International Journal of Pure and Applied Mathematics
Volume 13 No. 4 2004, 535-543

SOME NEW SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY A CERTAIN INTEGRAL OPERATOR

Khalida Inayat Noor
Department of Mathematics and Computer Science
College of Science
United Arab Emirates University
P.O. Box 17551, Al-Ain, UNITED ARAB EMIRATES
e-mail: khalidan@uaeu.ac.ae

Abstract: Let $\mathcal{A}$ be the class of analytic functions defined on the open unit disc E. An integral operator $I_\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ is defined using the convolution $\star$. Let $f_\alpha(z) = \frac{z}{(1-z)^{\alpha+1}}, \quad \alpha > -1$ and let $f_\alpha^{(-1)}$ be defined such that $(f_\alpha \star f_\alpha^{(-1)})(z) = \frac{z}{1-z}$. Then, for $f \in \mathcal{A}, \quad \alpha > -1$, we define

$$I_\alpha f(z) = (f_\alpha^{(-1)} \star f)(z) = [\frac{z}{(1-z)^{\alpha+1}}]^{(-1)} \star f(z)$$
$$= [z_2F_1(1,1;\alpha+1,z] \star f(z),$$

where ${}_2F_1$ is the hypergeometric function. Using this operator, certain new subclasses of analytic functions are defined and studied. Some inclusion results and radii problems are investigated and it is shown that these classes are closed under convolution with convex functions.

AMS Subject Classification: 30C45, 30C50
Key Words: convolution, convex functions, Noor integral operator, radius

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ in the unit disk $E = \{z : |z| < 1\}$. Let $S, K, S^\star$ and C be the subclasses of $\mathcal{A}$ that consist of univalent, close-to-convex, starlike and convex in E respectively.

Received: April 14, 2004

Let $f \in \mathcal{A}$. Denote by $D^\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ the operator defined by

$$D^\alpha f(z) = \frac{z}{(1-z)^{\alpha+1}} \star f(z), \quad \alpha > -1.$$

The symbol $(\star)$ stands for the Hadamard product or convolution. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. The operator D^α is called the Ruscheweyh Derivative. Analogous to D^α, we define here the integral operator $I_\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ as follows.

Let $f_\alpha(z) = \frac{z}{(1-z)^{\alpha+1}}, \quad \alpha > -1$ and let $f_\alpha^{(-1)}$ be defined such that

$$(f_\alpha \star f_\alpha^{(-1)})(z) = \frac{z}{1-z}. \tag{1.1}$$

Then

$$I_\alpha f(z) = (f_\alpha^{(-1)} \star f)(z) = [\frac{z}{(1-z)^{\alpha+1}}]^{(-1)} \star f(z). \tag{1.2}$$

Using (1.1), (1.2) and the well-known identity for D^α, we have

$$(\alpha+1) I_\alpha f(z) - \alpha I_{\alpha+1} f(z) = z(I_{\alpha+1} f(z))'. \tag{1.3}$$

Using the hypergeometric functions $_2F_1$, we can write (1.2) as

$$I_\alpha f(z) = [z {}_2F_1(1,1;\alpha+1,z)] \star f(z), \quad (\alpha > -1).$$

We note that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$. The operator I_n defined by (1.2) is known as the Noor integral operator of nth order, see [2, 4, 5]. Several classes of analytic functions, defined by using Noor integral operator, have been introduced and investigated by many authors, see [2-11].

Let $k \geq 2$. Let P_k be the class of analytic functions p in E with integral representation

$$p(z) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t), \tag{1.4}$$

where $\mu(t)$ is a function of bounded variation on $[-\pi, \pi]$ which satisfies the conditions

$$\int_{-\pi}^{\pi} d\mu(t) = 2, \quad \int_{-\pi}^{\pi} |d\mu(t)| \leq k.$$

We note that $P_2 \equiv P$ is the class of analytic functions p with positive real part in E satisfying $p(0) = 1$.

