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HOMOGENOUS BALANCE METHOD AND NEW EXACT SOLUTIONS TO THE (3+1)-DIMENSIONAL JIMBO-MIWA EQUATION

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Abstract: In this paper, various exact solutions, including soliton, multisoliton and rational-type solutions, of the (3+1)-dimensional Jimbo-Miwa equation are obtained by using homogenous balance method.

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1. Introduction

Up to now, there exist many powerful methods to construct exact solutions of nonlinear partial differential equations (PDEs). For example, inverse scattering transformation method [1], Darboux transformation method [4], Hirota method [5], homogeneous balance method [7], hyperbolic function method [2]. By using these methods, many kinds of exact solutions of some PDEs have been obtained. However, due to the importance of the exact solutions of PDEs in physics and

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mathematics, it is still a hot-spot to seek new methods for obtaining new exact solutions.

The main purpose of this paper is to make a further study of the question how to seek exact solutions of the (3+1)-dimensional Jimbo-Miwa equation. With the aid of symbolic computation software such as *Mathematica* or *Maple*, we obtain a great deal of new types of exact solutions of (3+1)-dimensional Jimbo-Miwa equation, including soliton, multisoliton and rational-type solutions, by using the homogenous balance method.

The rest of this paper is organized as follows. Section 2 gives the brief formulation of the homogenous balance method. In Section 3 we present some new types of exact solutions of the (3+1)-dimensional Jimbo-Miwa equation by using the homogenous balance method stated in Section 2. Section 4 concludes the paper with some comments and discussion.

2. Review on the Research of the Homogeneous Balance Method

Aiming at the characteristic of the PDEs in mathematics and physics, in 1995, Wang Ming-liang [7] put forward an effective homogeneous balance method to obtain solitons. The main steps are as follows:

Step 1. For the given nonlinear PDE, without loss generality, assume that it contains two variables x, t

$$Au = A(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \cdots) = 0, \tag{1}$$

Suppose that the solution of equation (1) can be expressed as a function of a new variable $\psi = \psi(x, t)$

$$u(x,t) = \sum_{i=0,j=0}^{m,n} a_{ij} \partial_t^i \partial_x^j f(\psi) + b, \tag{2}$$

where m and n are nonnegative integers which can be determined by making the balance between the highest order derivative and the highest nonlinear terms of (1).

Step 2. Substituting (2) into (1) and setting the coefficient of the highest order derivative and the highest power equal to zero, we obtain an ODE of $F(\psi)$, from which we get the function $f(\psi)$, and write it as

$$f = F(\psi). \tag{3}$$

Step 3. Applying (3), we can change the nonlinearities of derivatives of f obtained in Step 2 into the linearities of derivatives of f. Then setting the coefficients of all order derivatives of f to zero, leads to an over-determined system of partial differential equations of ψ

$$D(\psi) = 0. \tag{4}$$

Step 4. Wang supposed that equation (4) has the special solution in the form

$$\psi = 1 + e^{\alpha x + \beta t}, \tag{5}$$

where α, β are the constants to be determined later. Substituting (5) into (4), we can obtain α and β. Then the solitary wave solutions of (1) can be found, by substituting (3) and (5) into (2).

A further development made by Fan and Zhang [3] improved considerably the key steps of the HBM. In their recent paper they made the method much more lucid and straightforward to apply to a class of nonlinear PDEs and obtained many new types of exact solutions [2]. The motivation of the present paper is to utilize their ideas [3] to explore some new solutions of (3+1)-dimensional Jimbo-Miwa equation by using the homogenous balance method.

3. The New Types of Exact Solutions of the (3+1)-Dimensional Jimbo-Miwa Equation

The (3+1)-dimensional Jimbo-Miwa equation reads

$$u_{xxxy} + 3u_{xy}u_x + 3u_y u_{xx} + 2u_{yt} - 3u_{xz} = 0. \tag{6}$$

M. Senthilvelan studied the (3+1)-dimensional Jimbo-Miwa equation by using the tanh-function method and obtained the travelling wave solutions as follows [6]:

$$u = a_0 + 2k\alpha \tanh[k(\alpha x + \beta y + \gamma z - \lambda t)], \tag{7}$$

$$u = a_0 + 2k\alpha \coth[k(\alpha x + \beta y + \gamma z - \lambda t)], \tag{8}$$

$$u = a_0 - 2k\alpha \tanh[k(\alpha x + \beta y + \gamma z - \lambda t)], \tag{9}$$

$$u = a_0 - 2k\alpha \coth[k(\alpha x + \beta y + \gamma z - \lambda t)]. \tag{10}$$

Now let us construct solutions for this equation once more. To begin with, let us make a transformation of (6)

$$u = \partial_x^i \partial_y^j \partial_z^k h[\omega(x, y, z, t)] + h_0, \tag{11}$$

where $h(\omega)$, $h_0(x,y,z,t)$, $\omega(x,y,z,t)$, i, j and k are to be determined later.

