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THE FACTORIALITY OF THE LOCAL RING OF GERMS OF QUASI-ANALYTIC FUNCTIONS ON A p-NORMED BANACH SPACE, $0 < p < 1$

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Abstract: Let $(V, \| \, \|)$ be a p-Banach space over $\mathbb{R}$ or $\mathbb{C}$ which is separated by its topological dual V'. Here we prove that the local ring of all germs at 0 of quasi-analytic functions in a neighborhood of 0 in V is a unique factorization domain.

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1. The Statements

Let $\mathbb{K}$ be either the real field or the complex field. Let $(V, \| \, \|)$ be a p-Banach space over $\mathbb{K}$ which is separated by its topological dual V'. Let $\mathcal{Q}_{V,0}$ the germ at 0 of all quasi-analytic functions in the sense of [1], Definition 49, (also called Bayoumi analytic maps) defined in a neighborhood of 0 in V. Here we will study an algebraic property (the factoriality) of the local ring $\mathcal{Q}_{V,0}$ extending the classical case of a Banach space proved in [2], Theorem I.1.4.4. As in the classical case we prove the following results.

Theorem 1. *Let $(V, \| \, \|)$ be a p-Banach space over $\mathbb{K}$ which is separated by its topological dual V'. Let $\mathbb{K}\{V\}$ denote the local ring of all convergent power series on V. Then $\mathbb{K}\{V\}$ is a unique factorization domain.*

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Corollary 1. *Let $(V, \| \|)$ be a p-Banach space over $\mathbb{K}$ which is separated by its topological dual V'. Then $\mathcal{Q}_{V,0}$ is a unique factorization domain.*

As in the classical case done in [2], Chapter I, Theorem 1 will easily follow from the following form of Weierstrass Preparation Theorem.

Theorem 2. *Let $(V, \| \|)$ be a p-Banach space over $\mathbb{K}$ which is separated by its topological dual V'. Fix $g \in \mathbb{K}\{V\}$ such that $g(0) = 0$ and $g \neq 0$. Call $n > 0$ the order of g, i.e. the order of the first non-zero term of its Taylor expansion. There is a topological decomposition $V = A \oplus \mathbb{K}e$, $e \neq 0$, a distinguish polynomial $P \in \mathbb{K}\{A\}[z]$ with respect to this decomposition such that P has degree n in z and $g = hP$ with $h \in \mathbb{K}\{V\}$ and $h(0) \neq 0$, i.e. h invertible in $\mathbb{K}\{V\}$. For every $f \in \mathbb{K}\{V\}$ there are uniquely determined $q \in \mathbb{K}\{V\}$ and $r \in \mathbb{K}\{A\}[z]$ such that $f = qg + r$ and r has degree at most $n-1$ in z.*

2. The Proofs

Proof of Theorem 2. Choose $e \in V\backslash\{0\}$ such that $g_n(e) \neq 0$, where g_n is the n-degree homogeneous polynomial appearing in the Taylor expansion of g. It is easy to check that for any p-Banach space E' separates E if and only if every one-dimensional linear subspace $\mathbb{K}e$, $e \neq 0$, of E has a topological supplement. Choose any topological supplement A of $\mathbb{K}e$ in V. As in [2], pp. 11-12, everything is reduced to check the convergence of some scalar formal series. Repeat the proof of [2], pp. 12–13, using the p-norm $\| \|$ instead of the norm used in [2]. □

Proof of Theorem 1. With the notation of Theorem 1 and Theorem 2 the ring $\mathbb{K}\{A\}$ is completely integrally closed in its quotient field. Hence the proof of [2], pp. 15–16, works verbatim. □

Proof of Corollary 1. By the very definition of quasi-analyticity given in [1], Definition 49, we have $\mathcal{Q}_{V,0} \cong \mathbb{K}\{V\}$ and hence the result is just a reformulation of Theorem 1. □

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