [16], Rachev [23], Senatov [25] and Zolotarev [30] for definitions). Bounds (1.1) with $\mu = \rho$ and $\gamma = 1/2$ were obtained in Kalashnikov [16] for geometric sums Z and $\tilde{Z}$ under the condition

$$EX_1 = EY_1. \tag{1.3}$$

Certain bounds of $\rho(Z, \tilde{Z})$ in Beirlant et al [3] and Rachev [23] are given in terms of *the difference pseudomoments*

$$\mathbf{k}_r(X,Y) := r\int_{-\infty}^{\infty} |x|^{r-1}|F_X(x) - F_Y(x)|dx, \quad r > 0 \tag{1.4}$$

(see the discussion in Section 3.1). The paper Krajka et al [18] deals with comparisons of distributions of two random sums more general than those denoted above by Z. In the paper Gordienko [9] we derived particularly the following inequality

$$\rho(Z, \tilde{Z}) \le c \max\left\{\rho(X_1, Y_1), \frac{1}{2}\mathbf{k}_2(X_1, Y_1)\right\}, \tag{1.5}$$

which holds under (1.3) and certain restrictions on characteristic functions of X_1 and Y_1. We use (1.5) in this paper. Moreover, we prove and apply in the stability problem the inequality:

$$\rho(Z, \tilde{Z}) \le cE(\nu^{-1/2}) \max\left\{\rho(X_1, Y_1), \frac{1}{6}\mathbf{k}_3(X_1, Y_1)\right\}, \tag{1.6}$$

which extends the relevant estimate in the central limit theorem (see, Senatov [25] and Senatov [26]). Bound (1.6) is obtained under (1.3), $EX_1^2 = EY_1^2$ and the additional "smoothness" conditions imposed on the known distribution G of Y_1.

The primary goal of the paper is to show that bounds (1.5), (1.6) can provide reasonable stability estimates in several important applied stochastic models involving, particularly, parameter optimization procedures. The few examples that we have chosen are used to fix the idea. Other applications could arise from the robustness examination in queues, reliability or inventory models.

First, we apply (1.6) to get a new robustness estimate of a surplus of the insurance company in the S. Andersen risk model. Then making use of bounds (1.5), we estimate the stability of optimization of the initial capital that guarantees a prescribed level of ruin probability in the classical risk model.

Second, we give two versions of stability bounds for renewal functions associated with bounded random variables. In turn, we use these bounds to evaluate the so called "stability index" in the problem of planned replacement times optimization in the reliability block-replacement model. Finally, by means of the same bounds we test the stability of the average length of time interval with a zero release rate in the dam model with compound Poisson inputs.

2. The Proximity Inequalities for Distributions of Sums of Random Variables

Let $\{X_1, X_2, \dots\}$, $\{Y_1, Y_2, \dots\}$ be two sequences of i.i.d. random variables (r.v.'s.) with X and Y standing for generic of $X_1, X_2, \dots$ and of $Y_1, Y_2, \dots$, respectively, and let ν be a random variable taking natural values, independent of both $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$. We are going to use the notation:

(i) $Z = \sum_{k=1}^{\nu} X_k, \quad \tilde{Z} = \sum_{k=1}^{\nu} Y_k$;

(ii) $S_n = \sum_{k=0}^{n} X_k, \quad \tilde{S}_n = \sum_{k=0}^{n} Y_k$ (here $n = 0, 1, 2, \dots; X_0 = Y_0 = 0$);

(iii) $F = F_X$ and $G = F_Y$ are the distribution functions of X and Y, respectively;

(iv) $\mathbf{V}(X,Y) := 2 \sup\{|P(X \in B) - P(Y \in B)| : B$ is a Borel subset of $\mathbb{R}\}$;

(v) *the uniform metric* ρ and *the difference pseudomoment* $\mathbf{k}_r$ (of order r) were defined in (1.2) and (1.4), respectively;

(vi) the metric μ is defined as follow:

$$\mu(X,Y) = \max\{\rho(X,Y), \frac{1}{6}\mathbf{k}_3(X,Y)\}. \tag{2.1}$$

The next assumption refers to the properties of the "known" distribution G of the r.v. Y. Let $s \geq 1$ be an integer fixed in what follows.

Assumption 1. The random variable $Y_1 + Y_2 + \dots + Y_s$ possesses a bounded density p. The density p has two continuous and bounded derivatives on $\mathbb{R}$ and a bounded third derivative $p^{(3)}$ almost everywhere on $\mathbb{R}$ and, for some $\alpha > 0$,

$$\int_{|x| > n\alpha} |p^{(3)}(x)| dx = O(n^{-3/2}), \quad \text{as } n \to \infty. \tag{2.2}$$

Remark 1. One might need to use $s > 1$ in Assumption 1 in order not to take out of consideration such important (in applications to queuing or risk theory) densities as the uniform density, the exponential one, some Gamma densities, etc.

Theorem 1. *Suppose that* $EX = EY$, $\mathrm{Var}(X) = \mathrm{Var}(Y) =: \sigma^2 > 0$, $E|X|^3$, $E|Y|^3 < \infty$, *and Assumption 1 holds. There exist constants* $c, \overline{c}, \overline{\overline{c}}$ *(depending on* s, σ *and* G*) such that, if*

$$\overline{c}\boldsymbol{V}(X,Y) + \overline{\overline{c}}\mu(X,Y) \le 1/3, \tag{2.3}$$

then

$$\rho(S_n, \tilde{S}_n) \le c\mu(X,Y)n^{-1/2}, \qquad n = 1, 2, \dots, \tag{2.4}$$

where μ *is defined in (2.1).*

Corollary 1. *Under the condition of Theorem 1*

$$\rho(Z, \tilde{Z}) \le c\mu(X,Y)E(\nu^{-1/2}). \tag{2.5}$$

The proof of Theorem 1 is outlined in Section 4.

Remark 2. Inequalities (2.4) extend the known estimates of the rate of convergence in the central limit theorem. The latter is the case of normally distributed r.v.'s $Y_1, Y_2, \dots$, see Senatov [25] and Senatov [26]. On the other hand, these inequalities can be considered as a kind of "a non central limit theorem". Indeed, suppose we need to extract some information on the distribution of the sum $S_n = X_1 + X_2 + \cdots + X_n$ when a distribution F of X_1 is uncertain, but it is known that F is "close" to the exponential distribution G. Why do we have to approximate the distribution of S_n by means of the normal law (CLT) in such situation, whereas the readily calculated distribution of $Y_1 + Y_2 + \cdots + Y_n$ (with Y_i being exponential) can provide a considerably better approximation with the accuracy controlled by (2.4)?

