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CONTINUITY OF OPERATORS ON TOPOLOGICAL VECTOR SPACES

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Abstract: In this paper we characterize linear and continuous operators from real Frechet spaces X into real topological vector spaces Y as those additive operators which apply bounded sets in X onto bounded sets in Y. As a consequence, additive measurable functions between real finite dimensional spaces are continuous. Furthermore we remove the homogeneity hypothesis from the classical Closed Graph Theorem.

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1. Introduction and Preliminaries

A well know result due to Steinhauss states that for a measurable subset E of the real line with positive Lebesgue measure, 0 belongs to $\mathrm{int}(E-E)$. On the other hand, an old result of Frechet states that additive measurable functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous. An elegant extension of Frechet Theorem to additive functions $f : \mathbb{R}^n \to \mathbb{R}$ using Steinhauss Theorem in found in [3]. Furthermore, Tineo (see [7]) has proved that hypothesis of homogeneity can be removed from the classical Closed Graph Theorem.

It has been inspired in the works of Galaz Fontes [3] and Tineo [7] that we

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prove that for real topological vector spaces, X an $F-$space, additive operators $\Lambda : \mathrm{X} \to \mathrm{Y}$ are continuous if only if they apply bounded sets onto bounded sets, getting as a corollary that for the particular case in which X is a Banach space, the additive operator $\Lambda : \mathrm{X} \to \mathrm{Y}$ is continuous if and only if $\Lambda(V)$ is bounded for some bounded open subset V of X. Furthermore some characterizations of continuity for additive operators between real finite dimensional spaces are given.

We finish the paper providing examples to show that some hypotheses like metrizability and finite dimensionality cannot be removed in the correponding results.

Now, we state some definitions we are using in this paper.

Let X be a vector space with topology τ. The topological space (X, τ) is called a *topological vector space*, if each point in X is closed and the vector addition and scalar multiplication are continuous operation. We say that X is *metrizable* if the topology of X is compatible with a metric d. If d is translation invariant and (X, d) is complete, then the topological vectors space is called an *F-space*. A topological vector space is called *locally convex* if the origin, hence each point of X, contains a local basis of convex sets. A *Frechet space* is a locally convex F-space; a set $E \subset \mathrm{X}$ is *bounded* if there is $t_0 > 0$ such that $E \subset tV$ for every $t > t_0$.

2. The General Case

We recall that for vector spaces X and Y, an operator $\Lambda : \mathrm{X} \to \mathrm{Y}$ is said to be additive if $\Lambda(x + y) = \Lambda(x) + \Lambda(y)$ for each $x, y \in \mathrm{X}$.

Theorem 1. *Let* X *and* Y *be real topological vector spaces with* X *an $F-$space and $\Lambda : \mathrm{X} \to \mathrm{Y}$ an additive operator. Then the following statements are equivalent.*

(a) Λ is continuous.

(b) Λ applies bounded sets onto bounded sets.

(c) If $x_n \to 0$, then $\{\Lambda(x_n)\}_{n=1}^{\infty}$ is bounded.

(d) If $x_n \to x_n$, then $\Lambda(x_n) \to 0$.

Proof. $(a) \Rightarrow (b)$. If Λ is additive and continuous, then it is linear and the proof follows as in the implications $(a) \Rightarrow (b)$ of Theorem 2.32 in [6].

$(b) \Rightarrow (c)$. Because every convergent sequence is bounded.