From the integral representation (1.4), it is immediately clear that $p \in P_k$ if, and only if there exist analytic functions $p_1, p_2 \in P$ such that

$$p(z) = (\frac{k}{4} + \frac{1}{2})p_1(z) - (\frac{k}{4} - \frac{1}{2})p_2(z). \tag{1.5}$$

Definition 1.1. Let $g \in \mathcal{A}$ and $\alpha > -1$. Then $g \in N^\star(\alpha)$, if and only if, $I_\alpha g \in S^\star$ for $z \in E$.

It is clear that $N^\star(0) = C$ and $N^\star(1) = S^\star$. That is, for $\alpha = 0$, and $\alpha = 1$, $N^\star(\alpha) \subset S$. The classes $N^\star(\alpha)$ for $\alpha = n \in N_0 = \{0, 1, 2, \ldots\ldots\}$ have been introduced and discussed in some details in [7].

Definition 1.2. Let $f \in \mathcal{A}$ and $k \geq 2$. Then $f \in T_k^\star(\alpha)$, $\alpha > -1$, if and only if, there exists a function $g \in N^\star(\alpha)$ such that, for $z \in E$, $k \geq 2$,

$$\frac{z(I_\alpha f(z))'}{I_\alpha g(z)} \in P_k.$$

We note that for $\alpha = n \in N_0$ and $k = 2$, we obtain the class $T_2^\star(\alpha)$ that was introduced and studied in [10].

2. Preliminary Results

Lemma 2.1. (see [12]) *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u, v)$ be a complex-valued function satisfying the conditions:*

(i) $\Psi(u, v)$ *is continuous in a domain* $D \subset \mathbf{C}^2$,

(ii) $(1, 0) \in D$ *and* $\Psi(1, 0) > 0$.

(iii) $\operatorname{Re}\Psi(iu_2, v_1) \leq 0$, *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.

If $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *is an analytic function in* E *such that* $(h(z), zh'(z)) \in D$ *and* $\operatorname{Re}\Psi(h(z), zh'(z)) > 0$ *for* $z \in E$, *then* $\operatorname{Re} h(z) > 0$ *in* E.

Lemma 2.2. (see [3]) *Let* $p \in P$. *Then*

$$(i).\quad \frac{1-r}{1+r} \leq \operatorname{Re} p(z) \leq |p(z)| \leq \frac{1+r}{1-r}, \quad |z| = r, \quad z \in E$$

$$(ii).\quad |p'(z)| \leq \frac{2\operatorname{Re} p(z)}{1-r^2}.$$

All these bounds are sharp.

Lemma 2.3. (see [1]) *Let ϕ be convex and g be starlike in E. Then if F is an analytic function in E with $F(0) = 1$,* $\frac{\phi\star Fg}{\phi\star g}$ *is contained in the convex hull of $F(E)$.*

Lemma 2.4. *Let, for $g \in N^\star(\alpha)$, $z \in E$,* $\operatorname{Re}\{\frac{z(I_\alpha g(z))'}{I_\alpha g(z)}\} > 0$. *Then*

$$\operatorname{Re}\{\frac{z(I_{\alpha+1}g(z))'}{I_{\alpha+1}g(z)}\} > \rho \quad \text{in} \quad E,$$

where

$$\rho = \rho_\alpha = 2/\{(2\alpha+1) + \sqrt{4\alpha^2+4\alpha+9}\} \quad (\alpha > -1). \tag{2.1}$$

In particular, we note that $N^\star(\alpha) \subset N^\star(\alpha+1)$.

Proof. Set

$$\frac{z(I_{\alpha+1}g(z))'}{I_{\alpha+1}g(z)} = h(z), \tag{2.2}$$

where h is an analytic function in E and $h(0) = 1$.

