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A CONTRIBUTION TO RESULTS ON RANDOM PARTITIONS OF THE SEGMENT

Milena Bieniek[1 §], Dominik Szynal[2]

[1]Department of Statistics and Economics
Faculty of Economics
Maria Curie–Skłodowska University
Pl. Marii Curie-Skłodowskiej 5, 20-031, Lublin, POLAND
e-mail: milena@ramzes.umcs.lublin.pl

[2]Institute of Mathematics
Maria Curie–Skłodowska University
Pl. Marii Curie-Skłodowskiej 1, 20-031, Lublin, POLAND
e-mail: szynal@golem.umcs.lublin.pl

Abstract: Let $X_1, \ldots, X_{k-1}$ be a sequence of independent random variables uniformly distributed on the interval $[0, 1]$ and let Δ_k and δ_k denote the lengths of the greatest and the smallest interval of a partition of $[0, 1]$ by the points $X_1, \ldots, X_{k-1}$. In the paper we give the moments and the central moments for Δ_k, δ_k, δ_k/Δ_k and $\Delta_k - \delta_k$. The moments for $D_k = \max\{X_1, \ldots, X_k\}$ and for the quasi-range $W_{k,r}$ from the exponential distribution are also discussed. Moreover, we investigate the asymptotic behaviour of $(D_t^*/\log t)$, where D_t^* is the diameter of a partition of the interval $[0, t]$ by renewal moments of a standard Poisson process.

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§Correspondence author

1. Introduction

Let $X_1, \ldots, X_{k-1}$ be a sequence of independent uniformly distributed random variables on the interval $[0,1]$. Here Δ_k and δ_k stand for the length of the greatest and the smallest interval, respectively, obtained in a partition of the segment $[0,1]$ by the random points $X_1, \ldots, X_{k-1}$. When (X_k) is a sequence of independent random variables with probability density function $f(x) = e^{-x}$ then we write $D_k = \max\{X_1, \ldots, X_k\}$ and $d_k = \min\{X_1, \ldots, X_k\}$.

We refer to results presented in Darling [5], Pyke [18], Jajte [14], Kopocinska and Kopocinski [16] and Kaniowski [15]. The probability density function of (δ_k, Δ_k) was found in Darling [5]. The convergence in probability of $(k\Delta_k/\log k)$ to 1 was proved in Jajte [14]. The connection between the Poisson process and a random partition of the segment was studied in Pyke [18], Feller [7] and in Kaniowski [15], where it was shown that $D_t^*/\log t \xrightarrow{P} 1$ with D_t^* being the diameter of a partition of the interval $[0,t]$ by renewal moments of a standard Poisson process. Moreover, in Kaniowski [15] the distribution function of δ_k/Δ_k was established and it was noted that δ_k/Δ_k and d_k/D_k are identically distributed for $k \geq 3$.

We discuss the following problems connected with the above quantities. In Section 2 we give formulae for the moments of Δ_k, δ_k and δ_k/Δ_k. Also, we determine the distribution of the range $R_k = \Delta_k - \delta_k$ and its moments. The moments and the central moments of D_k, $W_{k,r} = X_{k-r:k} - X_{r+1:k}$ and recurrence relations for the moments are discussed in Section 3.

Let N_t, $t \geq 0$, be a standard Poisson process. Write $D_t^* = \max\{\sigma_1, \sigma_2 - \sigma_1, \ldots, t - \sigma_{N_t}\}$ and $d_t^* = \min\{\sigma_1, \sigma_2 - \sigma_1, \ldots, t - \sigma_{N_t}\}$, $t > 0$, where $\sigma_1, \sigma_2, \ldots$ are successive renewal moments. In Section 4 we investigate the asymptotic properties of functions of Δ_k, δ_k, D_k and d_k. Moreover, we give the moments for D_t^* and d_t^* and we show that $D_t^*/\log t \xrightarrow{L_p} 1$ and $d_t^* \log t \xrightarrow{L_p} 0$. Finally, we prove that $(D_t^*/\log t)$ and $(d_t^* \log t)$ converge almost surely to one and to zero, respectively.

2. The Moments of Functions of Maximal and Minimal Uniform Spacings

Let $X_1, \ldots, X_{k-1}$, $k \geq 2$, be independent random variables uniformly distributed on $[0,1]$. The corresponding order statistics are $0 \leq X_{1:k-1} \leq X_{2:k-1} \leq \ldots \leq X_{k-1:k-1} \leq 1$. Set $X_{0:k-1} = 0$, $X_{k:k-1} = 1$. The distances between the order statistics

$$Y_{l+1} = X_{l+1:k-1} - X_{l:k-1}, \quad 0 \le l \le k-1,$$

are called "spacings" (or "gaps"). The quantities Δ_k and δ_k are defined by

$$\Delta_k = \max\{Y_1, Y_2, \dots, Y_k\} \quad \text{and} \quad \delta_k = \min\{Y_1, Y_2, \dots, Y_k\}.$$

Proposition 1. *The distribution function of Δ_k for $k \ge 2$ is*

$$F_{\Delta_k}(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1-jx)_+^{k-1} \quad \text{(cf. Feller [7], p. 28)},$$

where $y_+ = \max\{0, y\}$. The rate hazard function, i.e. $r(t) = \frac{f(t)}{1-F(t)}$ is

$$r(t) = \frac{(k-1)\sum_{j=1}^{k-i}(-1)^{j+1}\binom{k}{j} j(1-jt)^{k-2}}{\sum_{j=1}^{k-i}(-1)^{j+1}\binom{k}{j}(1-jt)^{k-1}},$$

for $\frac{\min\{i,1\}}{k+1-i} < t < \frac{1}{k-i}$, $i = 0, 1, \dots k-1$.

Now we give some moment properties of Δ_k. It was shown in Jajte [14] that

$$E\Delta_k^r = \frac{r!}{(k)_r}\, \gamma_k^r, \tag{1}$$

where

$$\gamma_k^r = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} \frac{1}{i^r}, \quad r \in \mathbb{N}, \tag{2}$$

and $(x)_r$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_r := \frac{\Gamma(x+r)}{\Gamma(r)} = \begin{cases} 1 & r = 0, \\ x(x+1)\dots(x+r-1) & r \in \mathbb{N}, \end{cases}$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the Gamma function.

Let $H_k^{(r)}$, $r \ge 1$, be the harmonic number of order r, i.e.

$$H_k^{(r)} = \sum_{i=1}^{k} \frac{1}{i^r}, \quad r \ge 1 \quad \text{(cf. Graham et al [12], Chapter 6.3).} \tag{3}$$

In what follows r is a positive integer. For $r > 1$ we use the Riemman's $\zeta(r)$ and $\zeta(r,q)$ functions, i.e.

$$\zeta(r) = \sum_{i=1}^{\infty} \frac{1}{i^r}, \quad \text{and} \quad \zeta(r;q) = \sum_{i=0}^{\infty} \frac{1}{(q+i)^r} = \zeta(r) - H_{q-1}^{(r)} \tag{4}$$

(cf. Gradstein and Ryzyk [11], 7.422.1., 7.421.1.). The quantity γ_k^r can be represented by

$$\gamma_k^1 = \sum_{i=1}^{k} \frac{1}{i}, \quad \gamma_k^r = \sum_{1 \le i_1 \le \ldots \le i_r \le k} \frac{1}{i_1 \ldots i_r}, \quad r = 2, 3, \ldots \quad \text{(Jajte [14]).} \tag{5}$$

We write the numbers γ_k^r in (5) using the harmonic numbers.

Define $a_r \equiv a_r(\alpha_1, \ldots, \alpha_k)$, $r = 1, 2, \ldots, k$, the *elementary symmetric function* of weight r, and $h_r \equiv h_r(\alpha_1, \ldots, \alpha_k)$, $r = 1, 2, \ldots$, the so called *homogeneous product sum symmetric function* of weight r (cf. Riordan [20], pp. 47, 93) by the equations

$$\begin{aligned} &1/(1-\alpha_1 x)(1-\alpha_2 x) \ldots (1-\alpha_k x) \\ &= 1/(1 - a_1 x + a_2 x^2 + \ldots + (-1)^k a_k x^k) = 1 + h_1 x + h_2 x^2 + \ldots + h_r x^r + \ldots . \end{aligned}$$

For instance we have

$$\begin{aligned} a_1(\alpha_1, \ldots, \alpha_n) &= \alpha_1 + \alpha_2 + \ldots + \alpha_k, \\ a_2(\alpha_1, \ldots, \alpha_k) &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_{n-1}\alpha_k \, , \end{aligned}$$

and

$$\begin{aligned} h_1(\alpha_1, \ldots, \alpha_k) &= \alpha_1 + \alpha_2 + \ldots + \alpha_k, \\ h_2(\alpha_1, \ldots, \alpha_k) &= \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_k^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_{k-1}\alpha_k). \end{aligned}$$

The generating function of the sequence $\{\gamma_k^r, \, r \ge 0\}$ in (5) has the form

$$G_k(z) = \sum_{r=0}^{\infty} \gamma_k^r z^r = \frac{1}{(1-z)\left(1-\frac{z}{2}\right) \ldots \left(1-\frac{z}{k}\right)}$$

(cf. Graham et al [12], Chapter 7, problems). Thus

$$\gamma_k^r = h_r\left(1, \frac{1}{2}, \ldots, \frac{1}{k}\right). \tag{6}$$

It is known that $h_r(\alpha_1, \ldots, \alpha_k)$ satisfies

$$r! h_r(\alpha_1, \ldots, \alpha_k) = C_r(s_1, \ldots, s_r) \quad \text{(cf. Riordan [20], p. 119)}, \tag{7}$$

where s_i denotes the so called *power sum symmetric function* given by

$$s_i = \sum_{j=1}^{k} \alpha_j^i, \tag{8}$$

and C_r is the so called *cycle indicator* of the symmetric group defined by

$$C_r(s_1, \ldots, s_r) = r! \sum_{a_1+2a_2+\ldots+ra_r=r} \prod_{i=1}^{r} \frac{(s_i)^{a_i}}{i^{a_i} a_i!}, \quad \text{(cf. Riordan [20], p. 68)}. \tag{9}$$

The sum in (9) is over all non-negative integers a_i, $1 \leq i \leq r$, such that $a_1 + 2a_2 + \ldots + ra_r = r$, or equivalently, over all partitions of k. For instance,

$$C_1(s_1) = s_1, \quad C_2(s_1, s_2) = s_1^2 + s_2, \quad C_3(s_1, s_2, s_3) = s_1^3 + 3s_1 s_2 + 2s_3,$$
$$C_4(s_1, s_2, s_3, s_4) = s_1^4 + 6s_1^2 s_2 + 3s_2^2 + 8s_1 s_3 + 6s_4$$

(cf. Riordan [20], p. 69). Letting in (8)

$$\alpha_j = \frac{1}{j}, \quad 1 \leq j \leq k,$$

we write s_r, $r \geq 1$, as the harmonic number $H_k^{(r)}$ ((3)). Combining (6), (7), (8) and (9) we see that the quantity γ_k^r can be written as

$$\gamma_k^r = \sum_{a_1+2a_2+\ldots+ra_r=r} \prod_{i=1}^{r} \frac{\left(H_k^{(i)}\right)^{a_i}}{i^{a_i} a_i!}, \tag{10}$$

where a_i, $1 \leq i \leq r$, are non-negative integers. Moreover, one can see using $G_k(z)$ that

$$\gamma_k^{r+1} = \frac{1}{r+1} \sum_{j=0}^{r} H_k^{(j+1)} \gamma_k^{r-j}, \quad r = 0, 1, 2 \ldots \tag{11}$$

(cf. Bieniek and Szynal [3]).

