International Journal of Pure and Applied Mathematics
Volume 13 No. 3 2004, 381-389

M-GROUP AND SEMI-DIRECT PRODUCT

Liguo He
Department of Mathematics
Shenyang University of Technology
Shenyang, 110023, P.R. CHINA
e-mail: heliguo@online.ln.cn

Abstract: A finite group is said to be an Mim-group if all of its subgroups are M-groups. We show that a group is an M-group if it is the semidirect product of an Abelian group with an Mim-group. A group is an M-group if it is the semidirect product of a Sylow tower group with an Mim-group such that their orders are coprime and the Sylow subgroups of the Sylow tower group are Abelian. We also show that if G is an inner supersolvable group, then $G/\Phi(\mathrm{G})$ ($\Phi(G)$, Frattini subgroup) is an M-group. Generally, the inner supersolvable groups need not be M-groups.

AMS Subject Classification: 20C15
Key Words: M-group, Mim-group, semi-direct product of groups, supersolvable group

1. Introduction

A finite group is an M-group if every irreducible complex character is induced from a linear (i.e. degree 1) character of a subgroup. It is known that all M-groups are solvable (see Corollary 5.13 of [5] for this result of Taketa), but one can say nothing about the class of all subgroups of M-groups other than they

Received: March 27, 2004

are all solvable. The subgroups of M-groups need not be M-groups. An M-group must have an M-subgroup because M-group is solvable; also there exist M-groups that have non-M-subgroups (see [3, 13]). Hence, for the purpose of this note we define a group to be an *Mim-group* if all of its subgroups are M-groups.

Now, in order to state our principal results, a few more definitions are needed. Recall that a group is said to be inner supersolvable if all of its proper subgroups are supersolvable, but it is not supersolvable. A group is said to be outer supersolvable if all of its nontrivial homomorphic images are supersolvable, but it is not supersolvable. A group is said to be minimal nonsupersolvable if it is inner and outer supersolvable.

The following result [4, Satz V.18.4] is well known for M-groups.

Proposition. *Let $N \trianglelefteq G$ be such that N is solvable and all of its Sylow subgroups are Abelian and G/N is supersolvable, then G is an M-group.*

By the above proposition, we may obtain some results, for instance, both metabelian groups and nilpotent groups are M-groups; supersolvable groups are M-groups; minimal nonsupersolvable groups are M-groups, and solvable outer supersolvable groups are M-groups (these results can also be found in [2], [14] and [12] respectively). A question naturally arises: Must the inner supersolvable groups be M-groups?

The answer to this question is the first goal of this note.

We investigate detailed the question, and find that the alternating group A_4 is an affirmative example, but the special linear group $SL(2,3)$ is negative, thus we have a conclusion as follows.

Theorem A. *Inner supersolvable groups need not be M-groups.*

Furthermore, we get the following result.

Theorem B. *Let G be an inner supersolvable group, then $G/\Phi(G)$ ($\Phi(G)$, Frattini subgroup) is an M-group.*

Recall that for a finite group G if there exist a normal subgroup A and a subgroup H such that $G = AH$ with $A \cap H = 1$, then G is said to be the semidirect product of A with H and denoted $G = A \rtimes H$.

Using the previous proposition, it follows that the semidirect product of an Abelian group with a supersolvable group is an M-group (also see [12, Exercise 8.10]). The semidirect product of an Abelian group with a minimal nonsupersolvable group is an M-group (also see [14, Theorem 5]). Both supersolvable groups and minimal nonsupersolvable groups are Mim-groups, thus we are motivated to prove the following conclusion that generalizes the above two results.

Theorem C. *Let $G = N \rtimes H$, suppose that N is Abelian and H is an Mim-group. Then G is an M-group.*

The following theorem, in a sense, is an extension of Theorem C.

Theorem D. *Let $G = N \rtimes H$ be such that $(|N|, |H|) = 1$, suppose that H is an Mim-group and N is a Sylow tower group and all of Sylow subgroups of N are Abelian. Then G is an M-group.*

The other goal of this note is to prove Theorem C and Theorem D, and gives an example for Theorem C to show that the generalization is nontrivial. In the end, we discuss the converse of Theorem D.

We only consider finite groups and complex characters. The notation and terminology follow Isaacs' book [5].

2. The Case of Inner Supersolvable Groups

Similar to the definition of the inner supersolvable groups, a group is said to be inner nilpotent if all of its proper subgroups are nilpotent but it is not nilpotent. Let G be a finite group, let $\pi(G)$ denote all prime divisors of the order of G , and write G' to denote the commutator subgroup of G. To prove Theorem A, we need the following key lemma [1] as summarized by Y. Berkovich.