The sharpness and, so, the applicability of inequalities (2.4) and (2.5) depend on numerical values of the constants $c, \overline{c}$ and $\overline{\overline{c}}$. These constants are entirely determined by variance σ^2 and certain attributes of the known distribution function G. It follows from the proof of Theorem 1 that

$$c = \max\left\{(2s-1)^{3/2}, \frac{3}{2}\frac{b}{\sigma^3}\left(\sqrt{2} + c_1(s)\right)\right\}, \tag{2.6}$$

$$\overline{c} = c_3(s), \quad \overline{\overline{c}} = \frac{d}{\sigma^3}c_2(s), \tag{2.7}$$

where the following "absolute" constants $c_1(s), c_2(s)$ and $c_3(s)$ are easily calculated by computer (below $[x]$ stands for the integer part of x).

$$c_1(s) = \sup_{n \ge 2s} \frac{\left([\frac{n}{2}] + 1\right)}{(n-1)^{3/2}}\, n^{1/2}; \tag{2.8}$$

$$c_2(s) = \begin{cases} \sup\limits_{n \geq 2s} \sum\limits_{j=s}^{[n/2]} \left(\dfrac{n}{n-j-1} \right)^{1/2} j^{-3/2}, \text{ if } s > 1, \\ \sup\limits_{n \geq 2s+1} \sum\limits_{j=s}^{[n/2]} \left(\dfrac{n}{n-j-1} \right)^{1/2} j^{-3/2}, \text{ if } s = 1; \end{cases} \tag{2.9}$$

$$c_3(s) = \sup_{n \geq 2s} \sum_{j=0}^{s-1} \left(\frac{n}{n-j-1} \right)^{1/2}. \tag{2.10}$$

To write the expressions for b and d in (2.6) and (2.7) we denote by f_k the density of the r.v. $\dfrac{Y_1 + \cdots + Y_k}{\sigma\sqrt{k}}$ $(k \geq s)$. Then

$$b := \sup_{k \geq s} \sup_{x \in \mathbb{R}} |f_k''(x)|, \tag{2.11}$$

$$d := \sup_{k \geq s} \int_{-\infty}^{\infty} |f_k^{(3)}(x)| dx. \tag{2.12}$$

We shall see that Assumption 1 guarantees finiteness of b and d. For some important particular classes of distributions G we can give tight upper bonds for the constants b and d in (2.11) and (2.12). To do this we make use of the following circumstances:

(i) In several cases of interest there are analytical formulas for the densities f_k.

(ii) The quantities under the "sup" sign in (2.11) and (2.12) can be evaluated by computer for a few values of k $(k = s, k = s+1, \dots)$.

(iii) These quantities

$$\sup_{x \in \mathbb{R}} |f_k''(x)| \text{ and } \int_{-\infty}^{\infty} \left| f_k^{(3)}(x) \right| dx$$

converge (rather fast) to the corresponding quantities for the standard normal distribution.

The results of calculations are summarized in the below tables (s indicates an integer for which Assumption 1 is satisfied). The values (bounds) of "c-constants" in Tables 1–4 are calculated for $\sigma = 1$ (using (2.6) and (2.7)). To illustrate the dependence of these constants on $\sigma^2 = \mathrm{Var}(X)$ we write their bounds for $\sigma = 5$ in Table 5.

s	b	d	c	$\overline{c}$	$\overline{\overline{c}}$
1	.3990	1.5101	2.5392	1.4143	3.9447

Table 1: The distribution G is normal

s	b	d	c	$\overline{c}$	$\overline{\overline{c}}$
4	1.3455	4.4778	18.53	4.9029	4.7722

Table 2: The distribution G is exponential

s	b	d	c	$\overline{c}$	$\overline{\overline{c}}$
4	.4010	2.3095	18.53	4.9029	2.4613

Table 3: The distribution G is uniform

s	b	d	c	$\overline{c}$	$\overline{\overline{c}}$
1	1.3455	4.4778	8.5628	1.4143	11.6968
2	.5327	2.3128	5.1962	2.5690	3.7463

Table 4: The distribution G has the gamma density: $\dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $x \geq 0$ with $\alpha = 4$

In the last case, we obtain by (2.3), (2.4) (with $s = 2$) the following stability inequality

$$\rho(S_n, \tilde{S}_n) \leq \max\{5.2, \frac{2.0527}{\sigma^3}\}\mu(X,Y)n^{-1/2}, \quad n \geq 1,$$

which are valued under the condition:

$$2.569\mathbf{V}(X,Y) + \frac{3.7463}{\sigma^3}\mu(X,Y) \leq 1/3.$$

	c	$\overline{c}$	$\overline{\overline{c}}$
Uniform distribution ($s = 4$)	18.53	4.9029	.0197
Gamma distribution with $\alpha = 4$ ($s = 1$)	1	1.4143	.09358

Table 5: ($\sigma = 5$).

To display the procedure of a numerical calculation of the above constants in the case, say, of the exponential distribution (Table 2) we present in Table 6 some members of the sequences

$$b_k := \sup_{x\in\mathbb{R}} |f_k''(x)|, d_k := \int_{-\infty}^{\infty} \left|f_k^{(3)}(x)\right| dx; \quad k = 4, 5, \dots .$$

k	4	5	7	10	20	60	120	170
b_k	1.3455	.8625	.5649	.5002	.4447	.4133	.4060	.4039
d_k	4.4779	3.2810	2.4918	2.0982	1.7616	1.5865	1.5475	1.5361

Table 6

Observe that for $k = 170$ the above quantities are rather close to their limit values coming from the normal distribution and giving in Table 1 (for the normal r.v. Y, $b_k = b$, $d_k = d$, $k = 1, 2, \dots$)

For the convenience of references we cite a particular result from Gordienko [9] . Let $\sigma^2 := \mathrm{Var}(X)$, $\tilde{\sigma}^2 := \mathrm{Var}(Y)$ and $\varphi, \tilde{\varphi}$ denote the characteristic functions of the random variables X/σ and $Y/\tilde{\sigma}$.

Assumption 2. There exist an integer $m \geq 1$ and a real number $r < \infty$ such that

$$|\varphi(t) \leq r|t|^{-3/m}, \qquad |\tilde{\varphi}(t)| \leq r|t|^{-3/m} \qquad \text{for} \quad |t| \geq 1.$$

It is easy to see that Assumption 2 is satisfied for the normal, uniform, triangular, Gamma distributions and their mixtures.