Now putting (10) into (1) we see that the r-th moment of Δ_k is

$$E\Delta_k^r = \frac{r!}{(k)_r} \sum_{a_1+2a_2+\ldots+ra_r=r} \prod_{i=1}^{r} \frac{\left(H_k^{(i)}\right)^{a_i}}{i^{a_i} a_i!}. \tag{12}$$

The central moment $E(\Delta_k - E\Delta_k)^r$ is given by

$$E(\Delta_k - E\Delta_k)^r = \sum_{m=0}^{r} \frac{\left(H_k^{(1)}\right)^{r-m}}{k^r} \sum_{j=m}^{r} \binom{r}{j} \frac{(-1)^{r-j} j! k^j}{(j-m)!(k)_j}$$
$$\sum_{2a_2+\ldots+ma_m=m} \prod_{i=2}^{m} \frac{\left(H_k^{(i)}\right)^{a_i}}{i^{a_i} a_i!} \quad \text{(cf. Bieniek and Szynal [3])}.$$

Here we observe that $E\Delta_k^r$ and $E(\Delta_k - E\Delta_k)^r$ can be represent in an alternative form using the relation between the harmonic numbers and Psi (or Digamma) function $\psi(x) := \psi^{(0)}(x)$ defined by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) \quad \text{(cf. Gradstein and Ryzyk [11])}$$

and the derivatives of Psi (or Polygamma) functions $\psi^{(r)}(x)$ defined by

$$\psi^{(r)}(x) = \frac{d^{r+1}}{dx^{r+1}} \log \Gamma(x) = \frac{d^r}{dx^r} \psi(x), \quad r = 0, 1, 2 \ldots$$

(cf. Gradstein and Ryzyk [11]). It is known that

$$H_k^{(1)} = \gamma + \psi(k+1) \quad \text{(cf. Hansen [13], (5.13.5))}, \tag{13}$$

where $\gamma := -\psi(1) = 0,5772156649\ldots$ is the Euler-Mascheroni constant and

$$\psi^{(r-1)}(k+1) = (-1)^r (r-1)! \sum_{i=0}^{\infty} \frac{1}{(k+1+i)^r}$$

(cf. Gradstein and Ryzyk [11], 6.356.), which by (4) can be written as

$$\psi^{(r-1)}(k+1) = (-1)^r (r-1)! \zeta(r, k+1).$$

Hence

$$\zeta(r, k+1) = \frac{(-1)^r}{(r-1)!} \psi^{(r-1)}(k+1) \tag{14}$$

and

$$\zeta(r) - H_k^{(r)} = \frac{(-1)^r}{(r-1)!}\psi^{(r-1)}(k+1),$$

which implies that for $r \geq 2$

$$H_k^{(r)} = \zeta(r) - \zeta(r, k+1) \quad \text{(cf. Hansen [13], (6.2.3))},$$

or

$$H_k^{(r)} = \frac{(-1)^r}{(r-1)!}\left(\psi^{(r-1)}(1) - \psi^{(r-1)}(k+1)\right) \qquad \text{(cf. Hansen [13], (6.2.4))}, \quad (15)$$

as

$$\zeta(r) = \frac{(-1)^r}{(r-1)!}\psi^{(r-1)}(1) \quad \text{(cf. Hansen [13], (6.2.6))}. \quad (16)$$

Hence the numbers γ_k^r in (10) can be written in the form

$$\gamma_k^r = \sum_{a_1+2a_2+\ldots+ra_r=r} \prod_{i=1}^{r} \frac{(-1)^{ia_i}}{(i!)^{a_i} a_i!}\left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+1)\right)^{a_i}, \quad (17)$$

where $\psi^{(0)}(x) = \psi(x)$, or

$$\gamma_k^r = \sum_{a_1+2a_2+\ldots+ra_r=r} \frac{(\gamma + \psi(k+1))^{a_1}}{a_1!} \prod_{i=2}^{r} \frac{(\zeta(i) - \zeta(i, k+1))^{a_i}}{i^{a_i} a_i!}.$$

Proposition 2. *The r-th moment of Δ_k is given by*

$$E\Delta_k^r = \frac{r!}{(k)_r} \prod_{i=1}^{r} \frac{(-1)^{ia_i}}{(i!)^{a_i} a_i!}\left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+1)\right)^{a_i}, \quad (18)$$

and

$$E(\Delta_k - E\Delta_k)^r = \frac{1}{k^r} \sum_{m=0}^{r} (\gamma + \psi(k+1))^{r-m} \sum_{j=m}^{r} \binom{r}{j} \frac{(-1)^{r-j} j!}{(m-j)!} \frac{k^j}{(k)_j}$$
$$\sum_{2a_2+\ldots+ma_m=m} \prod_{i=2}^{m} \frac{(-1)^{ia_i}}{(i!)^{a_i} a_i!}\left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+1)\right)^{a_i}. \quad (19)$$

Remark 3. In terms of Zeta function

$$E\Delta_k^r = \frac{r!}{(k)_r} \sum_{a_1+2a_2+\ldots+ra_r=r} \frac{(\gamma+\psi(k+1))^{a_1}}{a_1!} \prod_{i=2}^{r} \frac{(\zeta(i)-\zeta(i,k+1))^{a_i}}{i^{a_i} a_i!},$$

and

$$E(\Delta_k - E\Delta_k)^r = \frac{1}{k^r} \sum_{m=0}^{r} (\gamma+\psi(k+1))^{r-m} \sum_{2a_2+\ldots+ma_m=m} \prod_{i=2}^{m} \frac{(\zeta(i)-\zeta(i,k+1))^{a_i}}{i^{a_i} a_i!} \sum_{j=m}^{r} \binom{r}{j} \frac{(-1)^{r-j} j!}{(m-j)!} \frac{k^j}{(k)_j},$$

respectively.

Corollary 4. *The mean residual life function of Δ_k, i.e. $m(t) = E(X - t|X > t) = \int_0^\infty \frac{(x-t)f(x)dx}{1-F(t)}$ is*

$$m(t) = \frac{\sum_{j=1}^{k-i}(-1)^{j+1}\binom{k}{j}\left\{\left(t-\frac{1}{k-i}\right)\left(1-\frac{j}{k-i}\right)^{k-1} - \frac{\left(1-\frac{j}{k-i}\right)^k}{jk} + \frac{(1-jt)^k}{jk}\right\}}{\sum_{j=1}^{k-i}(-1)^{j+1}\binom{k}{j}(1-jt)^{k-1}},$$

for $\frac{\min\{i,1\}}{k+1-i} < t < \frac{1}{k-i}$, $i = 0, 1, \ldots k-1$, *and the expectation, the variance, the third and fourth central moments of Δ_k are as follows*

$$E\Delta_k = \frac{\gamma+\psi(k+1)}{k}, \qquad \sigma^2\Delta_k = \frac{\frac{\pi^2}{6}k - k\psi'(k+1) - (\psi(k+1)+\gamma)^2}{k^2(k+1)},$$

$$E(\Delta_k - E\Delta_k)^3 = \frac{1}{(k)_3}\left\{\frac{4}{k^2}(\psi(k+1)+\gamma)^3 - \frac{6}{k}(\psi(k+1)+\gamma)\left(\frac{\pi^2}{6} - \psi'(k+1)\right) - \psi''(1) + \psi''(k+1)\right\},$$

$$E(\Delta_k - E\Delta_k)^4 = \frac{3(k-6)(\psi(k+1)+\gamma)^4}{k^3(k)_4} + \frac{\frac{\pi^4}{15} - \psi^{(3)}(k+1)}{(k)_4} - \frac{6(k-6)(\psi(k+1)+\gamma)^2\left(\frac{\pi^2}{6} - \psi'(k+1)\right)}{k^2(k)_4} + \frac{3\left(\frac{\pi^2}{6} - \psi'(k+1)\right)^2}{(k)_4} + \frac{12(\psi(k+1)+\gamma)\left(\psi''(1) - \psi''(k+1)\right)}{k(k)_4}.$$

Skewness and kurtosis are

$$\gamma = \frac{\frac{4}{k}(\psi(k+1)+\gamma)^3 - 6(\psi(k+1)+\gamma)\left(\frac{\pi^2}{6}-\psi'(k+1)\right) - k(\psi''(1)-\psi''(k+1))}{(k+1)^{-1/2}(k+2)\left(\frac{\pi^2}{6}k - k\psi'(k+1) - (\psi(k+1)+\gamma)^2\right)^{3/2}},$$

$$\kappa = \frac{k^2(k+1)}{(k+2)(k+3)\left(\frac{\pi^2}{6}k - k\psi'(k+1) - (\psi(k+1)+\gamma)^2\right)^2} \left\{\frac{3(k-6)(\psi(k+1)+\gamma)^4}{k^3} - \frac{6(k-6)(\psi(k+1)+\gamma)^2\left(\frac{\pi^2}{6}-\psi'(k+1)\right)}{k^2} + \frac{\pi^4}{15} - \psi^{(3)}(k+1) + 3\left(\frac{\pi^2}{6}-\psi'(k+1)\right)^2 + \frac{12(\psi(k+1)+\gamma)\left(\psi''(1)-\psi''(k+1)\right)}{k}\right\}.$$

Corollary 5. *For sufficiently large k*

$$E(\Delta_k - E\Delta_k)^{2r} \le \frac{A(r)}{(k)_{2r}} \quad \text{(cf. Bieniek and Szynal [3])},$$

where

$$A(r) = (2r)!\sum_{j=0}^{2r}\sum_{2a_2+\ldots+ja_j=j}\prod_{i=2}^{j}\frac{(-1)^{ia_i}\psi^{(i-1)}(1)^{a_i}}{(i!)^{a_i}a_i!} = (2r)!\sum_{j=0}^{2r}\sum_{2a_2+\ldots+ja_j=j}\prod_{i=2}^{j}\frac{\zeta(i)^{a_i}}{i^{a_i}a_i!}.$$

Now we can give the recurrence relation for γ_k^r in terms of Psi function using (11).

Lemma 6. *The numbers $\{\gamma_k^r,\ r \in \mathbb{N}\}$ satisfy the recurrence equation*

$$\gamma_k^{r+1} = \frac{1}{r+1}\sum_{j=0}^{r}\frac{(-1)^{j+1}}{j!}\left(\psi^{(j)}(1) - \psi^{(j)}(k+1)\right)\gamma_k^{r-j}, \qquad (20)$$

where $\gamma_k^1 = \gamma + \psi(k+1)$.

Remark 7. Using (16) we have the following recurrence

$$\gamma_k^{r+1} = \frac{\gamma+\psi(k+1)}{r+1}\gamma_k^r + \frac{1}{r+1}\sum_{j=1}^{r}\left(\zeta(j+1)-\zeta(j+1,k+1)\right)\gamma_k^{r-j},$$

where $\gamma_k^1 = \gamma + \psi(k+1)$.

The recurrence formula (20) for γ_k^r leads to the following result

Proposition 8. *The moments $E\Delta_k^r$ satisfy the recurrence relation*

$$E\Delta_k^{r+1} = \sum_{j=0}^{r}\frac{r!(-1)^{j+1}}{(r-j)!j!}\frac{\psi^{(j)}(1)-\psi^{(j)}(k+1)}{(k+r-j)_{j+1}}E\Delta_k^{r-j},$$

where $E\Delta_k = \frac{1}{k}(\gamma+\psi(k+1))$.

Remark 9. The formula (16) allows us to give the following recurrence in terms of Zeta function

$$E\Delta_k^{r+1} = \frac{\gamma+\psi(k+1)}{k+r}E\Delta_k^r + \sum_{j=1}^{r}\frac{r!}{(r-j)!}\frac{\zeta(j+1)-\zeta(j+1,k+1)}{(k+r-j)_{j+1}}E\Delta_k^{r-j},$$

where $E\Delta_k = \frac{1}{k}(\gamma+\psi(k+1))$.

Now we present similar results for δ_k.