Lemma 2.1. *Let G be a finite group, G is inner nilpotent if and only if:*

1. *$\pi(G) = \{p, q\}$ and $G = PQ$, where $Q \in \mathrm{Syl}_q(G)$, $G' = P \in \mathrm{Syl}_p(G)$,*
2. *Q is cyclic and $|Q : Q \cap Z(G)| = q$, and*
3. *$P/P \cap Z(G)$ is a minimal normal subgroup of $G/P \cap Z(G), P$ is special.*

Note that a p-group P is *special* if it is elementary Abelian or $P' = \Phi(P) = Z(P)$, in particular, the exponent of P is at most p^2. It is known that if $p > 2$, then $\exp(P)$ is p.

The following is Theorem A. The proof of Theorem A will be used later, although it seems to be standard.

Theorem 2.2. *Inner supersolvable groups need not be M-groups.*

Proof. We distinguish two cases.

Case 1. The special linear group $SL(2, 3)$ is an inner supersolvable non-M-group.

Since $SL(2, 3)$ is not an M-group (see page 67 of [5], for example), it is not supersolvable. To prove its inner supersolvability, it suffices to show that it is inner nilpotent. Hence we shall use Lemma 2.1.

It follows that $SL(2,3)$ is isomorphic to $P \rtimes C_3$, where P is a quaternion group of order 8 and C_3 is cyclic group of order 3 (see [10, §8.4]). Let $G = P \rtimes C_3$, we get $G' = P$. It is clear that $P \in \mathrm{Syl}_2(G)$, and $C_3 \in \mathrm{Syl}_3(G)$, hence the condition 1 of Lemma 2.1 is satisfied.

We denote P by the symbol set $\{\pm 1, \pm i, \pm j, \pm k\}$ such that $-1 = i^2 = j^2 = k^2, ij = k = -ji, jk = i = -kj, ki = j = -ik$. Let $C_3 = < c >$, by routine computation we have $Z(G) = \{\pm 1\}$ and $C_3 \cap Z(G) = 1$, thus $|C_3 : C_3 \cap Z(G)| = 3$. Thus, the condition 2 of Lemma 2.1 is satisfied.

Since all maximal subgroups of P are $< i >, < j >$ and $< k >$, we get $\Phi(P) = \{\pm 1\}$, therefore, since $Z(P) = \{\pm 1\} = P'$, it follows that P is special and $P \cap Z(G) = \{\pm 1\}$. The element $c \in < c >$ permutates the set $\{i, j, k\}$ cyclically by conjugation action, hence none subgroups of order four of P are normal in G, and $P/\{\pm 1\}$ is a minimal normal subgroup of $G/\{\pm 1\}$, and so the condition 3 of Lemma 2.1 is satisfied. Therefore, the group $SL(2,3)$ is inner nilpotent, as desired.

Case 2. The alternating group of degree four A_4 is an inner supersolvable M-group.

The group A_4 is not supersolvable (see [12, Section 8.4], for example), however, all of its proper subgroups are supersolvable since its order is 12, hence it is inner supersolvable. Since $A_4 = H \rtimes K$, where H and K are Abelian (see [12, §5.7], for example), it follows that A_4 is an M-group by the previous proposition, and the proof is complete. □

Although inner supersolvable groups need not be M-groups, we have the following result.

Theorem 2.3. *Let G be inner supersolvable. Then $G/\Phi(G)$ is an M-group.*

Proof. This is the special case of Corollary 3.7 (below). Note that supersolvable groups are M-groups. □

3. The Proofs of Theorem C and Theorem D

The purpose of this section is to prove Theorem C and Theorem D. Before we do it, however, we wish to collect some facts that will be used in our proofs.