Theorem 2. (Gordienko [9]) *Suppose that* $EX = EY$, EX^2, $EY^2 < \infty$ *and Assumption 2 holds. Then for any* ν

$$\rho(Z, \tilde{Z}) \leq \tilde{c} \max\left\{\rho(X, Y), \frac{1}{2}\mathbf{k}_2(X, Y)\right\}, \tag{2.13}$$

where

$$\begin{aligned} \tilde{c} \quad = \quad & \max\left\{2m - 1, \frac{1}{\pi}\left[\frac{1 + m^{-1}}{\sigma^2} + \frac{1}{\tilde{\sigma}^2}\right]\right. \\ & \left. \times \inf_{t: rt^{-3/m} < 1} t^2 \left[m\left(rt^{-3/m}\right)^m + \frac{4}{1 - (rt^{-3/m})^2}\right]\right\}. \end{aligned} \tag{2.14}$$

For example, for X and Y with the triangular density and $\sigma = 1.5$, $\tilde{\sigma} = 1.4$ we get $\tilde{c} < 3.031$.

3. Stability Estimates in Applied Stochastic Models

3.1. Robustness of Surpluses in the S. Andersen and the Classical Risk Model

Let us examine the risk process defined as follows (see, for instance, Grandell [13], Kass et al [15] and Kalashnikov [16])

$$X(t) = x + \gamma t - \sum_{k=1}^{N(t)} X_k \tag{3.1}$$

$\left(\sum_{k=1}^{0} := 0 \quad \text{by the convention}\right)$, where x is the initial capital of an insurance company, $\gamma > 0$ is a gross premium rate, $N(t)$ is the number of claims occurred within the time interval $[0, t]$ and nonnegative i.i.d. r.v's $X_1, X_2, \dots$ represent successive claim sizes. $N(t)$ is supposed to be a renewal process independent of $\{X_k,\ k \geq 1\}$.

We interpret $X(t)$ in (3.1) as "a real model" of fortune movement with known x and γ, but an unknown distribution F of X (generic of $X_1, X_2, \dots$). Let $Y_1, Y_2, \dots$ be a sequence of i.i.d. r.v.'s (with a known available distribution G) used to approximate $X_1, X_2, \dots$. In other words, the approximating risk process

$$\tilde{X}(t) = x + \gamma t - \sum_{k=1}^{\tilde{N}(t)} Y_k \tag{3.2}$$

is used as "a hypothetical model" to replace $X(t)$ in analysis of the latter. Being concerned with the stability aspect we seek for upper bounds of

$$\rho\left(X(t),\ \tilde{X}(t)\right) = \rho\left(\sum_{k=1}^{N(t)} X_k, \sum_{k=1}^{\tilde{N}(t)} Y_k\right).$$

We will consider only a simple case when $\tilde{N}(t) \stackrel{\mathrm{d}}{=} N(t)$. An application of inequality (2.13) with $EX_1 = EY_1$ considered in Gordienko [9] provides an upper bound of $\rho(X(t), \tilde{X}(t))$ independent of $t \geq 0$. Now we impose more moment restrictions:

$$EX_1 = EY_1,\ EX_1^2 = EY_1^2;\ E|X_1|^3,\ E|Y_1|^3 < \infty, \tag{3.3}$$

in order to get the rate $t^{-1/2}$ of vanishing $\rho\big(X(t), \tilde{X}(t)\big)$ as $t \to \infty$.

Letting

$$\nu^{-1} := \begin{cases} 0, & \text{if} \quad N(t) = 0, \\ [N(t)]^{-1}, & \text{if} \quad N(t) > 0, \end{cases}$$

and accepting Assumption 1 for the "theoretical" r.v.'s Y_k, $k \geq 1$ we obtain from (2.5) the inequality:

$$\rho\left(\sum_{k=1}^{N(t)} X_k,\ \sum_{k=1}^{N(t)} Y_k\right) \leq c\mu(X,Y)\sum_{n=1}^{\infty} n^{-1/2} P(N(t)=n) \tag{3.4}$$

valid under restriction (2.3). Here $\mu = \max\left\{\rho, \frac{1}{6}\mathbf{k}_3\right\}$ and c is defined in (2.6).

Let us suppose now that $N(t)$ is a Poisson process with parameter λ, i.e. (3.1) represents the classical risk process. By easy calculations given below we get for all $t > 0$

$$\sum_{n=1}^{\infty} n^{-1/2}\frac{(\lambda t)^n}{n!}e^{-\lambda t} \leq \left(\frac{2}{\lambda t}\right)^{1/2}. \tag{3.5}$$

Thus (3.4) turns into the inequality

$$\rho\left(\sum_{k=1}^{N(t)} X_k, \sum_{k=1}^{N(t)} Y_k\right) \leq c\left(\frac{2}{\lambda}\right)^{1/2}\frac{1}{\sqrt{t}}\mu(X,Y), \tag{3.6}$$

with the right-hand part vanishing at the correct rate dictated by the central limit theorem.

On the other hand, assuming conditions (3.3) and the existence of a bounded density $g_t(x)$ of the r.v. $\sum_{k=1}^{N(t)} Y_k$, the results of Rachev [23], Chapter 16 afford the following stability bound for the classical process:

$$\begin{aligned}\rho\left(\sum_{k=1}^{N(t)} X_k, \sum_{k=1}^{N(t)} Y_k\right) &\leq [1 + \lambda t \sup_x g_t(x)] \\ &\quad \times 20^{1/2} 7^{1/8} (\lambda t)^{-1/2} [\mathbf{k}_3(X,Y)]^{1/4}.\end{aligned} \tag{3.7}$$

Surely, inequality (3.7) works under less restrictive conditions than (3.6), but the factor expressing the proximity of Y to X enters in (3.7) as a power function of the conjectually small distance $\mathbf{k}_3(X,Y)$, whereas the right-hand side of (3.6) is a linear function of the distance $\mu(X,Y)$. Moreover, it follows from the central limit theorem for densities that

$$\sup_x g_t(x) \geq c' t^{-1/2}, \tag{3.8}$$

for all sufficiently large t ($c' > 0$). Consequently, the time depend term on the right-hand side of (3.7) behaves as a constant as $t \to \infty$.

Remark 3. Combining bound (3.6) with the results of the paper Roos et al [24] on comparison between distributions of random sums $\sum_{k=1}^{\nu} X_k$ and $\sum_{k=1}^{\tilde{\nu}} X_k$, one can get bounds for $\rho\left(\sum_{k=1}^{N(t)} X_k, \sum_{k=1}^{\tilde{N}(t)} Y_k\right)$ with different processes $N(t)$ and $\tilde{N}(t)$.

Let us outline the proofs of inequalities (3.5) and (3.8). To simplify notations we assume without less of generality that $\lambda = 1$, $\sigma = 1$, $EY_1 = 0$ and denote by φ the density of the standard normal distribution.