In order to obtain i, j, k, substituting (11) into (6). We make the leading-order analysis as follows. For equation (6) the possible highest-power terms are $\omega_x^{i+3}\omega_y^{j+1}\omega_z^{k}$ and $\omega_x^{2i+2}\omega_y^{2j+1}\omega_z^{2k}$, which are, contributed by u_{xxxy}, $3u_{xy}u_x$ and $3u_y u_{xx}$, respectively. Then the balancing act requires that the two terms have the same power, i.e.

$$i+3=2i+2,\quad j+1=2j+1,\quad k=2k. \tag{12}$$

From (12) we get

$$i=1,\quad j=0,\quad k=0. \tag{13}$$

Thus (11) becomes

$$u=h'\omega_x+h_0. \tag{14}$$

With the aid of symbolic computation software such as *Mathematica* or *Maple*, substituting (14) into (6), and collecting all homogeneous terms in partial derivatives of $\omega(x,y,z,t)$, we get

$$\begin{aligned}
&(6h''h^{(3)}+h^{(5)})\omega_x^4\omega_y+(4h^{(4)}+6(h'')^2+3h^{(3)}h')\omega_x^3\omega_{xy}\\
&+(6h^{(4)}+12(h'')^2+3h^{(3)}h')\omega_x^2\omega_y\omega_{xx}+12h^{(3)}\omega_x\omega_{xx}\omega_{xy}\\
&+3h^{(3)}\omega_y\omega_{xx}^2+6h^{(3)}\omega_x^2\omega_{xxy}+4h^{(3)}\omega_x\omega_y\omega_{xxx}\\
&+3h''h'\omega_x^2\omega_{xxy}+3h''h'\omega_y\omega_{xx}^2+6h''h'\omega_x\omega_{xx}\omega_{xy}+3h^{(3)}h_{0x}\omega_y\omega_x^2\\
&+3h''h'\omega_x\omega_y\omega_{xxx}+9h''h'\omega_x\omega_{xx}\omega_{xy}+3h^{(3)}h_{0y}\omega_x^3+2h^{(3)}\omega_x\omega_y\omega_t\\
&-3h^{(3)}\omega_z\omega_x^2+8h''\omega_{xx}\omega_{xxy}+4h''\omega_{xy}\omega_{xxx}+4h''\omega_x\omega_{xxxy}\\
&+h''\omega_y\omega_{xxxx}+3h''h_{0xy}\omega_x^2+3(h')^2\omega_{xx}\omega_{xxy}+6h''h_{0x}\omega_x\omega_{xy}\\
&+3h''h_{0x}\omega_y\omega_{xx}+3h''h_{0xx}\omega_x\omega_y+3(h')^2\omega_{xy}\omega_{xxx}+9h''h_{0x}\omega_x\omega_{xx}\\
&+2h''\omega_x\omega_{yt}+2h''\omega_y\omega_{xt}+2h''\omega_t\omega_{xy}-6h''\omega_x\omega_{xz}-3h''\omega_z\omega_{xx}\\
&+h'\omega_{xxxxy}+3h'h_{0xy}\omega_{xx}+3h'h_{0x}\omega_{xxy}+3h'h_{0xx}\omega_{xy}\\
&+3h'h_{0y}\omega_{xxx}+2h'\omega_{xyt}-3h'\omega_{xxz}+h_{0xxxy}\\
&+3h_{0xy}h_{0x}+3h_{0xx}h_{0y}+2h_{0yt}-3h_{0xz}=0.
\end{aligned} \tag{15}$$