Lemma 3.1. *Let $N \lhd G$. Then:*

1. *If χ is a character of G and $N \le Ker\chi$, then χ is constant on cosets of N in G and the function $\overline{\chi}$ on G/N defined by $\overline{\chi}(Ng) = \chi(g)$ is a character of G/N.*

2. *If $\overline{\chi}$ is a character of G/N, then the function χ defined by $\chi(g) = \overline{\chi}(Ng)$ is a character of G.*

3. *In both 1 and 2, $\chi \in \mathrm{Irr}\,(G)$ iff $\overline{\chi} \in \mathrm{Irr}\,(G/N)$*

Proof. See Lemma 2.22 of [5], for example. □

Lemma 3.2. (Gallagher) *Let $N \lhd G$ and let $\chi \in \mathrm{Irr}\,(G)$ be such that $\chi_N = \theta \in \mathrm{Irr}\,(N)$. Then the characters $\beta\chi$ for the associated characters $\overline{\beta} \in \mathrm{Irr}\,(G/N)$ are irreducible, distinct for distinct $\overline{\beta}$ and are all of the irreducible constituents of θ^G.*

Proof. See Corollary 6.17 of [5], for example. □

The following map is usually referred to as Clifford correspondence which is of fundamental importance in the character theory of normal subgroups.

Lemma 3.3. *Let $H \lhd G$, $\theta \in \mathrm{Irr}\,(H)$, and $T = I_G(\theta)$ (the inertia group of θ in G). Let $\mathcal{A} = \{\psi \in \mathrm{Irr}\,(T) | [\psi, \theta^T] \neq 0\}$, and $\mathcal{B} = \{\chi \in \mathrm{Irr}\,(G) | [\chi, \theta^G] \neq 0\}$. Then:*

1. *If $\psi \in \mathcal{A}$, then ψ^G is irreducible;*

2. *The map $\psi \mapsto \psi^G$ is a bijection of $\mathcal{A}$ onto $\mathcal{B}$.*

Proof. This is essentially Theorem 6.11 of [5]. □

The next lemma is useful, indeed, we only need its sufficient condition.

Lemma 3.4. *Assume the situation of Lemma 3.1. Let $\chi \in \mathrm{Ch}\,(G)$, $H < G$, and $\eta \in \mathrm{Ch}\,(H)$, then $\chi = \eta^G$ iff $\overline{\chi} = (\overline{\eta})^{\overline{G}}$.*

Proof. Let $N \lhd G$, and $N \leq \mathrm{Ker}\ \chi$. By Lemma 5.11 of [5], we get $N \leq \mathrm{Ker}\ \eta$, thus $\overline{\eta}$ is well defined. If $\overline{\chi} = (\overline{\eta})^{\overline{G}}$, then, by Lemma 3.1,

$$\begin{aligned}\chi(s) &= \overline{\chi}(Ns)\\ &= \frac{1}{|G/N|} \sum_{gN \in G/N} \overline{\eta}^{\circ}(gsg^{-1}N) = \frac{1}{|G|} \sum_{g \in G} \eta^{\circ}(gsg^{-1}) = \eta^G(s),\end{aligned}$$

as desired.

The converse, similarly, can be obtained and the proof is complete. □

Lemma 3.5. *Assume A acts coprimely on a π-separable group G by automorphisms. Let $C_G(A)$ denote the fixed-point subgroup of G acted on by A. Then:*

1. *There exists an A-invariant Hall π-subgroup.*

2. *The A-invariant Hall π-subgroups of G are conjugate to each other by elements of $C_G(A)$.*

3. *Each A-invariant π-subgroup of G is contained in an A-invariant Hall π-subgroup of G.*

Proof. See [9, Lemma 3.4]. It can also follows immediately from the Glauberman Lemma (Theorem 13.8 and Theorem 13.9 of [5]). □

We proceed now with proving Theorem C.

Theorem 3.6. *Let $G = N \rtimes H$, where N is an Abelian normal subgroup and H is an Mim-subgroup, then G is an M-group.*

Proof. We distinguish two steps.

Step 1. Let $\chi \in \mathrm{Irr}\,(G)$, we show that χ can be written in the form $(\lambda\hat{\theta})^G$, where $\hat{\theta}, \lambda \in \mathrm{Irr}\,(T)$, T is some subgroup of G.

Let θ be an irreducible constituent of χ_N. Let T denote the inertia group of θ in G. Then we may extend the class function θ from N to T by setting $\hat{\theta}(ah) = \theta(a)$ for all $ah \in T, a \in N$, and $h \in H$. It follows that $\hat{\theta}$ is a character of T, since θ is linear. By Lemma 3.2, the characters $\lambda\hat{\theta}$ for $\overline{\lambda} \in \mathrm{Irr}\,(T/N)$ are all of the irreducible constituents of θ^T. Hence, by Lemma 3.3 and $[\chi, \theta^G] \neq 0$, we have $\chi = (\lambda\hat{\theta})^G$ for some $\overline{\lambda} \in \mathrm{Irr}\,(T/N)$.

Step 2. We show that $\lambda\hat{\theta}$ is monomial, that is, $\lambda\hat{\theta}$ is induced by some linear character of some subgroup of T.