Letting f_ξ be the density of a r.v. ξ, we learn from the supplement to Ch. VII in Petrov [21] (assuming boundedness of the density of Y_1) that for a positive constant c''

$$\sup_x \left| f_{\frac{\tilde{S}_n}{\sqrt{n}}}(x) - \varphi(x) \right| \le c'' n^{-1/2}, \qquad n = 1, 2, \dots .$$

Thus

$$f_{\tilde{S}_n}(0) \ge \frac{\varphi(0)}{\sqrt{n}} - \frac{c''}{n} \ge \frac{c'}{\sqrt{n}}, \qquad n = m, m+1 \dots ,$$

for appropriately chosen m and $c' > 0$, and

$$\begin{aligned} \sup_x g_t(x) &= \sup_x \sum_{n=1}^{\infty} f_{\tilde{S}_n}(x) \frac{t^n}{n!} e^{-t} \ge \sum_{n=m}^{\infty} f_{\tilde{S}_n}(0) \frac{t^n}{n!} e^{-t} \\ &\ge c' \sum_{n=m}^{\infty} \frac{1}{\sqrt{n}} \frac{t^n}{n!} e^{-t} = c' \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \frac{t^n}{n!} e^{-t} - c' \sum_{n=1}^{m-1} \frac{1}{\sqrt{n}} \frac{t^n}{n!} e^{-t}. \end{aligned}$$

The first term on the right-hand side of this inequality is

$$c' t \sum_{n=0}^{\infty} \frac{1}{(n+1)^{3/2}} \frac{t^n}{n!} e^{-t},$$

and it is greater than $c' t (t+1)^{-3/2}$ by Jensen inequality. The second term is of order $t^{m-1} e^{-t}$ as $t \to \infty$, thus (3.8) follows.

As for (3.5), using the notation

$$\psi(x) = \begin{cases} 0, & x = 0, \\ x^{-1/2}, & x > 0, \end{cases}$$

we write

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \frac{t^n}{n!} e^{-\lambda t} = E\psi(N(t)) \le \{E\psi^2(N(t))\}^{1/2},$$

and

$$E\psi^2(N(t)) = \frac{1}{t} \sum_{n=1}^{\infty} \frac{n+1}{n} \frac{t^{n+1}}{(n+1)!} e^{-t} \le \frac{2}{t} \sum_{n=2}^{\infty} \frac{t^n}{n!} e^{-t} \le \frac{2}{t}.$$

To get a rough idea of the accuracy of bound (3.6), we consider the following simple illustrative example. Let Y be a r.v. uniformly distributed on [0,3] and ξ be a r.v. independent of Y taking two values a and b with equal probabilities, where $a = 1.5 - \sqrt{0.75} \approx 0.633974596$ and $b = 1.5 + \sqrt{0.75} \approx 2.366025404$.

Set

$$X = \begin{cases} Y & \text{with probability } 0.99, \\ \xi & \text{with probability } 0.01. \end{cases}$$

Then $EX = EY,\ EX^2 = EY^2$ and the r.v. Y satisfies Assumption 1 with $s = 4$. By elementary calculations we obtain the following data for this example (see Table 3 in Section 2).

$$\begin{aligned} \rho(X,Y) &\approx 0.002886751, & \mathbf{V}(X,Y) &= 0.02, \\ \mathbf{k}_3(X,Y) &\approx 0.033485595, & [\mathbf{k}_3(X,Y)]^{1/4} &\approx 0.427774115, \\ \mu(X,Y) &\approx 0.005580932, & \overline{c} &\approx 4.9029 \le 5, \\ \overline{\overline{c}} \approx 3.7895 < 3.8, \quad \sigma^2 &= 0.75, & c &= \max\{7^{3/2}, 2.016870311\} = 7^{3/2}. \end{aligned}$$

Comparing the magnitudes of $\overline{c}, \overline{\overline{c}}, \mathbf{V}(X,Y)$ and $\mu(X,Y)$ we see that condition (2.3) of Theorem 1 is satisfied, so we can apply inequality (2.5), which turns out to be inequality (3.6) in this example.

Letting $\lambda = 1$, $\Delta_t = \rho\left(\sum_{k=1}^{N(t)} X_k, \sum_{k=1}^{N(t)} Y_k\right)$ and considering three cases: $t = 3,\ t = 30,\ t = 365$ we get by computer calculations:

$$\begin{aligned} \Delta_3 &\le 11.09534\mu(X,Y) < 0.0635; \\ \Delta_{30} &\le 3.42568\mu(X,Y) < 0.0194; \\ \Delta_{365} &\le 1.37096\mu(X,Y) < 0.00768. \end{aligned}$$

One can contrast these bounds with those calculated by (3.7).

3.2. Stability of Selection of the Initial Capital Securing a Prescribed Risk

Again, let us consider the risk process (3.1) and its approximation (3.2) supposing that $N(t) = \tilde{N}(t)$ is a Poisson process with parameter λ.

Let $\psi(x)$ denote a ruin probability given initial capital x and $\psi^* > 0$ be a tolerable level of this probability.

In Kalashnikov [16], Chapter 6 the value

$$x^* := \inf\{x : \psi(x) \leq \psi^*\} \tag{3.9}$$

was evaluated in the case when the relative safety loading $\rho = \dfrac{\gamma}{\lambda a} - 1 \to 0$. The method of Kalashnikov [16] uses appropriate exponential bounds for ruin probability which operate well in the small loading limit $\rho \to 0$.

We consider another approach to approximation of x^* in (3.9) which works in certain circumstances without special conditions on ρ. Denote by $\tilde{\psi}(x)$ the ruin probability in the available (approximating) model (3.2) and suppose that $\tilde{\psi}(x)$ can be calculated for any $x \geq 0$. This is the case, for example, when the distribution of Y_1 in (3.2) is a mixture of exponential distributions. Now we suppose that for a given ε, $0 < \varepsilon < \psi^*$ the following stability bound of ruin probability is available:

$$\sup_{x \geq 0} |\psi(x) - \tilde{\psi}(x)| \leq \varepsilon. \tag{3.10}$$

Define

$$\tilde{x}^* = \inf\{x : \tilde{\psi}(x) \leq \psi^* - \varepsilon\}. \tag{3.11}$$

Then $\psi(\tilde{x}^*) \leq \psi^*$. Let us display bound (3.10) and show how to bound $|x^* - \tilde{x}^*|$ in terms of a suitable distance between the distributions of X_1 and Y_1 in (3.1) and (3.2).

The inequality supplying (3.10) is obtained in Gordienko [9].