Proposition 10. *The probability distribution function of δ_k is given by*

$$F_{\delta_k}(x) = 1-(1-kx)_+^{k-1}, \quad x>0 \quad \text{(cf. Darling [5]).} \tag{21}$$

The rate hazard function is given by

$$r(t) = \frac{k(k-1)}{1-kt}, \quad 0<t<\frac{1}{k}.$$

Proposition 11. *The r-th moment of δ_k is given by*

$$E\delta_k^r = \frac{r!}{(k)_r k^r}, \tag{22}$$

and

$$E(\delta_k - E\delta_k)^r = \frac{r!}{k^{2r}}\sum_{j=0}^{r}\frac{(-1)^{r-j}}{(r-j)!}\frac{k^j}{(k)_j}. \tag{23}$$

Proof. The formula (22) follows from the well known equality

$$EX^p = p\int_0^\infty y^{p-1}P(X \geq y)dy, \quad p > 0, \quad \text{(cf. Serfling [21])}. \tag{24}$$

To prove (25) we use the binomial formula

$$E\left(\delta_k - E\delta_k\right)^r = \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}E\delta_k^j(E\delta_k)^{r-j}\,,$$

which by (22) gives (23). □

Corollary 12. *The mean residual life function of δ_k is*

$$m(t) = \frac{1-kt}{k^2}, \quad 0 < t < \frac{1}{k},$$

and the expectation, the variance, the third and fourth central moments of δ_k are as follows

$$E\delta_k = \frac{1}{k^2}, \quad \sigma^2\delta_k = \frac{k-1}{k^3(k)_2}, \quad E(\delta_k - E\delta_k)^3 = \frac{2(k-1)(k-2)}{k^5(k)_3},$$
$$E(\delta_k - E\delta_k)^4 = \frac{3(k-1)(3k^2-7k+6)}{k^7(k)_4}.$$

Skewness and kurtosis are

$$\gamma = 2\frac{k-2}{k+2}\sqrt{\frac{k+1}{k-1}} \quad \text{and} \quad \kappa = \frac{3(k+1)(3k^2-7k+6)}{(k-1)(k+2)(k+3)}.$$

Corollary 13. *For sufficiently large k*

$$E(\delta_k - E\delta_k)^{2r} = O\left(\frac{1}{k^{4r}}\right). \tag{25}$$

Proof. By (23) we have

$$E(\delta_k - E\delta_k)^{2r} = \frac{(2r)!}{k^{4r}}\sum_{j=0}^{2r}\frac{(-1)^{2r-j}}{(2r-j)!}\frac{k^j}{(k)_j}$$
$$= \frac{(2r)!}{k^{2r}(k)_{2r}}\sum_{j=0}^{2r}\frac{(-1)^j(k+j)_j}{j!k^j} = \frac{(2r)!a(k)}{k^{2r}(k)_{2r}},$$

where $a(k) = \sum_{j=0}^{2r}\frac{(-1)^j(k+j)_j}{j!k^j}$ and $a(k) = O(1)$, which implies (25). □

Now we consider the distribution of the quotient δ_k/Δ_k and the range $\Delta_k - \delta_k$ and their moments. Taking into account that the random variables δ_k/Δ_k and d_k/D_k are identically distributed we get the following result.

Proposition 14. *The distribution function of δ_k/Δ_k, $k \geq 3$, is given by*

$$F_{\delta_k/\Delta_k}(x) = F_{d_k/D_k}(x) = 1 - \frac{xk}{1-x}\frac{(k-1)!}{\left(\frac{x}{1-x}k\right)_k}, \quad x \in (0,1)$$

(cf. Kaniowski [15]).

The rate hazard function of δ_k/Δ_k is

$$r(t) = \frac{(1-t)\left(\frac{t}{1-t}k\right)_k}{t(k-1)!}\sum_{j=1}^{k-1}(-1)^{j+1}\binom{k-1}{j}\frac{j}{((k-j)t+j)^2}, \quad t \in (0,1).$$

Proof. Taking into account that

$$\frac{k!}{(y)_{k+1}} = \sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{1}{y+j}$$

(cf. Graham et al [12], Chapter 6.4) , (26)

we get

$$\begin{aligned} F_{\delta_k/\Delta_k}(x) &= 1 - \frac{kx}{1-x}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\frac{1}{\frac{x}{1-x}k+j} \\ &= 1 - \sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j kx}{(k-j)x+j}. \end{aligned} \tag{27}$$

Hence the probability density function of δ_k/Δ_k is

$$f_{\delta_k/\Delta_k}(x) = \sum_{j=1}^{k-1}\binom{k-1}{j}(-1)^{j+1}\frac{kj}{((k-j)x+j)^2},$$

which gives the formula for $r(t)$. □

Proposition 15. *The r-th moment of δ_k/Δ_k is*

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = E\left(\frac{d_k}{D_k}\right)^r$$
$$= 1 + r\sum_{j=1}^{k-1}(-1)^j\binom{k}{j}\left(\sum_{i=1}^{r}\frac{\left(\frac{-j}{k-j}\right)^{i-1}}{r-i+1} + \left(\frac{-j}{k-j}\right)^r \log\frac{k}{j}\right). \tag{28}$$

Proof. By (24) and (27) we have

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = 1 + r\sum_{j=1}^{k-1}(-1)^j\binom{k-1}{j}k\int_0^1 \frac{x^r}{(k-j)x+j}dx.$$

Using the formula

$$\int_0^1 \frac{x^n dx}{a+bx} = \sum_{i=1}^{n}\frac{(-1)^{i-1}a^{i-1}}{(n-i+1)b^i} + \frac{(-1)^n a^n}{b^{n+1}}\log\frac{a+b}{a}$$

(cf. Gradstein and Ryzyk [11], 2.111.3.), we obtain

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = 1$$
$$+ r\sum_{j=1}^{k-1}(-1)^j\binom{k-1}{j}k\left(\sum_{i=1}^{r}\frac{(-1)^{i-1}j^{i-1}}{(r-i+1)(k-j)^i} + \frac{(-1)^r j^r \log\frac{k}{j}}{(k-j)^{r+1}}\right),$$

which ends the proof. □

To investigate the asymptotic behaviour of $E\left(\frac{\delta_k}{\Delta_k}\right)^r$ as $k \to \infty$ we need another formula for $E\left(\frac{\delta_k}{\Delta_k}\right)^r$.

Proposition 16. *For $k \geq 3$*

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = E\left(\frac{d_k}{D_k}\right)^r = rr!k!\sum_{i=k}^{\infty}\frac{S(i,k)}{(i)_{r+1}}\left(\frac{1}{k}\right)^i, \tag{29}$$

where $S(n,k)$ denote the Stirling numbers of the second kind, or

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = E\left(\frac{d_k}{D_k}\right)^r = rr!\sum_{i=k}^{\infty}\frac{\Delta^k 0^i}{(i)_{r+1}}\left(\frac{1}{k}\right)^i,$$

where Δ denotes the forward difference operator, i.e. $\Delta f(x) = f(x+1) - f(x)$, $\Delta^n 0^i = \Delta^n x^i|_{x=0}$ and $\Delta^n 0^i := \Delta(\Delta^{n-1}0^i)$, $n \geq 2$.

Proof. Using the formula (28) we have

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = 1$$

$$+ r\sum_{j=1}^{k-1}(-1)^j\binom{k}{j}\left(\sum_{i=1}^{r}\frac{(-1)^{i-1}j^{i-1}}{(r-i+1)(k-j)^{i-1}} + \frac{(-1)^r j^r}{(k-j)^r}\log\frac{k}{j}\right) = 1$$

$$+ r\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k}{k-j}\left(\sum_{i=1}^{r}\frac{(-1)^{i-1}(k-j)^{i-1}}{(r-i+1)j^{i-1}} + \frac{(-1)^r(k-j)^r}{j^r}\log\frac{k}{k-j}\right)$$

$$= 1 + r\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k}{j}\left(\sum_{i=1}^{r}\frac{(-1)^{i-1}\left(\frac{k}{j}-1\right)^{i-1}}{r-i+1}\right.$$

$$\left. + \frac{(-1)^r\left(\frac{k}{j}-1\right)^{r-1}\left(1-\frac{j}{k}\right)}{\frac{j}{k}}\log\frac{1}{1-\frac{j}{k}}\right).$$

By the equality

$$\frac{1-x}{x}\log\frac{1}{1-x} = 1 - \sum_{i=1}^{\infty}\frac{x^i}{i(i+1)}, \quad x^2 < 1$$

(cf. Gradstein and Ryzyk [11], 1.513.5.),
we obtain

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = 1 + r\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k}{j}\left(\sum_{i=1}^{r}\frac{(-1)^{i-1}\left(\frac{k}{j}-1\right)^{i-1}}{r-i+1} + (-1)^r\left(\frac{k}{j}-1\right)^{r-1}\right)$$

$$- (-1)^r r\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k}{j}\left(\frac{k}{j}-1\right)^{r-1}\sum_{i=1}^{\infty}\frac{\left(\frac{j}{k}\right)^i}{i(i+1)}$$

$$= 1 + (-1)^{r+1} r\sum_{i=1}^{r-1}\frac{(-1)^i}{i+1}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k}{j}\left(\frac{k}{j}-1\right)^{r-i-1}$$

$$+ (-1)^{r+1} r\sum_{i=1}^{\infty}\frac{1}{i(i+1)}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k}{j}\left(\frac{k}{j}-1\right)^{r-1}\left(\frac{j}{k}\right)^i.$$

For $r = 1$

$$E\frac{\delta_k}{\Delta_k} = 1 + \sum_{i=1}^{\infty} \frac{(-1)^k}{i(i+1)} \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \left(\frac{j}{k}\right)^i$$
$$= \sum_{i=1}^{\infty} \frac{\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^i}{i(i+1)k^i} = \sum_{i=k}^{\infty} \frac{k! S(i,k)}{i(i+1)k^i},$$

as

$$S(n,k) = \begin{cases} \frac{1}{k!} \sum_{r=1}^{k} \binom{k}{r} (-1)^{k-r} r^n, & \text{for } n \geq k, \\ 0, & \text{for } n < k \end{cases} \quad \text{(cf. Comtet [4], p. 204).}$$

For $r > 1$

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = (-1)^{r+1} r \sum_{i=1}^{r-1} \frac{(-1)^i}{i+1} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \left(\frac{k}{j} - 1\right)^{r-i-1}$$
$$+ (-1)^{r+1} r \sum_{i=1}^{\infty} \frac{1}{i(i+1)} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \left(\frac{k}{j} - 1\right)^{r-1} \left(\frac{j}{k}\right)^i.$$

Let

$$\eta_k^m = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \left(\frac{k}{j}\right)^m. \tag{30}$$

By the binomial formula

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = r \sum_{i=1}^{r-1} \frac{1}{i+1} \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} \sum_{j=0}^{r-i-1} \binom{r-i-1}{j} (-1)^j \left(\frac{k}{l}\right)^j$$
$$+ r \sum_{i=1}^{\infty} \frac{1}{i(i+1)} \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \left(\frac{j}{k}\right)^{i-j}$$

and by (30)