From (2.2) and the identity (1.3), we have

$$\operatorname{Re}\{\frac{z(I_\alpha g(z))'}{I_\alpha g(z)}\} = \operatorname{Re}\{h(z) + \frac{zh'(z)}{h(z)+\alpha}\} > 0, \quad z \in E. \tag{2.3}$$

Let $h(z) = (1-\rho)p(z) + \rho$ and define the functional $\Psi(u,v)$ by choosing $u = p$, $v = zp'$ in (2.3). Thus

$$\Psi(u,v) = (1-\rho)u + \rho + \frac{(1-\rho)v}{(1-\rho)u + (\rho+\alpha)}.$$

The first two conditions of Lemma 2.1 are clearly satisfied . We verify the condition (iii). Let

$$\operatorname{Re}\Psi(iu_2, v_1) = \rho + \frac{(1-\rho)(\rho+\alpha)v_1}{(\rho+\alpha)^2 + (1-\rho)^2u_2^2}.$$

If $v_1 \leq \frac{-(1+u_2^2)}{2}$, $1-\rho > 0$ and $\rho+\alpha > 0$, we obtain

$$\begin{aligned}\operatorname{Re}\Psi(iu_2, v_1) &\leq \rho - \frac{1}{2}\frac{(1-\rho)(\rho+\alpha)(1+u_2^2)}{(\rho+\alpha)^2+(1-\rho)^2u_2^2} \\ &= \frac{2\rho(\rho+\alpha)^2 + 2\rho(1-\rho)^2u_2^2 - (1-\rho)(\rho+\alpha) - (1-\rho)(\rho+\alpha)u_2^2}{2[(\rho+\alpha)^2+(1-\rho)^2u_2^2]} \\ &= \frac{A+Bu_2^2}{2C},\end{aligned}$$

where:

$$\begin{aligned} A &= 2\rho(\rho+\alpha)^2-(1-\rho)(\rho+\alpha)=(\rho+\alpha)[2\rho(\rho+\alpha)-(1-\rho)]\,, \\ B &= 2\rho(1-\rho)^2-(1-\rho)(\rho+\alpha)=(1-\rho)[2\rho(1-\rho)-(\rho+\alpha)]\,, \\ C &= [(\rho+\alpha)^2+(1-\rho)^2u_2^2]>0\,. \end{aligned}$$

We note that $\operatorname{Re}\Psi(iu_2,v_1)\le 0$ if and only if $A\le 0$ and $B\le 0$. From $A\le 0$, we obtain $\rho=\rho_\alpha$ as given by (2.1) and $B\le 0$ gives us $0<\rho_\alpha<1$. □

When we take $\alpha=0$, we obtain a well-known result that $\operatorname{Re}\{\frac{(zg'(z))'}{g'(z)}\}>0$ implies $\operatorname{Re}\{\frac{zg'(z)}{g(z)}\}>\frac{1}{2}$, $z\in E$.

3. Main Results

Theorem 3.1. *Let $k\ge 2$ and $\alpha>-1$. Then $T_k^\star(\alpha)\subset T_k^\star(\alpha+1)$.*

Proof. Let $f\in T_k^\star(\alpha)$. Then, for $z\in E$,

$$\frac{z(I_\alpha f(z))'}{I_\alpha g(z)}\in P_k,\quad \text{for some}\quad g\in N^\star(\alpha).$$

We define an analytic function p in E such that

$$\frac{z(I_{\alpha+1}f(z))'}{I_{\alpha+1}g(z)}=p(z)=(\frac{k}{4}+\frac{1}{2})p_1(z)-(\frac{k}{4}-\frac{1}{2})p_2(z), \tag{3.1}$$

and $p(0)=1$.

We shall show that $p\in P_k$ in E or equivalently we show that $p_i\in P$, $i=1,2$ for $z\in E$.