Let $q = \dfrac{\rho}{1+\rho}$, $a = EX_1$, $\xi_1, \xi_2, \dots$ be i.i.d. r.v.s. with the common distribution function

$$F_\xi(x) = \frac{1}{a} \int_0^x [1 - F(t)]dt, \quad x \geq 0, \tag{3.12}$$

and ν be the geometric distributed r.v. with parameter q independent of $\xi_1, \xi_2, \dots$. Then (see e.g. Kalashnikov [16], Chapter 6)

$$\psi(x) = (1 - q) P\left(\sum_{k=1}^{\nu} \xi_k > x\right). \tag{3.13}$$

Using this representation of ruin probability and the nice fact that for every F the distribution function F_ξ given in (3.12) satisfies Assumption 2 with $r = 2/a$ and $m = 3$, we deduce in Gordienko [9] from (2.13), (3.13) the following inequality:

$$\sup_{x\geq 0} |\psi(x) - \tilde{\psi}(x)| \leq \frac{\tilde{c}(1-q)}{a} \max\left\{\bar{\mu}(X,Y),\ \frac{1}{6}\,\mathbf{k}_3(X,Y)\right\}, \tag{3.14}$$

where

$$\bar{\mu}(X,Y) := \sup_{x>0}\left|\int_0^x [F(t) - G(t)]dt\right|,$$

and $\tilde{c}$ is calculated by (2.14) with $\sigma^2 = (3a)^{-1}EX^3 - (4a^2)^{-1}(EX^2)^2$ and $\tilde{\sigma}^2 = (3a)^{-1}EY^3 - (4a^2)^{-1}(EY^2)^2$. Inequality (3.14) is proven under the condition:

$$EX = EY; \qquad EX^2 = EY^2; \qquad E|X|^3, E|Y|^3 < \infty.$$

We can choose in (3.10)

$$\varepsilon = \frac{\tilde{c}(1-q)}{a} \max\left\{\bar{\mu}(X,Y),\ \frac{1}{6}\,\mathbf{k}_3(X,Y)\right\},$$

if only this quantity is less than the tolerable level ψ^* of ruin probability. To estimate $|x^* - \tilde{x}^*|$ (see (3.9), (3.11)) in terms of ε we need additional aprior information. Suppose that:

(i) a number L is known such that $x^*, \tilde{x}^* \leq L$;

(ii) there is $\alpha \geq \frac{1}{a}$ such that $1 - F(x) \geq e^{-\alpha x}$, $x \geq 0$; (i.e. the r.v. X is "stochastically greater" than the exponential r.v. with parameter α).

We get from (3.13) that

$$\psi'(x) = -(1-q)\sum_{n=1}^{\infty} f_{S_n}(x)q(1-q)^{n-1},$$

where f_{S_n} is the density of $S_n = \xi_1 + \cdots + \xi_n$. From (3.12) and (ii) we conclude that

$$f_{S_n}(x) \geq \frac{x^{n-1}}{a^n(n-1)!}e^{-\alpha x}, \qquad x \geq 0, \quad n = 1, 2, \ldots.$$

Therefore (see (i)), $\psi'(x) < 0$ and

$$|\psi'(x)| \geq \frac{(1-q)q}{a}e^{-(\alpha - \frac{1-q}{a})x} \geq \frac{\rho}{a(1+\rho)^2}e^{-(\alpha - \frac{1}{(1+\rho)a})L} := b(\rho).$$

Finally, we have the inequality

$$|x^* - \tilde{x}^*| \leq \varepsilon / b(\rho), \tag{3.15}$$

which states that $\tilde{x}^* \to x^*$ if $\bar{\mu}(X,Y) \to 0, \mathbf{k}_3(X,Y) \to 0$ (see (3.10), (3.14)). Remark that the "stability" bound (3.15) is rather poor in the case of small ρ. Indeed, from Kalashnikov [16], Chapter 6 we get that $x^* \sim \beta/\rho$ as $\rho \to 0$ for some $\beta > 0$. So, we should assume that the constant L in (i) increases as β/ρ with $\rho \to 0$. Consequently, there are constants $\tilde{\beta}, \tilde{\tilde{\beta}} > 0$ such that in (3.15)

$$1/b(\rho) \geq \tilde{\beta}\ \frac{1}{\beta} e^{\tilde{\tilde{\beta}}/\rho} \text{ as } \rho \to 0.$$

3.3. Stability Estimates of Renewal Functions

We suppose that the sequences $(X_k,\ k \geq 1)$, $(Y_k,\ k \geq 0)$ consist of positive r.v.'s such that

$$0 \leq \alpha \leq X_1, \quad Y_1 \leq \beta < \infty\,, \tag{3.16}$$

for some given numbers α and β.

Let us consider two renewal processes:

$$N(t) := \max\{n \text{ for which } S_n \leq t\},\ t \geq 0;$$

$$\tilde{N}(t) := \max\{n \text{ for which } \tilde{S}_n \leq t\},\ t \geq 0.$$

Let $M(t) = EN(t)$, $\tilde{M}(t) = E\tilde{N}(t) \quad (t \geq 0)$ be the renewal functions associated with $(X_k,\ k \geq 1)$ and $(Y_k,\ k \geq 1)$ respectively. Let $L > 0$ be arbitrary but fixed. Recall that $a = EX = EY$.

Proposition 1. *Suppose that the hypotheses of Theorem 1 hold. For every $\varepsilon \in (0,1)$ if*

$$\mu(X,Y) \leq \left(\frac{a}{2L}\right)^{\frac{1}{2\varepsilon}}, \tag{3.17}$$

then

$$\sup_{0 \leq t \leq L} \left|M(t) - \tilde{M}(t)\right| \leq c_\varepsilon \big[\mu(X,Y)\big]^{1-\varepsilon}, \tag{3.18}$$

where

$$c_\varepsilon = 2\left\{c + \left[\frac{\varepsilon a^2}{(1-\varepsilon)(\beta-\alpha)^2}\right]^{\frac{\varepsilon-1}{2\varepsilon}} e^{\frac{\varepsilon-1}{2\varepsilon}} \left[1 - \exp\left(-\frac{a^2}{2(\beta-\alpha)^2}\right)\right]^{-1}\right\},$$

and the constant c appeared in (2.6).

Now we are going to evaluate the difference between $M(t)$ and $\tilde{M}(t)$ without assuming equality of variances of X and Y. Let us denote the metric involved in Theorem 2 by $\tilde{\mu} := \max\left\{\rho,\ \frac{1}{2}\,\mathbf{k}_2\right\}$.

Proposition 2. *Suppose that the hypotheses of Theorem 2 hold. For every $\varepsilon \in (0,1)$ if*

$$\tilde{\mu}(X,Y) \le \left(\frac{a}{2L}\right)^{1/\varepsilon}, \tag{3.19}$$

then

$$\sup_{0\le t\le L} |M(t) - \tilde{M}(t)| \le \tilde{c}_\varepsilon [\tilde{\mu}(X,Y)]^{1-\varepsilon}, \tag{3.20}$$

where

$$\tilde{c}_\varepsilon = \left\{\tilde{c} + 2\left[\frac{\varepsilon a^2}{2(1-\varepsilon)(\beta-\alpha)^2}\right]^{\frac{\varepsilon-1}{\varepsilon}} e^{\frac{\varepsilon-1}{\varepsilon}} \left[1 - \exp\left(-\frac{a^2}{2(\beta-\alpha)^2}\right)\right]^{-1}\right\} \tag{3.21}$$

and $\tilde{c}$ is the constant from (2.14).