To determine the function $h(\omega)$, we set the coefficients of the term $\omega_y\omega_x^4$ in equation (15) to zero, and obtain an ODE with respect to $h(\omega)$:

$$6h''h'''+h^{(5)}=0, \tag{16}$$

which has the special solution in the form

$$h(\omega) = 2\ln(\omega), \tag{17}$$

and it implies that

$$h'h'' = -h''', \quad (h')^2 = -2h''. \tag{18}$$

Having seen the expression for $h(\omega)$ from (17), we investigate $\omega(x, y, z, t)$. Substituting (17) and (18) into (15), then it is reduced to

$$\begin{aligned}
&(12\omega_x\omega_{xx}\omega_{xy} + 3\omega_y\omega_{xx}^2 + 6\omega_x^2\omega_{xxy} + 4\omega_x\omega_y\omega_{xxx} - 3\omega_x^2\omega_{xxy}\\
&\quad - 3\omega_y\omega_{xx}^2 - 6\omega_x\omega_{xx}\omega_{xy} + 3h_{0x}\omega_y\omega_x^2 - 3\omega_x\omega_y\omega_{xxx} - 9\omega_x\omega_{xx}\omega_{xy}\\
&\quad + 3h_{0y}\omega_x^3 + 2\omega_x\omega_y\omega_t - 3\omega_z\omega_x^2)h''' + (6\omega_{xx}\omega_{xxy} + 4\omega_{xy}\omega_{xxx}\\
&\quad + 4\omega_x\omega_{xxxy} + \omega_y\omega_{xxxx} + 3h_{0xy}\omega_x^2 - 6\omega_{xx}\omega_{xxy} + 6h_{0x}\omega_x\omega_{xy}\\
&\quad + 3h_{ox}\omega_y\omega_{xx} + 3h_{0xx}\omega_x\omega_y - 6\omega_{xy}\omega_{xxx} + 9h_{0y}\omega_x\omega_{xx}\\
&\quad + 2\omega_x\omega_{yt} + 2\omega_y\omega_{xt} + 2\omega_t\omega_{xy} - 6\omega_x\omega_{xz} - 3\omega_z\omega_{xx})h''\\
&\quad + (\omega_{xxxxy} + 3h_{0xy}\omega_{xx} + 3h_{0x}\omega_{xxy} + 3h_{0xx}\omega_{xy}\\
&\quad + 3h_{0y}\omega_{xxx} + 2\omega_{xyt} - 3\omega_{xxz})h' + h_{0xxxy}\\
&\quad + 3h_{0xy}h_{0x} + 3h_{0xx}h_{0y} + 2h_{0yt} - 3h_{0xz} = 0.
\end{aligned} \tag{19}$$

Setting the coefficients of h''',h'' and h' to zero, we derive an over-determined system of partial differential equations w.r.t. ω:

$$\begin{aligned}
&12\omega_x\omega_{xx}\omega_{xy} + 3\omega_y\omega_{xx}^2 + 6\omega_x^2\omega_{xxy} + 4\omega_x\omega_y\omega_{xxx} - 3\omega_x^2\omega_{xxy}\\
&-3\omega_y\omega_{xx}^2 - 6\omega_x\omega_{xx}\omega_{xy} + 3h_{0x}\omega_y\omega_x^2 - 3\omega_x\omega_y\omega_{xxx}\\
&-9\omega_x\omega_{xx}\omega_{xy} + 3h_{0y}\omega_x^3 + 2\omega_x\omega_y\omega_t - 3\omega_z\omega_x^2 = 0,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&6\omega_{xx}\omega_{xxy} + 4\omega_{xy}\omega_{xxx} + 4\omega_x\omega_{xxxy} + \omega_y\omega_{xxxx} + 3h_{0xy}\omega_x^2 - 6\omega_{xx}\omega_{xxy}\\
&+6h_{0x}\omega_x\omega_{xy} + 3h_{ox}\omega_y\omega_{xx} + 3h_{0xx}\omega_x\omega_y - 6\omega_{xy}\omega_{xxx} + 9h_{0y}\omega_x\omega_{xx}\\
&+2\omega_x\omega_{yt} + 2\omega_y\omega_{xt} + 2\omega_t\omega_{xy} - 6\omega_x\omega_{xz} - 3\omega_z\omega_{xx} = 0,
\end{aligned} \tag{21}$$

$$\omega_{xxxxy} + 3h_{0xy}\omega_{xx} + 3h_{0x}\omega_{xxy} + 3h_{0xx}\omega_{xy} + 3h_{0y}\omega_{xxx} + 2\omega_{xyt} - 3\omega_{xxz} = 0, \tag{22}$$

$$h_{0xxxy} + 3h_{0xy}h_{0x} + 3h_{0xx}h_{0y} + 2h_{0yt} - 3h_{0xz} = 0. \tag{23}$$