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r = r \sum_{i=1}^{r-1} \frac{1}{i+1} \sum_{j=0}^{r-i-1} \binom{r-i-1}{j} (-1)^j \eta_k^j$$

$$+\,r\sum_{i=1}^{\infty}\frac{1}{i(i+1)}\sum_{l=1}^{k}(-1)^{k-l}\binom{k}{l}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{j}\left(\frac{l}{k}\right)^{i-j}$$
$$:= A(r)+B(r),$$

where $A(r)$ and $B(r)$ stand for the first and the second summand, respectively. But

$$B(r)=r\sum_{j=1}^{r-1}\binom{r-1}{j}(-1)^{j}\sum_{i=1}^{j}\frac{1}{i(i+1)}\sum_{l=1}^{k}(-1)^{k-l}\binom{k}{l}\left(\frac{k}{l}\right)^{j-i}$$
$$+\,r\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{j}\sum_{i=j+1}^{\infty}\frac{1}{i(i+1)}\sum_{l=1}^{k}(-1)^{k-l}\binom{k}{l}\left(\frac{l}{k}\right)^{i-j}$$
$$=r\sum_{j=1}^{r-1}\binom{r-1}{j}(-1)^{j}\sum_{i=1}^{j}\frac{1}{i(i+1)}\eta_k^{j-i}$$
$$+\,r\sum_{i=1}^{\infty}\frac{k!S(i,k)}{k^{i}}\left(\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^{j}}{i+j}-\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^{j}}{i+j+1}\right)$$
$$:= C(r)+rr!\sum_{i=1}^{\infty}\frac{k!S(i,k)}{k^{i}(i)_{r+1}},$$

where

$$C(r)=r\sum_{j=1}^{r-1}\binom{r-1}{j}(-1)^{j}\sum_{i=1}^{j}\frac{1}{i(i+1)}\eta_k^{j-i},$$

and (26) was used. Hence

$$E\left(\frac{\delta_k}{\Delta_k}\right)^{r}=A(r)+C(r)+rr!\sum_{i=1}^{\infty}\frac{k!S(i,k)}{k^{i}(i)_{r+1}}\,,$$

and it is enough to show that $A(r)=-C(r)$. The coefficient of η_k^{m} in $A(r)$ is

$$r\sum_{i=1}^{r-1-m}\binom{r-i-1}{m}\frac{(-1)^{m}}{i+1},\quad m=0,\dots,r-2\,, \tag{31}$$

and in $-C(r)$ is

$$\begin{aligned}
&r\sum_{i=m+1}^{r-1}\binom{r-1}{i}\frac{(-1)^{i+1}}{(i-m)(i-m+1)}\\
&=r\sum_{i=1}^{r-1-m}\binom{r-1}{i+m}\frac{(-1)^{i+m+1}}{i(i+1)}\\
&=r\sum_{i=2}^{r-m}\frac{(-1)^{i+m+1}}{i}\left(\binom{r-1}{i+m}+\binom{r-1}{i+m-1}\right)+r(-1)^m\binom{r-1}{m+1}\\
&=(-1)^m r\sum_{i=2}^{r-m}\binom{r}{i+m}\frac{(-1)^{i+1}}{i}+r(-1)^m\binom{r-1}{m+1}.
\end{aligned}$$

Now we see that for $m=0,1,\ldots,r-2$

$$\sum_{i=1}^{r-1-m}\binom{r-i-1}{m}\frac{1}{i+1}=\sum_{i=2}^{r-m}\binom{r}{i+m}\frac{(-1)^{i+1}}{i}+\binom{r-1}{m+1}. \tag{32}$$

If $r=2$ then $m=0$ and (32) holds. Assuming that (32) is true for r we have to show that

$$\sum_{i=1}^{r-m}\binom{r-i}{m}\frac{1}{i+1}=\sum_{i=2}^{r-m+1}\binom{r+1}{i+m}\frac{(-1)^{i+1}}{i}+\binom{r}{m+1}.$$

For the left-hand side of the above equality we have

$$\begin{aligned}
L&=\sum_{i=1}^{r-m}\frac{\binom{r-i}{m}}{i+1}=\sum_{i=1}^{r-1-m}\frac{\binom{r-i}{m}}{i+1}+\frac{1}{r-m+1}\\
&=\sum_{i=1}^{r-1-m}\frac{\binom{r-1-i}{m}(r-i)}{(r-i-m)(i+1)}+\frac{1}{r-m+1}\\
&=\frac{r+1}{r+1-m}\sum_{i=1}^{r-1-m}\frac{\binom{r-1-i}{m}}{i+1}+\frac{m}{r+1-m}\sum_{i=1}^{r-1-m}\frac{\binom{r-1-i}{m}}{r-i-m}+\frac{1}{r-m+1},
\end{aligned}$$

and by the induction assumption

$$L = \frac{r+1}{r+1-m}\sum_{i=2}^{r-m}\frac{(-1)^{i+1}\binom{r}{i+m}}{i} + \frac{\binom{r-1}{m+1}(r+1)}{r+1-m} + \frac{m\sum_{i=m+1}^{r-1}\frac{\binom{i-1}{m}}{i-m}+1}{r+1-m}$$

$$= \frac{1}{r+1-m}\sum_{i=2}^{r-m+1}\frac{(-1)^{i+1}\binom{r+1}{i+m}}{i}(r+1-i-m) + \frac{\binom{r-1}{m+1}(r+1)}{r+1-m}$$

$$= \sum_{i=2}^{r-m+1}\frac{(-1)^{i+1}\binom{r+1}{i+m}}{i} + \frac{\sum_{i=0}^{m+1}\binom{r+1}{i}(-1)^{m+1-i}}{r+1-m}$$

$$+ \frac{(-1)^{m+r+1}}{r+1-m}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r+1-i} + \frac{\binom{r-1}{m+1}(r+1)}{r+1-m} + \frac{\sum_{i=m-1}^{r-2}\binom{i}{m-1}}{r+1-m}.$$

Noting that

$$\sum_{l=k}^{n}\binom{l}{k} = \binom{n+1}{k+1} \quad \text{and} \quad \sum_{l=0}^{k}(-1)^{k-j}\binom{n}{l} = \binom{n-1}{k}$$

(cf. Graham et al [12], Chapter 5.1), we get

$$L = \sum_{i=2}^{r-m+1}\binom{r+1}{i+m}\frac{(-1)^{i+1}}{i}$$

$$+ \frac{\binom{r}{m+1}}{r+1-m}\left(\frac{(r+1)(r-m-1)}{r} + \frac{m+1}{r} + 1\right)$$

$$= \sum_{i=2}^{r-m+1}\binom{r+1}{i+m}\frac{(-1)^{i+1}}{i} + \binom{r}{m+1},$$

as required. The proof is complete. □

Corollary 17. *The mean residual life function of* δ_k/Δ_k *is*

$$m(t) = \frac{(1-t)\left(\frac{t}{1-t}k\right)_k}{t(k-1)!} \times \sum_{j=1}^{k-1}(-1)^{j+1}\binom{k-1}{j}\left(\frac{(t-1)j}{k(k-j)} + \frac{j\log\frac{k}{(k-j)t+j}}{(k-j)^2}\right)$$

for $t \in (0,1)$*, and the expectation and the variance are*

$$E\frac{\delta_k}{\Delta_k} = \sum_{i=k}^{\infty} \frac{k!S(i,k)}{i(i+1)k^i} \text{ and } \sigma^2\left(\frac{\delta_k}{\Delta_k}\right) = \sum_{i=k}^{\infty} \frac{k!S(i,k)}{(i)_3 k^i} - \left(\sum_{i=k}^{\infty} \frac{k!S(i,k)}{i(i+1)k^i}\right)^2.$$

Taking into account that $S(i,k) \sim \frac{k^i}{k!}$ as $i \to \infty$ we have the following corollary.

Corollary 18.

$$E\left(\frac{\delta_k}{\Delta_k}\right)^r \sim rr! \sum_{i=k}^{\infty} \frac{1}{(i)_{r+1}} \sim \frac{r!}{k^r}.$$

Now we give the distribution function of the range R_k, i.e. $R_k = \Delta_k - \delta_k$.

Proposition 19. *The distribution function of* R_k *is given by*

$$F_{R_k}(w) = 1 - \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \left(1 - \frac{j}{k}\right) (1 - jw)_+^{k-1}, \quad w \in (0,1). \tag{33}$$

The rate hazard function of R_k *is*

$$r(t) = \frac{(k-1)\sum_{j=1}^{k-i}(-1)^{j+1}\binom{k}{j}\left(1-\frac{j}{k}\right) j(1-jt)^{k-2}}{\sum_{j=1}^{k-i}(-1)^{j+1}\binom{k}{j}\left(1-\frac{j}{k}\right)(1-jt)^{k-1}},$$

for $\frac{\min\{i,1\}}{k+1-i} < t < \frac{1}{k-i}$, $i = 0, 1, \dots k-1$.

Proof. The joint probability density function of (δ_k, Δ_k) has the form

$$f(x,y) = (k-1)(k-2) \sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} j(k-j)(1-(k-j)x - jy)_+^{k-3}$$

(cf. Kaniowski [15]), for $0 < x < y < 1$ and $k \geq 3$. After some evaluations we get for $w \in (0,1)$

$$P(R_k > w) = \int_0^{1-w} \int_{x+w}^{1} f(x,y)\,dy\,dx = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \left(1 - \frac{j}{k}\right) (1 - jw)_+^{k-1},$$

which implies (33). □

Using (24) and (17) we derive the moments of the range R_k.

Proposition 20. *The r-th moment of the range R_k is given by*

$$\begin{aligned} ER_k^r &= \frac{r!}{(k)_r} \sum_{a_1+2a_2+\ldots+ra_r=r} \prod_{i=1}^{r} \frac{(-1)^{ia_i}}{(i!)^{a_i} a_i!} \left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+1)\right)^{a_i} \\ &+ \frac{r!}{k(k)_r} \sum_{a_1+a_2\ldots+(r-1)a_{r-1}=r-1} \prod_{i=1}^{r-1} \frac{(-1)^{ia_i}}{(i!)^{a_i} a_i!} \left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+1)\right)^{a_i}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} &E(R_k - ER_k)^r \\ &= \frac{(\gamma+\psi(k))^r}{k^r} \left\{ (-1)^r (1-r) + \sum_{j=2}^{r} \frac{(-1)^{r-j} r! k^{j-1} \left(k\gamma_k^j - \gamma_k^{j-1}\right)}{(r-j)!(k)_j (\gamma+\psi(k))^j} \right\}, \end{aligned} \tag{35}$$

where γ_k^j is given by (17).

Proof. Using (33) in (24) we get

$$ER_k^r = r \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \left(1 - \frac{i}{k}\right) \int_0^1 w^{r-1} (1-iw)_+^{k-1} dw.$$

Now substituting $u = iw$

$$\begin{aligned} ER_k^r &= r \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \frac{1-\frac{i}{k}}{i^r} \int_0^i u^{r-1} (1-u)_+^{k-1} du \\ &= r \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \frac{1-\frac{i}{k}}{i^r} \int_0^1 u^{r-1} (1-u)^{k-1} du \\ &= \frac{r!}{(k)_r} \left(\sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \frac{1}{i^r} - \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \frac{1}{i^{r-1}} \right), \end{aligned}$$

and by (2) we get (34).

Now by the binomial formula

$$E(R_k - ER_k)^r = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} ER_k^j (ER_k)^{r-j},$$

and using (34) we obtain (35). □

Remark 21. The r-th moment of the range R_k can be written as

$$ER_k^r = \frac{r!}{(k)_r} \sum_{a_1+2a_2+\ldots+ra_r=r} \frac{(\gamma+\psi(k+1))^{a_1}}{a_1!} \prod_{i=2}^{r} \frac{(\zeta(i)-\zeta(i,k+1))^{a_i}}{i^{a_i} a_i!}$$
$$+ \frac{r!}{k(k)_r} \sum_{a_1+2a_2+\ldots+(r-1)a_{r-1}=r-1} \frac{(\gamma+\psi(k+1))^{a_1}}{a_1!} \prod_{i=2}^{r-1} \frac{(\zeta(i)-\zeta(i,k+1))^{a_i}}{i^{a_i} a_i!}.$$

Remark 22. From the proof it follows that ER_k^r can be written in terms of γ_k^r

$$ER_k^r = \frac{r!}{k(k)_r}\left(k\gamma_k^r - \gamma_k^{r-1}\right), \quad k \geq 1.$$

We can express the moments of R_k in terms of the moments of Δ_k.