Note that the constant $\tilde{c}_\varepsilon$ in (3.20), (3.21) is greater than c_ε in (3.18) but inequality (3.20) holds under less restrictive conditions than (3.18) does.

Proof. Denoting

$$\delta_t = |M(t) - \tilde{M}(t)| = \left|\sum_{n=1}^{\infty} P(S_n \le t) - \sum_{n=1}^{\infty} P(\tilde{S}_n \le t)\right|,$$

we have for any $N \ge 2$ (see (2.4)):

$$\delta_t \le c\mu(X,Y)\sum_{n=1}^{N-1} n^{-1/2} + \sum_{n\ge N} P(S_n \le L) + \sum_{n\ge N} (\tilde{S}_n \le L). \tag{3.22}$$

Let us write $\mu \equiv \mu(X,Y)$ and choose $N = \left[\mu^{-2\varepsilon}\right] + 1$. In view of (3.17) $N \ge \frac{2L}{a}$.

Thus

$$\sum_{n\ge N} P(S_n \le L) = \sum_{n\ge N} P\left(-S_n + na \ge n(a - \frac{L}{n})\right)$$
$$\le \sum_{n\ge N} P\left(-S_n + na \ge \frac{na}{2}\right) \le \sum_{n\ge N} \exp\left[-n\frac{a^2}{2(\beta-\alpha)^2}\right], \tag{3.23}$$

where the last inequality follows from (3.16) and Hoeffding inequality (see, Petrov [21]). Denoting $\gamma = \dfrac{a^2}{2(\beta-\alpha)^2}$ we estimate the right-hand side of (3.23) by $b_\varepsilon \mu^{1-\varepsilon}/(1-e^{-\gamma})$, where $e^{-\gamma/\mu^{2\varepsilon}} \le b_\varepsilon \mu^{1-\varepsilon}$.

Simple calculations show that the last inequality holds, provided that

$$b_\varepsilon = \left(\frac{2\gamma\varepsilon}{1-\varepsilon}\right)^{\frac{\varepsilon-1}{2\varepsilon}} e^{\frac{\varepsilon-1}{2\varepsilon}}.$$

Thus

$$\sum_{n\ge N} P(S_n \le L) + \sum_{n\ge N} P(\tilde{S}_n \le L) \le 2\mu^{1-\varepsilon} b_\varepsilon (1-e^{-\gamma})^{-1},$$

and also $\sum_{n=1}^{N-1} n^{-1/2} < 2/\mu^\varepsilon$. Therefore the desired bound (3.18) follows from (3.22). Inequality (3.20) is proven similarly. □

3.4. Stability Estimation in the Problem of Selection of an Optimal Replacement Period

Now we pass to the block-replacement rule optimization in reliability following the notation and the description given in Tijms [28], Chapter 1. In that model some unit is exposed to stochastic breakdowns and it is replaced by a new one upon failure and upon planned times $T, 2T, 3T, \dots$. Let the sequence $(X_k, k \ge 1)$ represents the lifetimes of units and c_p, c_f denote the costs, respectively, of a planned replacement (at instants $kT,\ k = 1, 2, \dots$) and a failure replacement. The time T is a controlled parameter. For a given T, the average cost $Q(T)$ per unit time over infinite interval is calculated as follows (see Tijms [28], Chapter 1):

$$Q(T) = \frac{1}{T}[c_p + c_f M(T)],$$

where $M(t)$ is the renewal function associated with $(X_k,\ k \ge 1)$. The optimal replacement period T_* is defined as

$$Q(T_*) = \inf_{T>0} Q(T).$$

Again, supposing that the distribution function F of X is not available and that some distribution function G (common for $Y_1, Y_2, \dots$) is used as a known approximation to F, we denote by $\tilde{M}(t)$ the renewal function related to $(Y_k,\ k \geq 1)$ and let:

$$\tilde{Q}(T) = \frac{1}{T}[c_p + c_f \tilde{M}(T)], \quad \tilde{Q}(\tilde{T}_*) = \inf_{T>0} \tilde{Q}(T).$$

One can try $\tilde{T}_*$ as an available approximation for the optimal value T_*, i.e. to assign the replacement epoch $\tilde{T}_*$ in the original model determined by means of $(X_k,\ k \geq 1)$.

The following *stability index*

$$\Delta := Q(\tilde{T}_*) - Q(T_*) \geq 0$$

measures the raise in the average cost above the minimal value $Q(T_*)$, (see Gordienko [10], Gordieko et al [12], Hilgert et al [14] and Montes de Oca et al [19]).

We can use either (3.18) or (3.20) to bound Δ depending on the moment conditions we deal with.

Proposition 3. *Suppose the conditions of Proposition 2 and that $T_*, \tilde{T}_* \in [\ell, L]$ for some given, $\ell > 0$, $L < \infty$. Then for $\varepsilon \in (0,1)$ under (3.19) we get:*

$$\Delta \leq 2\tilde{c}_\varepsilon c_f \ell^{-1} [\tilde{\mu}(X,Y)]^{1-\varepsilon}\,, \tag{3.24}$$

with $\tilde{c}_\varepsilon$ being as in (3.21).

To show (3.24) it suffices to note that

$$\begin{aligned} \Delta &= Q(\tilde{T}_*) - \tilde{Q}(\tilde{T}_*) + \tilde{Q}(\tilde{T}_*) - Q(T_*) \\ &= Q(\tilde{T}_*) - \tilde{Q}(\tilde{T}_*) + \inf_{T\in[\ell,L]} \tilde{Q}(T) - \inf_{T\in[\ell,L]} Q(T) \\ &\leq 2 \sup_{T\in[\ell,L]} |Q(T) - \tilde{Q}(T)| = 2 \sup_{T\in[\ell,L]} \frac{c_f}{T} |M(T) - \tilde{M}(T)|\,, \end{aligned}$$

and then to use (3.20).