$(c) \Rightarrow (d)$. Since X is metrizable, Theorem 1.28 in [6] and its proof imply the existence of a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of natural numbers such that $\gamma_n \to \infty$ and $\gamma_n x_n \to 0$. Hence $\{\Lambda(\gamma_n x_n)\}_{n=1}^{\infty}$ is bounded. From the additivity of Λ along with the fact that $\gamma_n \in \mathbb{N}$, we deduce that $\{\gamma_n \Lambda(x_n)\}$ is bounded in Y, therefore

$$\Lambda(x_n) = \frac{1}{\gamma_n}\Lambda(\gamma_n x_n) \to 0.$$

$(d) \Rightarrow (a)$. If (a) is false, then there is a neighborhood W of 0 in Y such that $\Lambda^{-1}(W)$ does not contain any neighborhood of 0 in X. Let us put

$$B_{1/n}(0) = \left\{ x \in \mathrm{X} : \ d(x,0) < \frac{1}{n} \right\}.$$

Hence for each $n \in \mathbb{N}$, we can find $x_n \in B_{1/n}(0)$ such that $\Lambda(x_n) \notin W$. This means $x_n \to 0$ but $\Lambda(x_n) \not\to 0$. □

Corollary 2. *Let* X *be a Banach space and* Y *a topological vector space. An additive operator* $\Lambda : \mathrm{X} \longrightarrow \mathrm{Y}$ *is continuous if and only if there is a neighborhood* V *of 0 such that* $\Lambda(V)$ *is bounded.*

Proof. Let $\Lambda : \mathrm{X} \longrightarrow \mathrm{Y}$ be additive and continuous. Hence Λ is linear. Since X is a Banach space, 0 contains a bounded neighborhood V and, by the continuity of Λ, $\Lambda(V)$ is bounded.

To prove the other implication, we suppose that $\Lambda : \mathrm{X} \longrightarrow \mathrm{Y}$ is additive and V is a neighborhood of 0 with $\Lambda(V)$ bounded. Then there is $\varepsilon > 0$ such that $\Lambda(\mathbf{B}(0,\varepsilon))$ is bounded.

Let $x_n \to 0$. Then there is a sequence $\varepsilon_n \to 0$ and $N \in \mathbb{N}$ such that $\mathbf{B}(0,\varepsilon_n) \subset \mathbf{B}(0,\varepsilon)$, and $x_n \in \mathbf{B}(0,\varepsilon_n)$, $\forall n \in \mathbb{N}$. This implies that $\{\Lambda(x_n)\}_{n=1}^{\infty}$ is bounded and consequently, by Theorem 1, Λ is continuous. □

By using methods of non linear analysis, Tineo [7] has gotten the following version of the Closed Graph Theorem.

Theorem 3. *Let* X *and* Y *be real* $F-$*spaces and* $\Lambda : \mathrm{X} \longrightarrow \mathrm{Y}$ *an additive operator with closed graph. Then* Λ *is linear and continuous.*

Proof. Let $\Re = \{y \in \mathrm{Y} / \ \Lambda x = y\}$.

Since Λ is additive, it is not hard to conclude that

$$y_1, y_2 \in \Re \Longrightarrow y_1 + y_2 \in \Re.$$

On the other hand, applying the additivity of Λ one more time, we see that $\Lambda(\alpha x) = \alpha\Lambda(x)$, for each rational number α and each $x \in \mathrm{X}$ which implies

$\alpha y \in \Re$ for each α rational and each $y \in \mathrm{Y}$. In the general case, for $\alpha \in \mathbb{R}$ and $x \in \mathrm{X}$, there is a sequence of rational numbers (α_n) such that $\alpha_n \to \alpha$ and consequently $\alpha_n x \to \alpha x$. Therefore $\Lambda(\alpha_n x) = \alpha_n \Lambda(x) \to \alpha\Lambda(x)$. Since Λ has a closed graph, we conclude that $\Lambda(\alpha x) = \alpha\Lambda(x)$ and $\Re$ is a closed subspace of Y. So Λ is linear and by the classical Closed Graph Theorem, Λ is continuous. □

As an application of above result, we remove the homogeneity hypothesis in the theorem of Hellinger and Toeplitz:

Theorem 4. *In a real Hilbert space H, every additive symmetric operator $\Lambda : H \longrightarrow H$ is linear and continuous.*

The proof of this Theorem uses the following result.