Corollary 23. *For $r \in \mathbb{N}$*

$$ER_k^r = E\Delta_k^r - \frac{r}{k(k+r-1)} E\Delta_k^{r-1}.$$

Corollary 24. *The mean residual life function of R_k is*

$$m(t) = \frac{\sum_{j=1}^{k-i}(-1)^{j+1}\binom{k}{j}\left(1-\frac{j}{k}\right)\left\{\left(t-\frac{1}{k-i}\right)\left(1-\frac{j}{k-i}\right)^{k-1} - \frac{\left(1-\frac{j}{k-i}\right)^k}{jk} + \frac{(1-jt)^k}{jk}\right\}}{\sum_{j=1}^{k-i}(-1)^{j+1}\binom{k}{j}\left(1-\frac{j}{k}\right)(1-jt)^{k-1}},$$

for $\frac{\min\{i,1\}}{k+1-i} < t < \frac{1}{k-i}$, $i = 0, 1, \ldots k-1$, *and the expectation, the variance, the third and fourth central moments of R_k are given by*

$$ER_k = \frac{\gamma+\psi(k)}{k},$$
$$\sigma^2 R_k = \frac{2k(\gamma+\psi(k+1)) - k^2(\gamma+\psi(k+1))^2 + k^3\left(\frac{\pi^2}{6} - \psi'(k+1)\right) - k - 1}{k^3(k)_2},$$
$$E(R_k - ER_k)^3 = \frac{1}{k^5(k)_3}\left\{6k^3\left(\frac{\pi^2}{6} - \psi'(k+1)\right)(1 - k(\gamma+\psi(k+1)))\right.$$
$$-k^5\left(\psi''(1) - \psi''(k+1)\right) + 4k^3(\gamma+\psi(k+1))^3 - 12k^2(\gamma+\psi(k+1))^2$$
$$\left.+6k(k+2)(\gamma+\psi(k+1)) - 2(k^2+3k+2)\right\},$$

and

$$E(R_k-ER_k)^4=\frac{1}{k^7(k)_4}\Big\{3k^4(k-6)(\gamma+\psi(k+1))^4-12k^3(k-6)(\gamma+\psi(k+1))^3$$
$$+6k^2(k^2-3k-18)(\gamma+\psi(k+1))^2+12k(k^2+5k+6)(\gamma+\psi(k+1))$$
$$-6k^5(k-6)(\gamma+\psi(k+1))^2\left(\frac{\pi^2}{6}-\psi'(k+1)\right)+k^7\left(\frac{\pi^4}{15}-\psi^{(3)}(k+1)\right)$$
$$+12k^5(k-6)(\gamma+\psi(k+1))\left(\frac{\pi^2}{6}-\psi'(k+1)\right)+3k^7\left(\frac{\pi^2}{6}-\psi'(k+1)\right)^2$$
$$+12k^6(\gamma+\psi(k+1))^2\left(\psi''(1)-\psi''(k+1)\right)-12k^5\left(\psi''(1)-\psi''(k+1)\right)$$
$$-6k^3(k^2+k-6)\left(\frac{\pi^2}{6}-\psi'(k+1)\right)+\frac{3\pi^2}{20}k^7-3(k^3+6k^2+11k+6)\Big\}.$$

We have the following recurrence for the moments of R_k.

Proposition 25. *The moments of R_k satisfy the recurrence relation*

$$ER_k^r=r(r-2)!\sum_{j=0}^{r-2}\frac{(-1)^{j+1}\left(\psi^{(j)}(1)-\psi^{(j)}(k+1)\right)ER_k^{r-1-j}}{j!(k+r-j-1)_{j+1}(r-1-j)!}$$
$$-\frac{(r-2)!}{(k)_r}\sum_{j=0}^{r-2}\frac{(-1)^{j+1}}{j!}\left(\psi^{(j+1)}(1)-\psi^{(j)}(k+1)\right)\gamma_k^{r-1-j}$$
$$+\frac{(-1)^r}{(k)_r}\left(\psi^{(r-1)}(1)-\psi^{(r-1)}(k+1)\right),\quad r=2,3,\dots,$$

where $ER_k=\frac{1}{k}(\gamma+\psi(k))$ and γ_k^r is given by (17). In particular we have

$$ER_k^2=\frac{2(\gamma+\psi(k+1))}{k+1}ER_k-\frac{(\gamma+\psi(k+1))^2-\frac{\pi^2}{6}+\psi'(k+1)}{k(k+1)}.$$

Proof. Using (34) and (20) we have

$$ER_k^r=\frac{(r-1)!}{(k)_r}\sum_{j=0}^{r-1}H_k^{(j+1)}\gamma_k^{r-1-j}-\frac{r(r-2)!}{k(k)_r}\sum_{j=0}^{r-2}H_k^{(j+1)}\gamma_k^{r-2-j}$$
$$=\frac{r(r-2)!}{(k)_r}\sum_{j=0}^{r-2}H_k^{(j+1)}\left(\gamma_k^{r-1-j}-\frac{\gamma_k^{r-2-j}}{k}\right)-\frac{(r-2)!}{(k)_r}\sum_{j=0}^{r-2}H_k^{(j+1)}\gamma_k^{r-1-j}$$
$$+\frac{(r-1)!}{(k)_r}H_k^{(r)}$$
$$=\sum_{j=0}^{r-2}\frac{r(r-2)!H_k^{(j+1)}ER_k^{r-1-j}}{(k+r-j-1)_{j+1}(r-1-j)!}-\frac{(r-2)!}{(k)_r}\sum_{j=0}^{r-2}H_k^{(j+1)}\gamma_k^{r-1-j}+\frac{(r-1)!}{(k)_r}H_k^{(r)},$$

which by (13) and (15) implies the relation. □

Remark 26. The recurrence relation for R_k can be also written in terms of Zeta function

$$ER_k^r = \frac{r(\gamma+\psi(k+1))}{(r-1)(k+r-1)} ER_k^{r-1} + \frac{(r-1)!}{(k)_r}\left(\zeta(r)-\zeta(r,k+1)\right)$$
$$+r(r-2)!\sum_{j=1}^{r-2}\frac{(\zeta(j+1)-\zeta(j+1,k+1))\, ER_k^{r-1-j}}{(k+r-j-1)_{j+1}(r-1-j)!} - \frac{(r-2)!(\gamma+\psi(k+1))}{(k)_r}\gamma_k^{r-1}$$
$$-\frac{(r-2)!}{(k)_r}\sum_{j=1}^{r-2}\left(\zeta(j+1)-\zeta(j+1,k+1)\right)\gamma_k^{r-1-j}, \quad r=2,3,\dots,$$

where $ER_k = \frac{1}{k}(\gamma+\psi(k))$ and γ_k^r is given by (17). In particular we have

$$ER_k^2 = \frac{2(\gamma+\psi(k+1))}{k+1} ER_k - \frac{(\gamma+\psi(k+1))^2 - \frac{\pi^2}{6} + \psi'(k+1)}{k(k+1)}.$$

3. The Moments of Functions of the Extremal Statistics from Exponential Distribution

We give the moments of any order and the recurrence relations for the moments of $D_k = \max\{X_1,\dots,X_k\}$ and $W_{k,r} = X_{k-r:k} - X_{r+1:k}$ when $(X_1,\dots,X_k)$ is a sample from an exponential distribution. First we recall the known results on the distribution of D_k and $W_{k,r}$.

Proposition 27. *The distribution function of D_k is*

$$F_{D_k}(x) = \left(1-e^{-x}\right)^k, \quad x>0 \quad \text{(cf. Galambos [9], Chapter 1.3).}$$

The rate hazard function of D_k is given by

$$r(t) = \frac{k(1-e^{-t})^{k-1}e^{-t}}{1-(1-e^{-t})^k}, \quad t>0.$$

Proposition 28. *The distribution function of $W_{k,r}$ is*

$$F_{W_{k,r}}(w)$$
$$= \frac{\Gamma(k-r)}{\Gamma(k-2r-1)\Gamma(r+1)}\sum_{j=0}^{k-2r-2}\binom{k-2r-2}{j}(-1)^{j+1}\frac{e^{-(j+r+1)w}}{j+r+1},$$

for $w > 0$ (cf. Ghosal [10], Rider [19]). *The rate hazard function of* $W_{k,r}$ *is given by*

$$r(t) = \frac{\Gamma(k-r)(1-e^{-t})^{k-2r-2}e^{-(r+1)t}}{\Gamma(k-2r-1)\Gamma(r+1) - \Gamma(k-r)\sum_{j=0}^{k-2r-2}\binom{k-2r-2}{j}(-1)^{j+1}\frac{e^{-(j+r+1)t}}{j+r+1}}.$$

Now we give formulae for the moments of D_k and $W_{k,r}$.

Proposition 29. *The* r*-th moment of* D_k *is given by*

$$ED_k^r = \Gamma(r+1)\sum_{j=1}^{k}\binom{k}{j}(-1)^{j+1}\frac{1}{j^r},$$

and for $r \in \mathbb{N}$

$$ED_k^r = r! \sum_{a_1+2a_2+\ldots+ra_r=r}\prod_{i=1}^{r}\frac{(-1)^{ia_i}}{(i!)^{a_i}a_i!}\left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+1)\right)^{a_i}, \quad (36)$$

and

$$E\left(D_k - ED_k\right)^r = r! \sum_{2a_2+\ldots+ra_r=r}\prod_{i=2}^{r}\frac{(-1)^{ia_i}}{(i!)^{a_i}a_i!}\left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+1)\right)^{a_i}. \quad (37)$$

Proof. Using (24) and (2) we see that $ED_k^r = r!\gamma_k^r$, which by (17) gives (36).

Now using the binomial formula

$$\begin{aligned} E\left(D_k - ED_k\right)^r &= \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}ED_k^j\left(ED_k\right)^{r-j} \\ &= \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}(\gamma+\psi(k+1))^{r-j} \\ &\qquad \times \sum_{a_1+2a_2+\ldots+ja_j=j}\frac{(\gamma+\psi(k+1))^{a_1}}{a_1!}\prod_{i=2}^{j}w(i)\,, \end{aligned}$$

where
$$w(i) = \frac{(-1)^{ia_i}}{(i!)^{a_i} a_i!} \left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+1)\right)^{a_i}.$$
Summing with respect to a_1 we have
$$E\left(D_k - ED_k\right)^r = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} (\gamma + \psi(k+1))^{r-j}$$
$$\times \sum_{s=0}^{j} \sum_{s+2a_2+\ldots+ja_j=j} \frac{(\gamma+\psi(k+1))^s}{s!} \prod_{i=2}^{j} w(i).$$

Now by the identity
$$\sum_{j=0}^{r}\sum_{s=0}^{j} a(j,s) = \sum_{p=0}^{r}\sum_{j=s}^{r} a(j, j-s)\,,$$

we obtain
$$E(D_k - ED_k)^r = \sum_{s=0}^{r}\sum_{j=s}^{r} \binom{r}{j} (-1)^{r-j} (\gamma + \psi(k+1))^{r-j}$$
$$\times \sum_{j-s+2a_2+\ldots+ja_j=j} \frac{j!(\gamma+\psi(k+1))^{j-s}}{(j-s)!} \prod_{i=2}^{j} w(i)$$
$$= \sum_{s=0}^{r}\sum_{j=s}^{r} \binom{r}{j} \frac{(-1)^{r-j} j!}{(j-s)!} (\gamma+\psi(k+1))^{r-s} \sum_{2a_2+\ldots+ja_j=s} \prod_{i=2}^{j} w(i).$$