From (3.1), (1.3) and the fact from Lemma 2.4 that $N^\star(\alpha)\subset N^\star(\alpha+1)$ for $\alpha>-1$, we obtain

$$\frac{z(I_\alpha f(z))'}{I_\alpha g(z)}=[p(z)+\frac{zp'(z)}{p_0(z)+\alpha}]\in P_k, \tag{3.2}$$

where

$$p_0(z)=\frac{z(I_{\alpha+1}g(z))'}{I_{\alpha+1}g(z)}\in P,\quad z\in E.$$

Using (3.1) and (1.5), we can write (3.2) as

$$\frac{z(I_\alpha f(z))'}{I_\alpha g(z)}=(\frac{k}{4}+\frac{1}{2})[p_1(z)+\frac{zp_1'(z)}{p_0(z)+\alpha}]-(\frac{k}{4}-\frac{1}{2})[p_2(z)+\frac{zp_2'(z)}{p_0(z)+\alpha}],$$

and applying Lemma 2.1 in a slightly modified form, we can easily show that $p_i \in P$ for $i = 1, 2,\ z \in E$. This implies that $p \in P_k$ and consequently $f \in T_k^\star(\alpha + 1)$, $z \in E$. □

Theorem 3.2. *Let $f \in T_k^\star(\alpha + 1)$ for $z \in E$. Then $f \in T_k^\star(\alpha)$ for $|z| < r_\alpha = \frac{\alpha+1}{2+\sqrt{\alpha^2+3}}$. This result is sharp.*

Proof. Since $f \in T_k^\star(\alpha + 1)$, we can write

$$\begin{aligned}\frac{z(I_{\alpha+1}f(z))'}{I_{\alpha+1}g(z)} &= H(z) = (\frac{k}{4} + \frac{1}{2})h_1(z) - (\frac{k}{4} - \frac{1}{2})h_2(z),\\ &\qquad g \in N^\star(\alpha + 1), \quad \operatorname{Re} h_i(z) > 0, \quad i = 1, 2, \quad z \in E.\end{aligned}$$

Using (1.3) and some computation, we obtain

$$\begin{aligned}\frac{z(I_\alpha f(z))'}{I_\alpha g(z)} &= \{H(z) + \frac{zH'(z)}{h(z) + \alpha}\}\\ &= (\frac{k}{4} + \frac{1}{2})\{h_1(z) + \frac{zh_1'(z)}{h(z) + \alpha}\} - (\frac{k}{4} - \frac{1}{2})\{h_2(z) + \frac{zh_2'(z)}{h(z) + \alpha}\},\end{aligned}$$

where

$$h(z) = \frac{z(I_{\alpha+1}g(z))'}{I_{\alpha+1}g(z)}, \quad \text{and} \quad h_i, h \in P, \quad i = 1, 2, \quad z \in E.$$

We consider $\{h_i(z) + \frac{zh_i'(z)}{h(z)+\alpha}\}$, with $h_i, h \in P$ and we will use Lemma 2.2 to have

$$\begin{aligned}\operatorname{Re}\{h_i(z) + \frac{zh_i'(z)}{h(z) + \alpha}\} &\geq \operatorname{Re} h_i(z)\{1 - \frac{2r}{1 - r^2}\frac{1}{\frac{1-r}{1+r} + \alpha}\}\\ &= \operatorname{Re} h_i(z)\{\frac{(1 + \alpha) - 4r + (1 - \alpha)r^2}{(1 - r)^2 + \alpha(1 - r^2)}\}.\end{aligned}$$

The right hand side of the above inequality is positive for $|z| < r_\alpha$, $r_\alpha = \frac{1+\alpha}{2+\sqrt{\alpha^2+3}}$. The function $f_0 \in T_k^\star(\alpha + 1)$, defined by

$$z(I_{\alpha+1}f_0(z))' = \frac{z(1 + z^2 - kz)}{(1 - z)(1 + z)^3}$$

shows that the radius r_α is the best possible. □

Theorem 3.3. *Let $\phi \in C$ and $f \in T_k^\star(\alpha)$. Then $f \star \phi \in T_k^\star(\alpha)$.*

Proof. First we refer to a result proved in [7] that if $\phi \in C$ and $g \in N^\star$, then $g \star \phi \in N^\star(\alpha)$.