Taking h_0 =const., from (14) and (17), we can get the Backlund transformation of (11) in the form

$$u = \frac{2\omega_x}{\omega} + h_0, \tag{24}$$

where $\omega(x,y,z,t)$ is to be determined in (31), (38), (40), and (42), respectively. Substituting h_0 =const. into (20)–(23), they are reduced to

$$-3\omega_x\omega_{xx}\omega_{xy}+3\omega_x^2\omega_{xxy}+\omega_x\omega_y\omega_{xxx}+2\omega_x\omega_y\omega_t-3\omega_x^2\omega_z=0, \tag{25}$$

$$\begin{aligned}&6\omega_{xx}\omega_{xxy}+4\omega_{xxx}\omega_{xy}+4\omega_x\omega_{xxxy}+\omega_y\omega_{xxxx}-6\omega_{xx}\omega_{xxy}\\ &-6\omega_{xy}\omega_{xxx}+2\omega_x\omega_{yt}+2\omega_t\omega_{xy}+2\omega_y\omega_{xt}-6\omega_x\omega_{xz}-3\omega_z\omega_{xx}=0,\end{aligned} \tag{26}$$

$$\omega_{xxxxy}+2\omega_{xyt}-3\omega_{xxz}=0. \tag{27}$$

It is easy to see that the above equations are satisfied, providing

$$\omega_z=\omega_{xxxx}, \tag{28}$$

$$\omega_t=\omega_{xxx}, \tag{29}$$

$$\omega_y=\omega_{xx}. \tag{30}$$

From (28)-(30), we can find four kinds special solutions for equations (25)–(27). Now let us give them in details.

Case 1.

$$\omega(x,y,z,t)=\sum_{i=0}^{N}[a_ix+b_i+e^{(k_ix+k_i^2y+k_i^3t+k_i^4z+c_i)}], \tag{31}$$

where $a_i,b_i,k_i,c_i,i=0,1,2,\cdots,N$ are arbitrary constants and N is a nonnegative integer. It is easy to verify that (31) is a special solution of (25)–(27).

Substituting (31) into (24), we can obtain various exact solutions of equation (6):

$$u(x,y,z,t)=\frac{2\sum_{i=0}^{N}[a_i+k_ie^{(k_ix+k_i^2y+k_i^3t+k_i^4z+c_i)}]}{\sum_{i=0}^{N}[a_ix+b_i+e^{(k_ix+k_i^2y+k_i^3t+k_i^4z+c_i)}]}+h_0. \tag{32}$$

If we take $a_i=0,i=0,1,2,\cdots,N,\sum_{i=0}^{N}b_i=1$, the multisoliton solutions of equation (6) can be obtained in the form

$$u(x,y,z,t)=\frac{2\sum_{i=0}^{N}[k_ie^{(k_ix+k_i^2y+k_i^3t+k_i^4z+c_i)}]}{1+\sum_{i=0}^{N}e^{(k_ix+k_i^2y+k_i^3t+k_i^4z+c_i)}}+h_0. \tag{33}$$

Case 1 A. $N=0$. In this case, we get the kink-soliton solution

$$u_1(x,y,z,t)=k_0\tanh[\frac{1}{2}(k_0x+k_0^2y+k_0^3t+k_0^4z+c_0)]+k_0+h_0. \tag{34}$$

Case 1 B. $N = 1$. In this case, we get the two-soliton solution

$$u_2(x,y,z,t) = \frac{2k_0 e^{(k_0x+k_0^2y+k_0^3t+k_0^4z+c_0)} + 2k_1 e^{(k_1x+k_1^2y+k_1^3t+k_1^4z+c_1)}}{1+e^{(k_0x+k_0^2y+k_0^3t+k_0^4z+c_0)}+e^{(k_1x+k_1^2y+k_1^3t+k_1^4z+c_1)}} + h_0. \quad (35)$$