Note that
$$\sum_{2a_2+\ldots+ja_j=s} \prod_{i=2}^{j} w(i) = \sum_{2a_2+\ldots+sa_s=s} \prod_{i=2}^{s} w(i)\,,$$
as $a_{s+1} = \ldots = a_j = 0$. Therefore
$$E\left(D_k - ED_k\right)^r = \sum_{s=0}^{r} (\gamma+\psi(k+1))^{r-s} \sum_{2a_2+\ldots+sa_s=s} \prod_{i=2}^{s} w(i) \sum_{j=s}^{r} \binom{r}{j} \frac{(-1)^{r-j} j!}{(j-s)!}.$$

We see that
$$\sum_{j=s}^{r} \binom{r}{j} (-1)^{r-j} \frac{j!}{(j-s)!} = \frac{r!}{(r-s)!} \sum_{j=0}^{r-s} \binom{r-s}{j} (-1)^{r-j-s}.$$

Hence

$$E\left(D_k-ED_k\right)^r=\sum_{s=0}^{r}\frac{r!(\gamma+\psi(k+1))^{r-s}}{(r-s)!}$$
$$\times\sum_{2a_2+..+sa_s=s}\prod_{i=2}^{s}w(i)\sum_{j=0}^{r-s}\binom{r-s}{j}(-1)^{r-j-s}=r!\sum_{2a_2+...+ra_r=r}\prod_{i=2}^{r}w(i),$$

as

$$\sum_{j=0}^{r-s}\binom{r-s}{j}(-1)^{r-s-j}=\begin{cases}1, & s=r,\\ 0, & 1\le s<r,\end{cases}$$

which ends the proof. □

Remark 30. The r-th moment of D_k can be written in terms of Zeta function

$$ED_k^r=r!\sum_{a_1+2a_2+...+ra_r=r}\frac{(\gamma+\psi(k+1))^{a_1}}{a_1!}\prod_{i=2}^{r}\frac{(\zeta(i)-\zeta(i,k+1))^{a_i}}{i^{a_i}a_i!},$$

and

$$E\left(D_k-ED_k\right)^r=r!\sum_{2a_2+...+ra_r=r}\prod_{i=2}^{r}\frac{(\zeta(i)-\zeta(i,k+1))^{a_i}}{i^{a_i}a_i!}.$$

Corollary 31. *The moments of D_k in terms of the moments of Δ_k are as follows*

$$ED_k^r=(k)_rE\Delta_k^r,$$
$$E(D_k-ED_k)^r=(k)_rE(\Delta_k-E\Delta_k)^r-\frac{(k)_r}{k^r}\sum_{m=0}^{r-1}(\gamma+\psi(k+1))^{r-m}$$
$$\times\sum_{2a_2+...+ma_m=m}\prod_{i=2}^{m}\frac{(-1)^{ia_i}}{(i!)^{a_i}a_i!}\left(\psi^{(i-1)}(1)-\psi^{(i-1)}(k+1)\right)^{a_i}$$
$$\times\sum_{j=m}^{r}\binom{r}{j}\frac{(-1)^{r-j}j!}{(m-j)!}\frac{k^j}{(k)_j}.$$

Corollary 32. *The mean residual life function of D_k is given by*

$$m(t)=\frac{1}{1-(1-e^{-t})^k}\sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{e^{-jt}}{j},\quad t>0,$$

and the expectation, the variance, the third and fourth central moments of D_k are as follows

$$ED_k = \gamma + \psi(k+1), \quad \sigma^2 D_k = \frac{\pi^2}{6} - \psi'(k+1),$$
$$E\left(D_k - ED_k\right)^3 = \psi''(k+1) - \psi''(1),$$
$$E\left(D_k - ED_k\right)^4 = 3\left(\frac{\pi^2}{6} - \psi'(k+1)\right)^2 + \frac{\pi^4}{15} - \psi^{(3)}(k+1).$$

Skewness and kurtosis are

$$\gamma = \frac{\psi''(k+1) - \psi''(1)}{\left(\frac{\pi^2}{6} - \psi'(k+1)\right)^{\frac{3}{2}}} \text{ and } \kappa = \frac{3\left(\frac{\pi^2}{6} - \psi'(k+1)\right)^2 + \frac{\pi^4}{15} - \psi^{(3)}(k+1)}{\left(\frac{\pi^2}{6} - \psi'(k+1)\right)^2}.$$

Using the recurrence relation (11) we have the following

Proposition 33. *The moments ED_k^r satisfy the following recurrence relation*

$$ED_k^{r+1} = \sum_{j=0}^{r} \frac{r!(-1)^{j+1}}{j!(r-j)!}\left(\psi^{(j)}(1) - \psi^{(j)}(k+1)\right) ED_k^{r-j},$$

where $ED_k = \gamma + \psi(k+1)$.

Remark 34. Using (16) we can write the recurrence

$$ED_k^{r+1} = (\gamma + \psi(k+1))ED_k^r + \sum_{j=1}^{r} \frac{r!\left(\zeta(j+1) - \zeta(j+1,k+1)\right)}{(r-j)!} ED_k^{r-j},$$

where $ED_k = \gamma + \psi(k+1)$.

In Arnold et al [2], p. 73, ED_k^r is given as the linear combination of the r-th moment of $X_{k-1:k}$ and ED_k^{r-1}, whereas here the moments of D_k only are used.

Now we consider the quasi-range $W_{k,r} = X_{k-r:k} - X_{r+1:k}$, $k \geq 2r+2$, from the exponential distribution.

Proposition 35. *The p-th moment of $W_{k,r}$ is given by*

$$EW_{k,r}^p = \frac{\Gamma(k-r)\Gamma(p+1)}{\Gamma(k-2r-1)\Gamma(r+1)} \sum_{j=0}^{k-2r-2} \binom{k-2r-2}{j} \frac{(-1)^j}{(j+r+1)^{p+1}},$$

while for $p \in \mathbb{N}$

$$EW_{k,r}^p = p! \times \sum_{a_1+2a_2+\ldots+pa_p=p} \prod_{i=1}^{p} \frac{(-1)^{ia_i}}{(i!)^{a_i} a_i!} \left(\psi^{(i-1)}(r+1) - \psi^{(i-1)}(k-r)\right)^{a_i}, \quad (38)$$

$$E\,(W_{k,r} - EW_{k,r})^p = p! \sum_{2a_2+\ldots+pa_p=p} \prod_{i=2}^{p} \frac{(-1)^{ia_i}\left(\psi^{(i-1)}(r+1) - \psi^{(i-1)}(k-r)\right)^{a_i}}{(i!)^{a_i} a_i!}.$$

Proof. It is known that

$$EX^p = p! \sum_{a_1+2a_2+\ldots+pa_p=p} \prod_{i=1}^{p} \frac{\kappa_i^{a_i}}{(i!)^{a_i} a_i!} \quad \text{(cf. Shiryaev [22], p. 291)},$$

where κ_p is the cumulant of order p of X. Taking into account that the moment generation function of $W_{k,r}$ is

$$G_{k,r}(z) = \frac{1}{\left(1 - \frac{z}{r+1}\right) \cdot \ldots \cdot \left(1 - \frac{z}{k-r-1}\right)},$$

and the cumulant of order p is

$$\kappa_p = (p-1)! \sum_{i=r+1}^{k-r-1} \frac{1}{i^p} \quad \text{(cf. Rider [19])},$$

we get the first assertion of (38). In the same way as (37) we can prove the second statement of (38). □

Corollary 36. *The mean residual life function of $W_{k,r}$ is given by*

$$m(t) = \frac{\Gamma(k-r) \sum_{j=0}^{k-2r-2} \binom{k-2r-2}{j} (-1)^j \frac{e^{-(j+r+1)t}}{(j+r+1)^2}}{\Gamma(k-2r-1)\Gamma(r+1) - \Gamma(k-r) \sum_{j=0}^{k-2r-2} \binom{k-2r-2}{j} (-1)^{j+1} \frac{e^{-(j+r+1)t}}{j+r+1}}.$$

and the expectation, the variance,the third and fourth central moments of $W_{k,r}$ are (cf. Rider [19]):

$$EW_{k,r} = \psi(k-r) - \psi(r+1), \quad \sigma^2 W_{k,r} = \psi'(r+1) - \psi'(k-r),$$
$$E(W_{k,r} - EW_{k,r})^3 = \psi''(k-r) - \psi''(r+1),$$
$$E(W_{k,r} - EW_{k,r})^4 = 3(\psi'(r+1) - \psi'(k-r))^2 + \psi^{(3)}(r+1) - \psi^{(3)}(k-r).$$

Skewness and kurtosis are

$$\gamma = \frac{\psi''(k-r) - \psi''(r+1)}{(\psi'(r+1) - \psi'(k-r))^{\frac{3}{2}}},$$
$$\kappa = \frac{3(\psi'(r+1) - \psi'(k-r))^2 + \psi^{(3)}(r+1) - \psi^{(3)}(k-r)}{(\psi'(r+1) - \psi'(k-r))^2}.$$

From the generating function of $W_{k,r}$ (cf. the recurrence for γ_k^r) we have the following proposition.

Proposition 37. *The moments $EW_{k,r}^p$ satisfy the following recurrence relation*

$$EW_{k,r}^{p+1} = \sum_{j=0}^{p} \frac{p!(-1)^{j+1}}{j!(p-j)!} \left(\psi^{(j)}(r+1) - \psi^{(j)}(k-r)\right) EW_{k,r}^{p-j},$$

where $EW_{k,r} = \psi(k-r) - \psi(r+1)$.

4. Asymptotic Properties

We give limit properties of functions of δ_k, Δ_k, d_k and D_k. In Kaniowski [15] it was shown that the sequences (δ_k/Δ_k) and (d_k/D_k) converge in probability to 0. We find the rate of convergence in probability and we also see that they converge in L_p, $p > 0$, and completely ($X_k \xrightarrow{c} X$), i.e. $\sum_{k=1}^{\infty} P(|X_k - X| > \varepsilon) < \infty$.

Proposition 38. *The following statements hold*

$$\frac{\delta_k}{\Delta_k} \xrightarrow{L_p} 0, \quad \frac{d_k}{D_k} \xrightarrow{L_p} 0, \quad k \to \infty, \quad p > 0, \tag{39}$$

and

$$\frac{\delta_k}{\Delta_k} \xrightarrow{c} 0, \quad \frac{d_k}{D_k} \xrightarrow{c} 0, \quad k \to \infty. \tag{40}$$

Proof. By Corollary 18 we have $E\left(\frac{\delta_k}{\Delta_k}\right)^r = E\left(\frac{d_k}{D_k}\right)^r \sim \frac{r!}{k^r}$ as $k \to \infty$, which implies (39) as δ_k/Δ_k and d_k/D_k are identically distributed.