Now

$$\frac{z(I_\alpha(f\star\phi))'}{I_\alpha(g\star\phi)} = \frac{z[\phi\star I_\alpha f]'}{\phi\star I_\alpha g} = \phi\star\frac{z(I_\alpha f)'}{I_\alpha g}(I_\alpha g) = \frac{\phi\star p(I_\alpha g)}{\phi\star I_\alpha g}, \quad p\in P_k$$
$$= (\frac{k}{4}+\frac{1}{2})[\frac{\phi\star p_1(I_\alpha g)}{\phi\star I_\alpha g}] - (\frac{k}{4}-\frac{1}{2})[\frac{\phi\star p_2(I_\alpha g)}{\phi\star I_\alpha g}], \quad p_1,p_2\in P.$$

Since $p_1, p_2 \in P$, $\phi\in C$ and $I_\alpha g\in S^\star$, we use a result due to Ruscheweyh and Shiel-Small [13] to get

$$\frac{\phi\star p_i(I_\alpha g)}{\phi\star I_\alpha g}\in P, \quad i=1,2.$$

This implies that $\frac{z[I_\alpha(f\star\phi)]'}{I_\alpha(g\star\phi)}\in P_k$ and thus $f\star\phi\in T_k^\star(\alpha)$ for $z\in E$. □

Theorem 3.4. *Let, for $\alpha > -1$ and $0<\beta<1$,*

$$f(z) = \frac{1}{\beta}z^{\frac{1}{\beta}-1}\int_0^z t^{\frac{1}{\beta}-2}F(t)dt. \tag{3.3}$$

Let $F\in T_k^\star(\alpha)$. Then $f\in T_k^\star(\alpha)$, $z\in E$. Conversely, if $f\in T_k^\star(\alpha)$, then $F\in T_k^\star(\alpha)$ for $|z|<r_\beta$, where

$$r_\beta = \frac{1}{2\beta+\sqrt{4\beta^2-2\beta+1}}. \tag{3.4}$$

This result is sharp.

Proof. We can write (3.3) as

$$f(z) = (\phi_\beta\star F)(z),$$

where $F\in T_k^\star(\alpha)$ and $\phi_\beta(z) = \sum_{m=1}^\infty \frac{1}{\beta(m-1)+1}z^m$. ϕ_β is convex in E, see [1]. Thus, applying Theorem 3.3, we see that $f\in T_k^\star(\alpha)$ for $z\in E$.

On the other hand $F(z) = (\Psi_\beta\star f)(z)$, where $f\in T_k^\star(\alpha)$ and $\Psi_\beta(z) = \sum_{m=1}^\infty[\beta(m-1)+1]z^m$, $\Psi_\beta\in C$ for $|z|<r_\beta$ and the exact values of r_β is given by (3.4). Applying Theorem 3.3, we obtain the required result. □

Theorem 3.5. *Let $F\in T_k^\star(\alpha)$, $\alpha>-1$ and let f be defined as*

$$f(z) = (\alpha+1)z^{-\alpha}\int_0^z t^{\alpha-1}F(t)dt. \tag{3.5}$$

Then $f\in T_k^\star(\alpha+1)$.

The proof follows immediately by observing that $I_\alpha F(z) = I_{\alpha+1}f(z)$.

Remark 3.1. We can apply Theorem 3.3 to note that the classes $T_k^\star(\alpha)$ are invariant under the following operators:

$$\begin{aligned} f_1(z) &= \int_0^z \frac{f(t)}{t}dt, \\ f_2(z) &= \int_0^z \frac{f(t)-f(xt)}{t-xt}dt, \quad |x|\le 1, \quad x\ne 1, \end{aligned}$$

since we can write $f_i = f\star\phi_i$, $i=1,2$ with

$$\begin{aligned} \phi_1(z) &= \sum_{m=1}^{\infty}\frac{z^m}{m} = -\mathrm{Log}\,(1-z)\,, \\ \phi_2(z) &= \sum_{m=1}^{\infty}\frac{1-x^m}{m(1-x)}z^m = \frac{1}{1-x}\,\mathrm{Log}\,\frac{1-xz}{1-z} \quad (|x|\le 1, x\ne 1)\,. \end{aligned}$$

and ϕ_i is convex for $i=1,2$, see [1].