The character $\overline{\lambda} \in \mathrm{Irr}\,(T/N)$ is monomial since $G/N \cong H$ and H is an Mim-group, thus, there exists a subgroup T_1/N of T/N and a linear character $\overline{\eta} \in \mathrm{Irr}\,(T_1/N)$ such that $\overline{\lambda} = (\overline{\eta})^{\overline{T}}$, where $\overline{T} = T/N$. By Lemma 3.5, we get $\lambda = \eta^T$, and η is linear. Therefore, $\lambda\hat{\theta} = \eta^T\hat{\theta} = (\eta\,\hat{\theta}_{T_1})^T$, where $\eta\,\hat{\theta}_{T_1}$ is a linear character of T_1. Consequently, it follows that $\chi = ((\eta\,\hat{\theta}_{T_1})^T)^G = (\eta\,\hat{\theta}_{T_1})^G$, hence, χ is monomial and the proof is complete. □

By the above theorem, a natural question is left: If N is an Abelian normal subgroup of G, and G/N is an Mim-group, then must G be M-group? The answer is "No", because there exists a desired counterexample as follows.

Let $G = SL(2,3)$, it is known [5, page 87] that $G/Z(G) \cong A_4$, where $Z(G)$ denotes the center of G. By the proof of Theorem A, we know that A_4 is an Mim-group, but G is not an M-group.

Another interesting question is that if the hypothesis that N is Abelian is weakened to N nilpotent, is Theorem 3.6 true yet? Actually, the above counterexample is also a negative answer to this problem (see the proof of Theorem 2.2).

Corollary 3.7. *Let G be solvable and all of its proper subgroups be M-groups, then $G/\Phi(G)$ is an M-group.*

Proof. By [8, Theorem 1.12] or [4, III, 4.2, 4.3 and 4.4], we have

$$G/\Phi(G) = F(G)/\Phi(G) \rtimes H/\Phi(G).$$

Since $F(G)/\Phi(G) > 1$ is Abelian(because G is solvable), we get that $H/\Phi(G)$ is an Mim-group, thus $G/\Phi(G)$ is an M-group by Theorem 3.6, as wanted. □

From the previous proposition, it follows that if all of Sylow subgroups of the solvable group H are Abelian, then H is a Mim-group. Hence, from Theorem 3.6 we may obtain the next result.

Corollary 3.8. *Let $G = N \rtimes H$ be such that N is Abelian, H is solvable and all of Sylow subgroups of H are Abelian. Then G is an M-group.*

Corollary 3.9. *Let G be solvable and $\Phi(H) = 1$ for all subgroups H of G. Then G is an M-group.*

Proof. This easily follows using Theorem 1.12 of [8] and Theorem 3.6. □

To show that Theorem 3.6 is a nontrivial extension of the known results, we exhibit one example.

The symmetric group S_4 has a normal subgroup B_4 (Klein group), and B_4 is not in the center of S_4, thus there is nontrival conjugation action of S_4 on B_4, hence we get an outer semidirect product of B_4 and S_4 and write $G = B_4 \rtimes S_4$. The B_4 is Abelian. Since $|S_4/B_4| = 6$, it follows that S_4/B_4 is supersolvable, thus, by the previous proposition, we get S_4 is an M-group, consequently it easily follows that S_4 is an Mim-group. Hence G is an M-group by Theorem 3.6, as desired.

Note that S_4 is neither supersolvable nor minimal nonsupersolvable, hence the analogs of Theorem 3.6 can not apply to this example.

Recall that G is said to be a *Sylow tower group* if there is a series $1 = G_0 < G_1 < \cdots < G_s = G$ of normal subgroups of G such that each $i = 1, \ldots, s$, G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G, where $p_1, \ldots, p_s$ are the distinct prime divisors of $|G|$. We call the positive integer s the *length of Sylow tower* of G. It is well known that supersolvable groups are Sylow tower groups. The next result, in a sense, is an extension of Theorem 3.6.

Theorem 3.10. *Let $G = N \rtimes H$ be such that $(|N|, |H|) = 1$, suppose that H is an Mim-group and N is a Sylow tower group and all of Sylow subgroups of N are Abelian. Then G is an M-group.*

Proof. Use induction on the length s of Sylow tower of N. Let n denote the length of Sylow tower of the normal Hall subgroup of any group T satisfying the hypotheses. If $n = 1$, the desired result follows from Theorem 3.6.