3.5. Stability of the Average Length of the Time Interval with Zero Release Rate in a Finite Dam

In the papers, Abdel-Hammed [1] and Bae et al [2], the so-called $P^M_{\alpha,\tau}$ policy is studied in a dam with compound Poisson inputs. Let us consider a finite dam of the capacity V (Bae et al [2]) and denote by $\lambda > 0$ the rate of a compound Poisson process which models the input of water. The amounts of inputs are i.i.d. r.v.'s $X_1, X_2, \ldots$ with common distribution function F and mean a. Under the policy $P^M_{\alpha,\tau}$, the release rate $r \geq 0$ is kept until the level of water $Z(t)$ exceeds α $(0 < \tau < \alpha < V)$, then the release rate becomes M $(M > r)$ and this rate is kept until $Z(t)$ reaches τ. After that the rate is switched to r, and so on. In Bae et al [2], the long-run average cost per unit time is calculated. One component of the formula obtained is the expectation $E[L^r(\alpha,\tau)]$ of the part of a regeneration cycle when the realize rate is r (or 0 while $Z(t) = 0$). We estimate the stability of this quantity in the particular case $r = 0$. For this, together with the above dam model let us consider its approximation given by the compound Poisson process with the same rate λ, but with random inputs of water $Y_1, Y_2, \dots$ with common distribution function G. We aim to compare $E[L^0(\alpha,\tau)]$ with the corresponding quantity $E[\tilde{L}^0(\alpha,\tau)]$ in the approximating model. As it is shown in Abdel-Hammed [1] and Bae et al [2],

$$E[L^0(\alpha,\tau)] = 1 - \lambda^{-1} M(\alpha - \tau),$$

where M is the renewal function associated with the sequence $X_1, X_2, \dots$. Thus we can apply the inequalities from Section 3.3. For example, we can state the following.

Proposition 4. *Let $\varepsilon \in (0,1), EX = EY$, the conditions of Proposition 2 be satisfied and*

$$\tilde{\mu}(X,Y) \leq \left(\frac{a}{2V}\right)^{1/\varepsilon}.$$

Then

$$\sup_{0<\tau<\alpha<V} |E[L^0(\alpha,\tau)] - E[\tilde{L}^0(\alpha,\tau)]| \leq \lambda^{-1} \tilde{c}_\varepsilon [\tilde{\mu}(X,Y)]^{1-\varepsilon} \tag{3.25}$$

Remark 4. For $r > 0$ the expectation $E[L^r(\alpha,\tau)]$, and other components of the average long-run reward, are expressed in terms of geometric convolutions of the "equilibrium distribution function $F_*(x) = \dfrac{1}{a}\displaystyle\int_0^x [1 - F(t)]dt$ (see Bae et al [2]). Therefore Theorem 1 and Theorem 2 can be used to study the stability of the average reward.

We stress that the bounds like (3.25) are uniform with respect to the controlled parameters α and τ. This allows us to take advantage of the such bounds to estimate the stability index in the problem of optimization of the average reward in dam models (along the lines of Section 3.4).

4. On the Proof of Theorem 1

To prove inequalities (2.4) we follow the well-known probabilistic metric method of estimation of the rate of convergence in the central limit theorem (see e.g. Senatov [25] and Senatov [26]). However, we can not exploit the specific properties of the normal distribution as in Senatov [25] and Senatov [26]. So we need to invoke some alternative considerations (based on the asymptotic expansion of the distributions of sums). For similar reasons we have to calculate numerically some constants involved in the setting and the proof of Theorem 1 (see Section 2). Thus, omiting arguments and details relevant to those given in Senatov [25] and Senatov [26] for the case of normally distributed r.v.'s $Y_1, Y_2, \dots$ we only outline the proof of Theorem 1.

By the regularity of the metric ρ,

$$\rho(S_n, \tilde{S}_n) \le \sum_{k=1}^{n} \rho(X_k, Y_k) \le (2s-1)^{3/2} \mu(X,Y) n^{-1/2},$$

for $n = 1, 2, \dots, 2s-1$.

Let $n \ge 2s$, $0 \le j \le n$ and

- F_j denote the distribution of $\dfrac{X_0 + \cdots + X_j}{\sqrt{n}}$,

- G_j denote the distribution of $\dfrac{Y_0 + \cdots + Y_j}{\sqrt{n}}$ ($F_0 = G_0$ is the distribution of $X_0 \equiv Y_0 \equiv 0$),

- $m := [n/2] \ge s$, where $[x]$ stands for the integer part of x.

In Section 4.1 and Section 4.3 of Senatov [25], it is shown that

$$\begin{aligned} \rho(F_n, G_n) \quad \le \quad & \sum_{j=0}^{m} \rho(F_1 * G_j * F_{n-j-1}, G_1 * G_j * F_{n-j-1}) \\ & + \rho(G_1 * G_m * F_{n-m-1}, G_1 * G_m * G_{n-m-1}). \end{aligned} \tag{4.1}$$

The last term on the right-hand side of (4.1) denoted by I_n can be rewritten as follows.

$$\begin{aligned} I_n \quad = \quad \rho\Bigg(& \frac{Y_1 + \cdots + Y_{m+1}}{\sqrt{n}} + \frac{X_1 + \cdots + X_{n-m-1}}{\sqrt{n}}, \\ & \frac{Y_1 + \cdots + Y_{m+1}}{\sqrt{n}} + \frac{Y_1 + \cdots + Y_{n-m-1}}{\sqrt{n}} \Bigg). \end{aligned} \tag{4.2}$$

Let $\xi = \dfrac{Y_1 + \cdots + Y_{m+1}}{\sqrt{n}} = \sigma \left(\dfrac{m+1}{n}\right)^{1/2} \xi_{m+1}$, where

$$\xi_k = \frac{Y_1 + \cdots + Y_k}{\sigma\sqrt{n}} \quad (k \geq s). \tag{4.3}$$

In view of Assumption 1 the density of ξ has a bounded second derivative and

$$\sup_x |f''_\xi(x)| \leq \frac{1}{\sigma^3}\left(\frac{n}{[n/2]+1}\right)^{3/2} \sup_x \left|f''_{\xi_{m+1}}(x)\right| \leq \frac{2^{3/2}b}{\sigma^3}, \tag{4.4}$$

where f_k stands for the density of ξ_k and the constant b was introduced in (2.11). Without any loss of generality we can suppose that $EY_k = 0,\ k = 1, 2, \ldots$. Theorem 7, Chapter VI in Petrov [21] states that if $EY_1 = 0,\ EY_1^2 = \sigma^2 > 0,\ E|Y_1|^3 < \infty,\ f^{(3)}_{Y_1+\cdots+Y_s}$ exists and the characteristic function φ_{Y_1} is such that

$$|\varphi_{Y_1}(t)| \leq \frac{M}{|t|^\delta}, \quad (\delta > 0)\,, \tag{4.5}$$

for all sufficiently large t then, for $k \geq s,\ i = 0, 1, 2, 3$

$$f_k^{(i)}(x) = \frac{1}{\sqrt{2\pi}} \frac{d^{i+1}}{dx^{i+1}} \left[\int_{-\infty}^{x} e^{-y^2/2} dy - e^{-x^2/2} \frac{EY^3(x^2-1)}{6\sigma^3\sqrt{n}}\right] + o(n^{-1/2})\,, \tag{4.6}$$

as $n \to \infty$ uniformly on $\mathbb{R}$.