Case 1 C. $N = 2$. In this case, we get the three-soliton solution

$$u_3(x,y,z,t) = \frac{2k_0 e^{(k_0x+k_0^2y+k_0^3t+k_0^4z+c_0)} + 2k_1 e^{(k_1x+k_1^2y+k_1^3t+k_1^4z+c_1)}}{1+e^{(k_0x+k_0^2y+k_0^3t+k_0^4z+c_0)}+e^{(k_1x+k_1^2y+k_1^3t+k_1^4z+c_1)}}$$

$$\frac{+2k_2 e^{(k_2x+k_2^2y+k_2^3t+k_2^4z+c_2)}}{+e^{(k_2x+k_2^2y+k_2^3t+k_2^4z+c_2)}} + h_0. \quad (36)$$

Case 1 D. $k_i = 0, i = 0, 1, 2, \cdots, N, \sum_{i=0}^{N} a_i^2 \neq 0$. In this case, we can get the rational solution from (32)

$$u_4(x,y,z,t) = \frac{2\sum_{i=0}^{N} a_i}{\sum_{i=0}^{N}(a_ix+b_i+1)} + h_0. \quad (37)$$

Case 2.

$$\omega(x,y,z,t) = \sum_{i=0}^{N}[a_ix+b_i+\sin(k_ix+k_i^3t+c_i)e^{(-k_i^2y+k_i^4z+d_i)}$$

$$+\cos(l_ix+l_i^3t+c_i')e^{(-l_i^2y+l_i^4z+d_i')}]. \quad (38)$$

From (24), we get the second type of new exact solution

$$u_5(x,y,z,t) = \frac{2\sum_{i=0}^{N}[a_i+k_i\cos(k_ix+k_i^3t+c_i)e^{(-k_i^2y+k_i^4z+d_i)}]}{\sum_{i=0}^{N}[a_ix+b_i+\sin(k_ix+k_i^3t+c_i)e^{(-k_i^2y+k_i^4z+d_i)}]}$$

$$\frac{-2\sum_{i=0}^{N}[l_i\sin(l_ix+l_i^3t+c_i')e^{(-l_i^2y+l_i^4z+d_i')}]}{+\sum_{i=0}^{N}[\cos(l_ix+l_i^3t+c_i')e^{(-l_i^2y+l_i^4z+d_i')}]}. \quad (39)$$

Case 3.

$$\omega(x,y,z,t) = \sum_{i=0}^{N}[a_ix+b_i+\sinh(k_ix+k_i^3t+c_i)e^{(k_i^2y+k_i^4z+d_i)}$$

$$+\cosh(l_ix+l_i^3t+c_i')e^{(l_i^2y+l_i^4z+d_i')}]. \quad (40)$$

From (24), we get the third type of new exact solution

$$u_6(x,y,z,t) = \frac{2\sum_{i=0}^{N}[a_i + k_i\cosh(k_ix + k_i^3t + c_i)e^{(k_i^2y+k_i^4z+d_i)}]}{\sum_{i=0}^{N}[a_ix + b_i + \sinh(k_ix + k_i^3t + c_i)e^{(k_i^2y+k_i^4z+d_i)}]}$$

$$\frac{+2\sum_{i=0}^{N}[l_i\sinh(l_ix + l_i^3t + c_i')e^{(l_i^2y+l_i^4z+d_i')}]}{+\sum_{i=0}^{N}[\cosh(l_ix + l_i^3t + c_i')e^{(l_i^2y+l_i^4z+d_i')}]}. \tag{41}$$

Case 4. A direct computation shows that

$$\omega(x,y,z,t) = 4x^3 + 24t + 24xy + ax + b \tag{42}$$

satisfies (28)-(30). Hence we get a new kind of rational solution of (6):

$$u_7(x,y,z,t) = \frac{24x^2 + 48y + 2a}{4x^3 + 24t + 24xy + ax + b} + h_0, \tag{43}$$

where $a,\ b$ are arbitrary constants.

Remark. In the above, we have found some new exact solutions to the (3+1)-dimensional Jimbo-Miwa equation that are not obtained by the tanh method [2] and [6]. It is worth mentioning that the rational solution given by (43) might be reported for the first time in the literature, to our best knowledge.

4. Conclusion and Discussion

In summary, we have obtained many types of new exact solutions for the (3+1)-dimensional Jimbo-Miwa equation by using the symbolic computation and an improved homogeneous balance method. These solutions obtained may be of significance for the explanation of some practical physical problems. It is also shown that the homogeneous balance method is a powerful technique for investigating nonlinear wave equations, in particular, for seeking different kinds of exact solutions for PDEs in mathematics and physics.

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