Theorem 5. (cf. [5], Theorem 5.1) *Let* X *and* Y *be real Frechet spaces and $\Lambda : \mathrm{X} \longrightarrow \mathrm{Y}$ additive. If $y^*\Lambda x_n \to 0$, for each $y^* \in \mathrm{Y}^*$, whenever $x_n \to 0$, Λ is linear and bounded.*

Proof of Theorem 5. If $x_n \to x$ and $\Lambda(x_n) \to y$, then for each $y^* \in \mathrm{Y}$,

$$y^*\Lambda(x_n - x) \to 0 \Rightarrow \; y^*(y) = y^*\Lambda(x).$$

Since this happens for each $y^* \in \mathrm{Y}^*$ and Y is a Frechet space we have that $y = \Lambda x$. Therefore Λ has a closed graph and so Λ is continuous. □

Proof of Theorem 4. Suppose $\|x_n\| \to 0$. Then the Cauchy-Schwartz inequality implies that $\langle x_n, x\rangle \to 0$ for each $x \in H$. Hence for each $y \in H$ we have

$$\langle \Lambda y, x_n\rangle = \langle y, \Lambda x_n\rangle = \langle \Lambda x_n, y\rangle \to 0\,,$$

which implies $\Lambda x_n \xrightarrow{w} 0$, and by Theorem 5, Λ is linear and bounded. □

Remark to Theorem 5. The hypothesis Y is a Frechet space has been imposed in order to guarantee the existence of enough linear functionals in Y^* separating points of Y. A non-Frechet space with enough such a functionals can be found in [4].

3. The Finite Dimensional Case

In this section we extend to the finite dimensional case the Frechet Theorem mentioned in the introduction. In order to do that, we state the Steinhauss Theorem in the following way.

Theorem 6. (see Steinhauss, [4]) *Let G be a locally compact Hausdorff topological group with Haar measure μ. If E is Borel measurable with $\mu(E) > 0$, then* $\mathrm{int}(E - E) \neq \emptyset$.

Proof. It can be obtained by slight modifications of Theorem 3.1 of [3]. □

Theorem 7. (Frechet) *Let* X *and* Y *be real finite dimensional topological vector spaces and* $\Lambda : \mathrm{X} \longrightarrow \mathrm{Y}$ *additive and measurable. Then* Λ *is continuous and consequently linear.*

Proof. X being finite dimensional, it is locally compact. Furthermore both X and Y are normable because they are finite dimensional. Let $\Lambda : \mathrm{X} \longrightarrow \mathrm{Y}$ be additive and Borel measurable. Then for each $n \in \mathbb{N}$, the set

$$A_n = \{x : \|\Lambda x\| \leq n\}$$

is Borel measurable with $\mathrm{X} = \bigcup_{n=1}^{\infty} A_n$. This means that there is $n \in \mathbb{N}$ such that $\mu(A_n) > 0$ regarding the Haar measure; consequently, $\mathrm{int}(A_n - A_n) \neq \emptyset$. Since Λ is bounded on $\mathrm{int}(A_n - A_n)$, Corollary 2 implies that Λ is continuous and so linear. □

We finish this section with the following characterization of linear operator in finite dimensional spaces.

Theorem 8. *Let* X *and* Y *be real finite dimensional topological vector spaces and* $\Lambda : \mathrm{X} \longrightarrow \mathrm{Y}$ *an additive operator. The following statements are equivalent:*

(a) Λ *is linear.*

(b) Λ *is continuous.*

(c) Λ *is Borel measurable.*

(d) $\Lambda(\mathrm{X})$ *is closed.*

(e) $\Lambda(\mathrm{X})$ *is a vector subspace of* X.

Proof. (a)$\Leftrightarrow$ (b). In finite dimensional spaces, every linear operator is continuous. The other direction is clear.

(b)$\Leftrightarrow$ (c). One direction is clear. The other one is Frechet Theorem.