Let $f(z) = z+\sum_{m=2}^{\infty} a_m z^m$, then by sections of f we mean $f_n(z) = z + \sum_{m=2}^{n} a_m z^m$. It is known that, for $z\in E$, $f_n(z) = (f\star g_n)(z)$, where

$$g_n(z) = z+\sum_{m=2}^{n} z^m = \frac{z-z^{n+1}}{1-z}, \tag{3.6}$$

and g_n is starlike in $|z|<(\frac{1}{2n})^{\frac{1}{n}}$ for all $n\in N_0$, $n\ge 2$, see [14].

We now prove the following.

Theorem 3.6. *Let $F = zf'\in T_k^\star(\alpha)$. Then $f_n\in T_k^\star(\alpha)$ for $|z|<(\frac{1}{2n})^{\frac{1}{n}}$ for all $n\in N_0$, $n\ge 2$.*

Proof. Let $\Psi = z\phi'$ for some $\phi\in C$. Then

$$(\Psi\star f)(z) = (z\phi'\star f)(z) = (\phi\star zf')(z) = (\phi\star F)(z).$$

Since $\phi\in C$ and $F\in T_k^\star(\alpha)$, it follows from Theorem 3.3 that $\Psi\star f\in T_k^\star(\alpha)$ for $z\in E$. Now $f_n(z) = (f\star g_n)(z)$, where g_n, given by (3.6), is starlike in $|z|<(\frac{1}{2n})^{\frac{1}{n}}$ for all $n\in N_0$, $n\ge 2$. Hence $f_n = f\star g_n$ is in $T_k^\star(\alpha)$ for $|z|<(\frac{1}{2n})^{\frac{1}{n}}$ for all $n\in N_0$, $n\ge 2$. □

References

[1] R.W. Barnard, C. Kellog, Applications of convolution operators to problems in univalent function theory, *Michigan Math. J.*, **27** (1980), 81-94.

[2] N.E. Cho, The Noor integral operator and strongly close-to-convex functions, *J. Math. Anal. Appl.*, **283** (2003), 202-212.

[3] A.W. Goodman, *Univalent Functions*, Volume **I**, Washington, N.J., Polygonal Publishing House (1983).

[4] Y. Ling, F. Liu, The generalized Noor integral operator and a class of analytic functions, *Preprint* (2003).

[5] J.L. Liu, The Noor integral operator and strongly starlike functions, *J. Math. Anal. Appl.*, **261** (2001), 441-447.

[6] J.L. Liu, K. Inayat Noor, Some properties of Noor integral operator, *J. Natural Geometry*, **21** (2002), 81-90.

[7] K. Inayat Noor, On new classes of integral operators, *J. Nat. Geometry*, **16** (1999), 71-80.

[8] K. Inayat Noor, Some classes of p-valent analytic functions defined by certain integral operator, *Appl. Math. Computation* (2004).

[9] K. Inayat Noor, Properties of certain new classes of analytic functions, *Nonl. Funct. Anal. Appl.*, **9** (2004).

[10] K. Inayat Noor, M.A. Noor, On integral operators, *J. Math. Anal. Appl.*, **238** (1999), 341-352.

[11] K. Inayat Noor, M. Aslam Noor, On certain classes of analytic functions defined by Noor integral operator, *J. Math. Anal. Appl.*, **281** (2003), 244-252.

[12] S.S. Miller, Differential inequalities and Caratheodory functions, *Bull. Amer. Math. Soc.,* **81** (1975), 79-81.

[13] S. Ruscheweyh, T. Shiel-Small, Hadamard products of schlicht functions and the Polya-Schoenberg conjecture, *Comment. Math. Helv.*, **48** (1973), 119-135.

[14] H. Silverman, Radii problems for sections of convex functions, *Proc. Amer. Math. Soc.*, **104** (1988), 1191-1196.