Now assume that the desired result holds for all groups of n less than s. Since $N = P \rtimes Q$, where P is a normal Sylow subgroup of N and Q is a complement of P in N, we have $G = (P \rtimes Q) \rtimes H$; by Lemma 3.5, there exists a H-invariant subgroup N_1 conjugate to Q by some element of N, thus $G = P \rtimes (N_1 \rtimes H)$. Using the induction hypothesis, $N_1 \rtimes H$ is an M-group. Without loss of generality, we may suppose that $N_1 \rtimes H_1$ is any proper subgroup of $N_1 \rtimes H$, where $H_1 < H$; by the hypotheses, H_1 is an Mim-group, then $N_1 \rtimes H_1$ is an M-group from the induction hypothesis, thus $N_1 \rtimes H$ is an Mim-group, and hence the desired result follows from Theorem 3.6. □

Corollary 3.11. *Let $G = N \rtimes H$ be such that $(|N|, |H|) = 1$, suppose that H is an Mim-group and all of Sylow subgroups of N are cyclic. Then G is an M-group.*

Proof. Since all of Sylow subgroups of N is cyclic, it follows that N is solvable and so is supersolvable using the upper fitting series of N(e.g. see [10, page 160]). Hence N satisfies the hypotheses of Theorem 3.10, yielding the desired result. □

4. The Converse of Theorem D

Now we wish to consider the converse of Theorem 3.11, i.e., suppose that $G = N \rtimes H$ is an M-group and $(|N|, |H|) = 1$, then what conclusion will be achieved about N or H? For H, it was proved that in this situation $N_G(H)/H$ is an M-group ([7, Corollary 3.4]), furthermore, $N_G(H)/H'$ is an M-group ([6, Theorem A]). For N, we will prove the following result.

Theorem 4.1. *Suppose that $G = N \rtimes H$ is an M-group and $(|N|, |H|) = 1$, then N is an M-group.*

Proof. Let $\chi \in \mathrm{Irr}\,(N)$, if χ is H-invariant, then χ may be extended to G, say $\bar{\chi}$. Since G is an M-group, it follows that $\bar{\chi} = \lambda^G$ such that $\lambda \in \mathrm{Irr}\,(U)$ is linear and U is some subgroup of G. Thus $\chi = (\lambda^G)_N = (\lambda_{N\cap U})^N$, as required. If χ is not H-invariant, let $T = I_G(\chi)$ be stable subgroup of χ in G, then χ can be extendible to T, say $\bar{\chi}$; by Lemma 3.3, $(\bar{\chi})^G$ is irreducible and $(\bar{\chi})^G = \lambda^G$ such that $\lambda \in \mathrm{Irr}\,(U)$ is linear and U is some subgroup of G. Because $((\bar{\chi})^G)_N = (\lambda^G)_N$ is irreducible, we get that $\chi = (\lambda^G)_N = (\lambda_{N\cap U})^N$, as desired. □

References

[1] Y. Berkovich, *Some corollaries of Frobenius' normal p-complement theorem*, *Proc. Amer. Math. Soc.*, **127** (1999), 2505-2509.

[2] C.W. Curtis, I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, John Wiley and Sons, New York (1962).

[3] E.C. Dade, Normal subgroups of M-groups need not be M-groups, *Math. Z.*, **133** (1973), 313–315.

[4] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg-New York (1967).

[5] I.M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York (1976).

[6] I.M. Isaacs, Hall subgroup normalizers and character correspondences in M-groups, *Proc. Amer. Math. Soc.*, **109** (1990), 647-651.

[7] I.M. Isaacs, G. Navarro, Character correspondences and irreducible induction and restriction, *J. Algebra*, **140** (1991), 131-140.

[8] O. Manz, T.R. Wolf, *Representations of Solvable Groups*, Cambridge Univ. Press, Cambridge (1993).

[9] G. Navarro, Fong characters and correspondences in π-separable groups, *Can. J. Math.*, **43** (1991), 405-412.

[10] D.J.S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York (1982).

[11] John S. Rose, *A Course on group theory*, Cambridge Univ. Press, Cambridge, UK (1978).

[12] J.-P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, New York (1977).

[13] R.W. van der Waall, On the embedding of minimal non-M-groups, *Indag. Math.*, **36** (1974), 157-167.

[14] Qiu Weisheng, Irreducible representations of the groups relating to minimal unsupersolvable groups, *Acta Scientiarum Naturalium Universitatis Pekinensis*, **26**, No. 5 (1990), 592-601, In Chinese.