(a)$\Leftrightarrow$ (d). If Λ is linear, then $\Lambda(\mathrm{X})$ is a subspace of Y; hence $\Lambda(\mathrm{X})$ is closed because it is a finite dimensional linear space.

On the other hand, if $\Lambda(\mathrm{X})$ is closed in Y, then

$$G_r\Lambda = \{(x, \Lambda(x)) : x \in \mathrm{X}\}$$

is a closed subset of the Banach space $\mathrm{X} \times \mathrm{Y}$ with the norm

$$\|(x, y)\| = \|x\| + \|y\|,$$

which means that Λ has closed graph. This implies Λ continuous and consequently linear.

(a)⇔(e) It is contained in the proof of (a)⇔(d). □

4. Counter Examples

In this section we provide several examples to show the existence of additive functions which violate the conclusions of the last theorem of foregoing section. We also show by mean of examples that hypothesis of metrizability cannot be omited in our first section; incidently, the same example shows that it is possible to find non continuous additive measurable operators in infinite dimensional spaces.

Example 1. (see [3]) Consider the set of rational numbers $\mathbb{Q}$ as vector space on $\mathbb{Q}$ and define $\Lambda : \mathbb{Q} \longrightarrow \mathbb{Q}$ by $\Lambda(1) = a \neq 0$, and $\Lambda(\alpha) = \alpha a$ for $\alpha \in \mathbb{Q}$. Λ is clearly linear.

We now denote by $\mathbb{R}_{\mathbb{Q}}$ the set of set of real numbers regarded as a vector space on $\mathbb{Q}$, and we extended $\{1\}$ to a Hamel basis $\{r_i\}_{i\in I}$ of $\mathbb{R}_Q$. Let us extended $\Lambda : \mathbb{R}_{\mathbb{Q}} \longrightarrow \mathbb{R}_{\mathbb{Q}}$ in such a way that $\Lambda(r_{i_0}) = 0$ for some $i_0 \in I$, and $\Lambda(r_i)$ is defined arbitrarily for $i \neq i_o$. For $x \in \mathbb{R}$ with $x = \sum x_i r_i (x_i \in \mathbb{Q})$, we define

$$\Lambda(x) = \sum x_i \lambda(r_i).$$

Clearly, Λ is linear on $\mathbb{R}_{\mathbb{Q}}$.

We now consider $\mathbb{R}$ as a real vector space. The function Λ is additive but not linear; on the one hand $\frac{1}{r_{i_0}} f(r_{i_0}) = 0$. On the other hand,

$$f\left(\frac{1}{r_{i_0}} \cdot r_{i_0}\right) = f(1) \neq 0\,,$$

which shows that example violates all conclusion of the last theorem of former section.

Example 2. Let X be an infinite dimensional Banach space with norm $\|\cdot\|$ and weak topology τ. It is well known that bounded sets are norm bounded. So the identity $i : (\mathrm{X}, \tau) \longrightarrow (\mathrm{X}, \|\cdot\|)$ is bounded but it is not continuous. On the other hand if X is reflexive, it is weakly sequentially complete [2], which means that the metrizability condition can not be omited in the first theorem.

We also notice that $i : (\mathrm{X}, \tau) \longrightarrow (\mathrm{X}, \|\cdot\|)$ is weakly continuous and consequently weakly measurable. Furthermore, it is known (see [1]) that the notions of weak and strong measurability coincide for separable Banach spaces. So, in this case our example exhibits a linear strongly measurable function which is not continuous; which means that Frechet Theorem does not hold in infinite dimensional spaces.

It is natural to ask whether the results shown in this paper can be extended to complex spaces. The answer is not as shows the following example.

Example 3. Let $\mathbb{C}$ be the set of complex numbers and

$$\begin{aligned} \lambda \quad : \quad & \mathbb{C} \longrightarrow \mathbb{C}\,, \\ & \lambda \longrightarrow \overline{\lambda}\,, \end{aligned}$$

then Λ is additive continuous but not linear.

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