Now by Markov inequality and Corollary 18 we have

$$P\left(\frac{\delta_k}{\Delta_k} > \varepsilon\right) \le \frac{E\left(\frac{\delta_k}{\Delta_k}\right)^{2r}}{\varepsilon^{2r}} \le \frac{C}{k^{2r}},$$

where C is a positive constant independent of k. Hence

$$\sum_{k=1}^{\infty} P\left(\frac{\delta_k}{\Delta_k} > \varepsilon\right) \le C \sum_{k=1}^{\infty} \frac{1}{k^{2r}} < \infty,$$

which gives the rate of convergence (δ_k/Δ_k) to 0 in probability and proves (40). □

The connection between the Poisson process and random partitions of the segment was discussed in Pyke [18], Feller [7] and Kaniowski [15]. Let N_t, $t \ge 0$, be a standard Poisson process. Write $D_t^* = \max\{\sigma_1, \sigma_2 - \sigma_1, \ldots, t - \sigma_{N_t}\}$ and $d_t^* = \min\{\sigma_1, \sigma_2 - \sigma_1, \ldots, t - \sigma_{N_t}\}$, $t > 0$, where $\sigma_1, \sigma_2, \ldots$ are successive renewal moments. Let $X_1, \ldots, X_{k-1}$ be a sequence of independent random variables uniformly distributed on the interval $[0, t]$, $t > 0$. Let $\delta_k^{(t)}$ and $\Delta_k^{(t)}$ stand for the length of the greatest and the smallest interval, respectively, obtained by partitioning of the segment $[0, t]$ by the points $X_1, \ldots, X_{k-1}$. We note that $\Delta_k = \Delta_k^{(1)}$ and $\delta_k = \delta_k^{(1)}$. One can see that the random vector $(\Delta_k^{(t)}/t, \delta_k^{(t)}/t)$ has the same distribution as (Δ_k, δ_k). The conditional distribution of the random vector (D_t^*, d_t^*) given $N_t = k-1$ is the same as the distribution of $(\Delta_k^{(t)}, \delta_k^{(t)})$ (cf. Pyke [18], Feller [7]). The asymptotic value of ED_t^* and the limit behaviour of D_t^* were given in Kopocinska and Kopocinski [16] and Kopocinski [17], respectively. In Kaniowski [15] it was proved that $D_t^*/\log t \xrightarrow{P} 1$ and $d_t^* \log t \xrightarrow{P} 0$, $t \to \infty$. We note that $D_t^*/\log t \xrightarrow{L_p} 1$ and $d_t^* \log t \xrightarrow{L_p} 0$. Moreover, we prove that $(D_t^*/\log t)$ converges to 1 almost surely and $(d_t^* \log t)$ converges to 0 with probability one as $t \to \infty$.

Proposition 39. *The distribution function of D_t^* is given by*

$$F_{D_t^*}(x) = 1 - \sum_{j=1}^{\infty} \frac{(-1)^{j+1} e^{-jx}}{(j-1)!} (t - jx)_+^{j-1} \left(1 + \frac{1}{j}(t - jx)_+\right)$$

(cf. Kopocinska and Kopocinski [16]). *The rate hazard function of D_t^* is*

$$r(x) = \frac{\sum_{j=1}^{k} \frac{(-1)^{j+1}}{(j-1)!} e^{-jx} (t - jx)^{j-2} \left((j + t - jx)(t - jx + j - 1) - t + jx\right)}{\sum_{j=1}^{k} \frac{(-1)^{j+1}}{(j-1)!} e^{-jx} (t - jx)^{j-1} \left(1 + \frac{1}{j}(t - jx)\right)},$$

for $\frac{t}{k+1} < x < \frac{t}{k}$, $k = 1, 2, \ldots$.

Proposition 40. *The r-th moment of D_t^* is given by*

$$E\left(D_t^*\right)^r = \sum_{k=0}^{\infty} \frac{r! e^{-t} t^{k+r}}{(k+1)_r k!} \sum_{a_1+2a_2+\ldots+ra_r=r} \prod_{i=1}^{r} \frac{(-1)^{ia_i}}{(i!)^{a_i} a_i!} \left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+2)\right)^{a_i}, \tag{41}$$

and

$$\begin{aligned} E\left(D_t^* - ED_t^*\right)^r &= \sum_{k=0}^{\infty} \frac{r! e^{-t} t^{k+r}}{(k+1)^r k!} \sum_{m=0}^{r} (\gamma + \psi(k+2))^{r-m} \\ &\times \sum_{2a_2+\ldots+ma_m=m} \prod_{i=2}^{m} \frac{(-1)^{ia_i}}{(i!)^{a_i} a_i!} \left(\psi^{(i-1)}(1) - \psi^{(i-1)}(k+2)\right)^{a_i} \\ &\times \sum_{j=m}^{r} \binom{r}{j} \frac{(-1)^{r-j} j!}{(m-j)!} \frac{(k+1)^j}{(k+1)_j}. \end{aligned} \tag{42}$$

Proof. We have

$$\begin{aligned} &E\left(D_t^*/t\right)^r = E\left(E\{(D_t^*/t)^r \,|N_t\}\right) \\ &= E\left(\sum_{k=0}^{\infty} E\{(D_t^*/t)^r \,|N_t = k\} I_{\{N_t=k\}}\right) = \sum_{k=0}^{\infty} \frac{E\Delta_{k+1}^r t^k}{k!}, \end{aligned}$$

which by (18) gives (41). Now using (19)

$$\begin{aligned} &E(D_t^*/t - ED_t^*/t)^r = E(E\{(D_t^*/t - ED_t^*/t)^r | N_t\}) \\ &= E\left(\sum_{k=0}^{\infty} E\{(D_t^*/t - ED_t^*/t)^r | N_t = k\} I_{\{N_t=k\}}\right) \\ &= e^{-t} \sum_{k=0}^{\infty} \frac{E(\Delta_{k+1} - E\Delta_{k+1})^r t^k}{k!}, \end{aligned}$$

we see that (42) holds true. ∎

Remark 41. The r-th moment of D_t^* can be written in terms of Zeta function

$E\left(D_t^*\right)^r$

$$=\sum_{k=0}^{\infty}\frac{r!e^{-t}t^{k+r}}{(k+1)_r k!}\sum_{a_1+2a_2+..+ra_r=r}\frac{(\gamma+\psi(k+2))^{a_1}}{a_1!}\prod_{i=2}^{r}\frac{(\zeta(i)-\zeta(i,k+2))^{a_i}}{i^{a_i}a_i!},$$

and

$$E\left(D_t^*-ED_t^*\right)^r=\sum_{k=0}^{\infty}\frac{r!e^{-t}t^{k+r}}{(k+1)^r k!}\sum_{m=0}^{r}(\gamma+\psi(k+2))^{r-m}$$
$$\sum_{2a_2+\ldots+ma_m=m}\prod_{i=2}^{m}\frac{(\zeta(i)-\zeta(i,k+2))^{a_i}}{i^{a_i}a_i!}\sum_{j=m}^{r}\binom{r}{j}\frac{(-1)^{r-j}j!}{(m-j)!}\frac{(k+1)^j}{(k+1)_j}.$$

Corollary 42. *The mean residual life function of D_t^* is*

$$m(x)=\frac{\sum_{j=1}^{k}\frac{(-1)^j}{(j-1)!}}{\sum_{j=1}^{k}\frac{(-1)^{j+1}}{(j-1)!}e^{-jx}(t-jx)^{j-1}\left(1+\frac{1}{j}(t-jx)\right)}$$
$$\left\{\left(\frac{t}{k}-x\right)e^{\frac{-jt}{k}}\left(t-\frac{jt}{k}\right)^{j-1}\frac{kj+t(k-j)}{kj}+\frac{e^{\frac{-jt}{k}}\left(t-\frac{jt}{k}\right)^{j}}{j^2}\right.$$
$$\left.-\frac{e^{\frac{-jx}{k}}(t-jx)^j}{j^2}\right\},$$

for $\frac{t}{k+1}<x<\frac{t}{k}$, $k=1,2,\ldots$, and the expectation, the variance, the third and fourth central moments of D_t^* are as follows

$$ED_t^*=\gamma+\log t-Ei(-t),$$
$$\sigma^2 D_t^*=\frac{\pi^2}{6}-\frac{\pi^2(t+1)}{6e^t}-e^{-t}\sum_{k=2}^{\infty}\frac{\left(\psi'(k)+\frac{(\psi(k)+\gamma)^2}{k-1}\right)t^k}{k!},$$
$$E\left(D_t^*-ED_t^*\right)^3=\sum_{k=0}^{\infty}\frac{2e^{-t}t^{k+3}}{(k+1)_3 k!}\left\{\frac{4}{(k+1)^2}(\psi(k+2)+\gamma)^3\right.$$
$$\left.-\frac{6(\psi(k+2)+\gamma)}{(k+1)}\left(\frac{\pi^2}{6}-\psi'(k+2)\right)-\psi''(1)+\psi''(k+2)\right\},$$

and

$$E\left(D_t^* - ED_t^*\right)^4 = \sum_{k=0}^{\infty} \frac{3e^{-t}t^{k+4}}{(k+1)_4 k!} \left\{ \frac{3(k-5)(\psi(k+2)+\gamma)^4}{(k+1)^3} \right.$$

$$- \frac{6(k-5)(\psi(k+2)+\gamma)^2\left(\frac{\pi^2}{6} - \psi'(k+2)\right)}{(k+1)^2} + \frac{\pi^4}{15} - \psi^{(3)}(k+2)$$

$$\left. +3\left(\frac{\pi^2}{6} - \psi'(k+2)\right)^2 + \frac{12(\psi(k+2)+\gamma)\left(\psi''(1) - \psi''(k+2)\right)}{(k+1)} \right\}.$$

Skewness and kurtosis are

$$\gamma = \frac{\sum_{k=0}^{\infty} \frac{2e^{-t}t^{k+3}}{(k+1)_3 k!}}{\left(\frac{\pi^2}{6} - \frac{\pi^2(t+1)}{6e^t} - e^{-t}\sum_{k=2}^{\infty} \frac{t^k}{k!}\left(\psi'(k) + \frac{(\psi(k)+\gamma)^2}{k-1}\right)\right)^{3/2}}$$

$$\left\{ \frac{4(\psi(k+2)+\gamma)^3}{(k+1)^2} - \frac{6(\psi(k+2)+\gamma)}{(k+1)} \left(\frac{\pi^2}{6} - \psi'(k+2)\right) \right.$$

$$\left. -\psi''(1) + \psi''(k+2) \right\},$$

$$\kappa = \frac{1}{\left(\frac{\pi^2}{6} - \frac{\pi^2(t+1)}{6e^t} - e^{-t}\sum_{k=2}^{\infty} \frac{t^k}{k!}\left(\psi'(k) + \frac{(\psi(k)+\gamma)^2}{k-1}\right)\right)^2}$$

$$\sum_{k=0}^{\infty} \frac{3e^{-t}t^{k+4}}{(k+1)_4 k!} \left\{ \frac{3(k-5)(\psi(k+2)+\gamma)^4}{(k+1)^3} \right.$$

$$- \frac{6(k-5)(\psi(k+2)+\gamma)^2\left(\frac{\pi^2}{6} - \psi'(k+2)\right)}{(k+1)^2} + \frac{\pi^4}{15} - \psi^{(3)}(k+2)$$

$$\left. +3\left(\frac{\pi^2}{6} - \psi'(k+2)\right)^2 + \frac{12(\psi(k+2)+\gamma)\left(\psi''(1) - \psi''(k+2)\right)}{(k+1)} \right\}.$$

Proof. From (41) we have

$$ED_t^* = e^{-t} \sum_{k=0}^{\infty} \frac{(\gamma + \psi(k+2))\, t^{k+1}}{(k+1)!} = e^{-t} \sum_{k=1}^{\infty} \frac{(\gamma + \psi(k+1))\, t^k}{k!},$$

and by the identity

$$\sum_{k=1}^{\infty} \frac{(\gamma + \psi(k+1))\, x^k}{k!} = e^x(\gamma + \log x - Ei(-x))$$

(cf. Hansen [13], (55.7.1)), where $Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt$, $x < 0$, we get the formula for the expectation. □

Corollary 43.

$$E(D_t^* - ED_t^*)^{2r} = O(1), \quad t \to \infty.$$

Proof. By (42) and Corollary 5 we see that there exists such k_0 that for $k \geq k_0$ we have

$$\begin{aligned} E(D_t^* - ED_t^*)^{2r} &\leq e^{-t} \sum_{k=0}^{k_0-1} \frac{t^{k+2r}}{k!(k+1)_{2r}} + Ce^{-t} \sum_{k=k_0}^{\infty} \frac{t^{k+2r}}{k!(k+1)_{2r}} \\ &= e^{-t} \sum_{k=0}^{k_0-1} \frac{t^{k+2r}}{k!(k+1)_{2r}} + Ce^{-t} \sum_{k=2r+k_0}^{\infty} \frac{t^k}{k!} \leq C, \end{aligned}$$

where C is a positive constant independent of t. □

Proposition 44. *The distribution function of d_t^* is given by*

$$F_{d_t^*}(x) = 1 - \sum_{j=1}^{\infty} (t - jx)_+^{j-1} \frac{e^{-t}}{(j-1)!}.$$

The rate hazard function of d_t^* is

$$r(x) = \frac{\sum_{j=1}^{k} \frac{1}{(j-2)!} j(t-jx)^{j-2}}{\sum_{j=1}^{k} \frac{1}{(j-1)!} (t-jx)^{j-1}}, \quad \frac{t}{k+1} < x < \frac{t}{k}, \quad k = 1, 2, \ldots .$$

Proof. For $x > 0$ we have

$$P\left(\frac{d_t^*}{t} < x\right) = \sum_{k=0}^{\infty} P\left(\frac{d_t^*}{t} < x | N_t = k\right) \frac{e^{-t} t^k}{k!} = \sum_{k=0}^{\infty} P\left(\delta_{k+1} < x\right) \frac{e^{-t} t^k}{k!},$$

and by (21) we get the distribution function $F_{d_k^*}$. □

In the same way as Proposition 40 we can prove the following proposition.