Assumption 1 guarantees condition (4.5) since $f^{(3)}_{Y_1+\cdots+Y_s}(x)$ is an integrable function.

The finiteness of b follows immediately from (4.6).

We can apply to (4.2) the following inequality:

$$\rho(X+\xi,\ Y+\xi) \leq \sup_x \left|f''_\xi(x)\right| \zeta_3(X, Y), \tag{4.7}$$

where ζ_3 is the Zolotarev metric of order 3 (see Senatov [25] and Zolotarev [30] for definitions) (Inequality (4.7) is proven in Senatov [25] for a normal r.v. ξ, but it holds always when f''_ξ exists). Thus by the well-known "ideal property" of the metric ζ_3 (see, for instance Senatov [25] and Zolotarev [30]):

$$\zeta_3\left(a\sum_{i=1}^n X_i,\ a\sum_{i=1}^n \tilde{X}_i\right) \leq a^3 \sum_{i=1}^n \zeta_3(X_i,\ \tilde{X}_i),\ (a \geq 0),$$

we obtain from (4.2), (4.4) and (4.7) that

$$\begin{aligned} I_n &\le 2^{3/2}\frac{b}{\sigma^3}\zeta_3\left(\frac{X_1+\cdots+X_{n-m-1}}{\sqrt{n}},\ \frac{Y_1+\cdots+Y_{n-m-1}}{\sqrt{n}}\right) \\ &\le 2^{3/2}\frac{b}{\sigma^3}\frac{n-m-1}{n^{3/2}}\zeta_3(X_1,Y_1) \le \frac{\sqrt{2}b}{\sigma^3}\frac{1}{\sqrt{n}}\zeta_3(X_1,Y_1). \end{aligned} \tag{4.8}$$

Now, let us go back to inequality (4.1). Denote by J_j a summand in the sum on the right-hand side and let

$$\begin{aligned} P &= F_1 * G_j\ ; \quad U = F_{n-j-1}; \\ Q &= G_1 * G_j\ ; \quad H = G_{n-j-1}. \end{aligned}$$

It is well-known that (see Rachev [23] and Senatov [25])

$$\rho(P*U,\ Q*U) \le \rho(U,H)V(P,Q) + \rho(P*H,\ Q*H)\,,$$

for arbitrary distribution functions P, Q, U, H.

Thus

$$\begin{aligned} J_j \le\ & \rho(F_{n-j-1}, G_{n-j-1})V(F_1*G_j, G_1*G_j) \\ & +\rho(F_1*G_{n-1}, G_1*G_{n-1}) \end{aligned} \tag{4.9}$$

$(j = 0, 1, \ldots, m)$.

Proceeding along the lines of the above estimating of I_n we get

$$\rho(F_1*G_{n-1}, G_1*G_{n-1}) \le \frac{b}{\sigma^3(n-1)^{3/2}}\zeta_3(X_1,Y_1). \tag{4.10}$$

By the regularity property of $\mathbf{V}$ (see Senatov [25]) we get for $0 \le j \le s-1$

$$\mathbf{V}(F_1*G_j,\ G_1*G_j) \le \mathbf{V}(F_1,\ G_1) = \mathbf{V}(X_1,Y_1). \tag{4.11}$$

For $s \le j \le m$ we bound the term on the left-hand side of (4.11) in a different way.

Denoting (see (4.3))

$$\xi = \frac{Y_1+\cdots+Y_j}{\sqrt{n}} = \sigma\left(\frac{j}{n}\right)^{1/2}\xi_j,$$

we have

$$f_\xi^{(3)}(x) = \frac{1}{\sigma^4}\left(\frac{n}{j}\right)^2 f_{\xi_j}\left(\frac{x}{\sigma(j/n)^{1/2}}\right)$$

and

$$\int_{-\infty}^{\infty} \left| f_\xi^{(3)}(x) \right| dx \leq \frac{1}{\sigma^3} \left(\frac{n}{j} \right)^{3/2} d, \tag{4.12}$$

where the constant d was defined in (2.12). Under Assumption 1, d is finite. The proof using (2.2) and (4.6) is given in Lemma 3 in Gordienko et al [11].

We shall use the following known inequality (see Rachev [23] and Gordienko et al [11]):

$$\mathbf{V}(X+\xi,\ Y+\xi) \leq \zeta_3(X,Y) \int_{-\infty}^{\infty} \left| f_\xi^{(3)}(x) \right| dx \tag{4.13}$$

(ξ is independent of (X, Y)).

Since f_ξ is the density of G_j we can apply (4.12) and (4.13) to get the bound:

$$\begin{aligned} \mathbf{V}(F_1 * G_j,\ G_1 * G_j) &\leq \left(\frac{1}{\sigma^3} \right)^{3/2} \left(\frac{n}{d} \right)^{3/2} d\zeta_3 \left(\frac{X_1}{\sqrt{n}}, \frac{Y_1}{\sqrt{n}} \right) \\ &\leq \frac{d}{\sigma^3} j^{-3/2} \zeta_3(X_1, Y_1), \end{aligned} \tag{4.14}$$

$s \leq j \leq m$. Let

$$\mu_*(X,Y) = \max\{\rho(X,Y), \zeta(X,Y)\}.$$

Combining inequalities (4.1), (4.8), (4.9), (4.10), (4.11), (4.14) and making the induction assumption that

$$\rho(S_k,\ \tilde{S}_k) \leq c\mu_*(X,Y) k^{-1/2}, \tag{4.15}$$

$k = 1, 2, \ldots, n-1$, we obtain (noting, that $\rho(aX, aY) = \rho(X, Y), a \neq 0$) the following inequality

$$\begin{aligned} \rho(S_n, \tilde{S}_n) \leq \mu_*(X,Y) n^{-1/2} \Bigg\{ & \frac{\sqrt{2}b}{\sigma^3} + \frac{b}{\sigma^3} c_1(s) \\ & + cc_3(s) \mathbf{V}(X,Y) + c\frac{d}{\sigma^3} c_2(s) \mu_*(X,Y) \Bigg\} \end{aligned} \tag{4.16}$$

(see (2.8)–(2.10) for the definitions of $c_1(s) - c_3(s)$).

To get inequality (4.15) for $k = n$, we take

$$c = \max\left\{ (2s-1)^{3/2}, \frac{3}{2} \frac{b}{\sigma^3} \left(\sqrt{2} + c_1(s) \right) \right\},$$

and use (4.16) and (2.3).

Finally we apply the well-known bound for Zolotarev metrics:

$$\zeta_\tau(X,Y) \leq \frac{1}{r!} k_\tau (X,Y)$$

(see, for example Rachev [23]). □

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