Proposition 45. *The r-th moment of d_t^* is given by*

$$E\left(d_t^*\right)^r = r! e^{-t} \sum_{k=1}^{\infty} \frac{t^{k+r-1}}{k^{r-1} (k)_r k!}, \tag{43}$$

and

$$E\left(d_t^* - Ed_t^*\right)^r = r! e^{-t} \sum_{k=1}^{\infty} \frac{t^{k+r-1}}{k^{2r-1} k!} \sum_{j=0}^{r} \frac{(-1)^{r-1} k^j}{(r-j)!(k)_j}. \tag{44}$$

Proof. We have

$$\begin{aligned} E\left(d_t^*/t\right)^r &= E\left(E\{(d_t^*/t)^r\,|N_t\}\right) \\ &= E\left(\sum_{k=0}^{\infty} E\{(d_t^*/t)^r\,|N_t=k\}I_{\{N_t=k\}}\right) = e^{-t}\sum_{k=0}^{\infty}\frac{E\delta_{k+1}^r t^k}{k!}, \end{aligned}$$

which by (22) gives (43). Now using (23)

$$\begin{aligned} E(d_t^*/t - Ed_t^*/t)^r &= E(E\{(d_t^*/t - Ed_t^*/t)^r|N_t\}) \\ &= E\left(\sum_{k=0}^{\infty} E\{(d_t^*/t - Ed_t^*/t)^r|N_t=k\}I_{\{N_t=k\}}\right) \\ &= e^{-t}\sum_{k=0}^{\infty}\frac{E(\delta_{k+1}-E\delta_{k+1})^r t^k}{k!}, \end{aligned}$$

we see that (44) holds true. □

Corollary 46. *The life residual life function of d_t^* is*

$$m(x) = \frac{\sum_{j=1}^{k}\frac{1}{(j-1)!}\left\{\left(x-\frac{t}{k}\right)\left(t-\frac{jt}{k}\right)^{j-1} - \frac{\left(t-\frac{jt}{k}\right)^j}{j} + \frac{(t-jx)^j}{j}\right\}}{\sum_{j=1}^{k}\frac{1}{(j-1)!}(t-jx)^{j-1}},$$

for $\frac{t}{k+1} < x < \frac{t}{k}$, $k=1,2,\dots$, *and the expectation, the variance, the third and fourth central moments of d_t^* are as follows*

$$\begin{aligned} Ed_t^* &= e^{-t}(Ei(t)-\gamma-\log t), \quad \sigma^2 d_t^* = e^{-t}\sum_{k=2}^{\infty}\frac{t^k(k-2)}{(k-1)^3 k!}, \\ E\left(d_t^*-Ed_t^*\right)^3 &= e^{-t}\sum_{k=3}^{\infty}\frac{2(k-3)(k-4)t^k}{(k-2)^5 k!}, \\ E\left(d_t^*-Ed_t^*\right)^4 &= e^{-t}\sum_{k=4}^{\infty}\frac{3(k-3)(3(k-3)^2-7k-15)t^k}{(k-3)^7 k!}. \end{aligned}$$

Skewness and kurtosis are

$$\gamma = \frac{e^{t/2}\sum_{k=3}^{\infty}\frac{2(k-3)(k-4)t^k}{(k-2)^5 k!}}{\left(\sum_{k=2}^{\infty}\frac{t^k(k-2)}{k!(k-1)^3}\right)^{3/2}}, \quad \kappa = \frac{e^t\sum_{k=4}^{\infty}\frac{3(k-3)(3(k-3)^2-7k-15)t^k}{(k-3)^7 k!}}{\left(\sum_{k=2}^{\infty}\frac{t^k(k-2)}{k!(k-1)^3}\right)^2}.$$

Proof. To prove the formula for the expectation it is enough to note that

$$\sum_{k=1}^{\infty} \frac{x^k}{kk!} = Ei(x) - \gamma - \log x \quad \text{(cf. Hansen [13] (5.12.41))}. \tag{45}$$

□

Theorem 47. *With the above notation*

$$\frac{D_t^*}{\log t} \xrightarrow{L_p} 1, \quad \frac{D_t^*}{\log t} \xrightarrow{a.s.} 1, \quad t \to \infty, \tag{46}$$

$$d_t^* \log t \xrightarrow{L_p} 0, \quad d_t^* \log t \xrightarrow{a.s.} 0, \quad t \to \infty, \tag{47}$$

and

$$\frac{d_t^*}{D_t^*} \xrightarrow{a.s.} 0, \quad t \to \infty. \tag{48}$$

Proof. First we show that $ED_t^*/\log t \to 1$, $t \to \infty$. Noting that for $x > 0$

$$Ei(-x) = e^{-x}\left(-\frac{1}{x} + \int_0^{\infty} \frac{e^{-t}dt}{(x+t)^2}\right) \sim \frac{-e^{-x}}{x}\left(1 - \frac{1!}{x} + \frac{2!}{x^2} - \dots\right),$$

$x \to \infty$ (cf. Fichtenholz [8], p. 559, Gradstein and Ryzyk [11], 6.212.3.), we get

$$\lim_{t\to\infty} E\left(D_t^*/\log t\right) = 1.$$

Therefore, it is enough to state that $E(D_t^*/\log t - ED_t^*/\log t)^{2r}$ converges to 0 as $t \to \infty$. Thus Corollary 43 implies the first assertion in (46).

Now by Markov inequality and Corollary 43 we have

$$P\left(\left|\frac{D_t^*}{\log t} - E\frac{D_t^*}{\log t}\right| > \varepsilon\right) \le \frac{E(D_t^* - ED_t^*)^{2r}}{\varepsilon^{2r}\log^{2r} t} \le \frac{C}{\log^{2r} t},$$

which gives the rate of convergence of $(D_t^*/\log t)$ in probability to 1.

Using Etemadi method (cf. Etemadi [6]) we prove that for $D_n^*/\log n \xrightarrow{a.s.} 1$ as $n \to \infty$. Let $\alpha > 1$ and $m_n = \lceil \alpha^n \rceil$ for $n \ge 1$, where $\lceil x \rceil$ = the smallest integer greater than or equal to x (cf. Graham et al [12]), i.e. $\lceil x \rceil$ denotes the ceiling function of x. In what follows, C denotes a finite positive constant that can vary from step to step. Then

$$\begin{aligned}\sum_{n=1}^{\infty} P\Big(\Big|\frac{D_{m_n}^*}{\log m_n} - E\frac{D_{m_n}^*}{\log m_n}\Big| > \varepsilon\Big) &\le C\sum_{n=1}^{\infty}\frac{1}{\log^{2r} m_n} \\ &\le C\sum_{n=1}^{\infty}\frac{1}{n^{2r}} < \infty.\end{aligned}$$

By the Borel-Cantelli Lemma we have

$$\frac{D^*_{m_n}}{\log m_n} \xrightarrow{a.s.} 1, \quad n \to \infty.$$

Let $p(n)$ be such that $m_{p(n)} \leq n < m_{p(n)+1}$, for $n \geq 1$. Since D^*_n is nondecreasing, we have

$$\liminf_{n\to\infty} \frac{D^*_n}{\log n} \geq \liminf_{n\to\infty} \frac{D^*_{m_{p(n)}}}{\log m_{p(n)}} \frac{\log m_{p(n)}}{\log m_{p(n)+1}} \geq \lim_{n\to\infty} \frac{p(n)}{p(n)+1} \frac{D^*_{m_{p(n)}}}{\log m_{p(n)}} = 1.$$

Similarly,

$$\limsup_{n\to\infty} \frac{D(n)}{\log n} \leq \limsup_{n\to\infty} \frac{D^*_{m_{p(n)+1}}}{\log m_{p(n)+1}} \frac{\log m_{p(n)+1}}{\log m_{p(n)}} \leq \lim_{n\to\infty} \frac{p(n)+1}{p(n)} \frac{D^*_{m_{p(n)+1}}}{\log m_{p(n)+1}} = 1,$$

which implies that $D^*_n / \log n \xrightarrow{a.s.} 1$. Since for all $t > 0$ there exists a positive integer m such that $m \leq t < m+1$, we have

$$\frac{\log m}{\log(m+1)} \frac{D^*_m}{\log m} \leq \frac{D^*_t}{\log t} < \frac{\log(m+1)}{\log m} \frac{D^*_{m+1}}{\log(m+1)},$$

which ends the proof of the second assertion of (46).

Now noting that for $x > 0$

$$Ei(x) = e^x \left(\frac{1}{x} + \int_0^\infty \frac{e^{-t} dt}{(x-t)^2} \right) \sim \frac{e^x}{x} \left(1 + \frac{1!}{x} + \frac{2!}{x^2} + \dots \right), \, x \to \infty$$

(cf. Abrmowitz and Stegun [1], 5.1.51, Gradstein and Ryzyk [11], 6.212.2.) we get $Ed^*_t \log t \to 0$ as $t \to \infty$. Moreover,

$$E(d^*_t)^r = r! e^{-t} t^r \sum_{k=0}^\infty \frac{t^k}{\Gamma(k+r+1)(k+1)^r} \sim O\left(\frac{1}{t^r} \right), \qquad (49)$$

as

$$\sum_{k=0}^\infty \frac{t^k}{\Gamma(k+r+1)(k+1)^r} \sim O\left(\frac{e^t}{t^{2r}} \right), \; t \to \infty \text{ (cf. Hansen [13], (6.12.1)).}$$

and $E(d_t^* \log t)^r \to 0$ as $t \to \infty$, which implies the first assertion in (47).

Now by Markov inequality and (49) we have

$$P(|d_t^* \log t - Ed_t^* \log t| \geq \varepsilon) \leq \frac{E(d_t^*)^{2r} \log^{2r} t}{\varepsilon^{2r}} \leq C \frac{\log^{2r} t}{t^{2r}},$$

where a constant C depends on ε, which gives the rate of convergence of $(d_t^* \log t)$ in probability to 0. Hence for $n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} P(|d_n^* \log n - Ed_n^* \log n| \geq \varepsilon) \leq C \sum_{n=1}^{\infty} \frac{\log^{2r} n}{n^{2r}} < \infty,$$

and it proves that $d_n^* \log n \xrightarrow{a.s.} 0$. For all $t > 0$ there exists a positive integer m such that $m \leq t < m+1$, and the inequality

$$d_{m+1}^* \log(m+1) \frac{\log m}{\log(m+1)} \leq d_t^* \log t < d_m^* \log m \frac{\log(m+1)}{\log m},$$

completes the proof of (47).

Finally, taking into account that

$$\frac{d_t^*}{D_t^*} = \frac{d_t^*}{\log t} : \frac{D_t^*}{\log t},$$

we get (48). □

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