**International Journal of Pure and Applied Mathematics**
**Volume 13 No. 4 2004, 481-486**

# A NOTE ON $\mid \bar{N}, p_n \mid_k$ SUMMABILITY FACTORS

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**Abstract:** In this paper two theorems of Lal [3] on $\mid \bar{N}, p_n \mid$ summability methods have been generalized for $\mid \bar{N}, p_n \mid_k$ summability methods by using different and general summability factors.

**AMS Subject Classification:** 40D25, 40F05
**Key Words:** absolute summability, summability factors

## 1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1}$$

The sequence-to-sequence transformation

Received: March 16, 2004 

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence $(t_n)$ of the $(\bar{N}, p_n)$ means of the sequence $(s_n)$, generated by the sequence of coefficients $(p_n)$ (see [2]).

The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{3}$$

Let $f(t)$ be a periodic function with period $2\pi$ and integrable $(L)$ over $(-\pi, \pi)$. The Fourier series of $f(t)$ is

$$f(t) \cong \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{4}$$

The following theorems are known.

**Theorem A.** (see [3]) *If the sequence $(s_n)$ is bounded and $(\lambda_n)$ is a sequence such that*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \mid \lambda_n \mid = O(1) \quad as \quad m \to \infty, \tag{5}$$

$$\sum_{n=1}^{m} \mid \Delta\lambda_n \mid = O(1) \quad as \quad m \to \infty, \tag{6}$$

*then the series $\sum a_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid$.*

**Theorem B.** (see [3]) *The summability $\mid \bar{N}, p_n \mid$ of the series $\sum A_n(t)\lambda_n$ at a point is a local property of the generating function if the conditions (5)-(6) of Theorem A are satisfied.*

In this paper we shall generalize Theorem A under different conditions by using a general summability factors for $\mid \bar{N}, p_n \mid_k$ summability methods.

Now, we shall prove the following theorems.

**Theorem 1.** *Let $k \geq 1$. If the sequence $(s_n)$ is bounded and the sequences $(\lambda_n)$ and $(p_n)$ satisfy the following conditions*

$$\sum_{n=1}^{m} p_n \mid \lambda_n \mid = O(1) \quad as \quad m \to \infty, \tag{7}$$

$$\sum_{n=1}^{m} P_n \mid \Delta\lambda_n \mid = O(1) \quad as \quad m\to\infty, \tag{8}$$

$$p_{n+1} = O(p_n), \tag{9}$$

*then the series* $\sum a_n\lambda_n P_n$ *is summable* $|\bar{N},p_n|_k$.

**Theorem 2.** *Let* $k\geq 1$. *The summability* $|\bar{N},p_n|_k$ *of the series* $\sum A_n(t)\lambda_n P_n$ *at a point is a local property of the generating function if the conditions (7) and (8) are satisfied.*

We need the following lemma for the proof of our theorem.

**Lemma.** *If the sequences* $(\lambda_n)$ *and* $(p_n)$ *satisfy the conditions (7) and (8) of Theorem 1, then* $P_m \mid \lambda_m \mid = O(1) \quad as \quad m\to\infty.$

*Proof.* By Abel partial summation formula, we have

$$\sum_{n=1}^{m} p_n\lambda_n = \sum_{n=1}^{m-1} P_n\Delta\lambda_n + P_m\lambda_m\,,$$

$$\mid P_m\lambda_m \mid = \mid \sum_{n=1}^{m} p_n\lambda_n - \sum_{n=1}^{m-1} P_n\Delta\lambda_n\,, \mid$$

$$P_m \mid \lambda_m \mid < \sum_{n=1}^{m} p_n \mid \lambda_n \mid + \sum_{n=1}^{m-1} P_n \mid \Delta\lambda_n \mid = O(1)\,..$$

Hence $P_m \mid \lambda_m \mid = O(1)$ as $m\to\infty$. □

*Proof of Theorem 1.* Without any loss of generality we may assume that $a_0 = s_0 = 0$. Let $(T_n)$ denotes the $(\bar{N},p_n)$ means of the series $\sum a_n\lambda_n P_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r P_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v P_v\lambda_v.$$

Then

$$T_n - T_{n-1} \quad = \quad \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}P_v a_v\lambda_v, \quad n\geq 1, \quad (P_{-1}=0).$$

By Abel transformation, we have

$$T_n - T_{n-1} = -\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v p_v s_v \lambda_v + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v \Delta\lambda_v P_v s_v$$

$$- \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v p_{v+1} s_v \lambda_{v+1} + p_n s_n \lambda_n$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, .$$

To complete the proof of Theorem 1, by Minkowski inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1} \mid T_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{10}$$

Now, applying Hölder inequality, we have that

$$\sum_{n=2}^{m+1}(P_n/p_n)^{k-1} \mid T_{n,1} \mid^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n-1} P_v \mid \lambda_v \mid\mid s_v \mid^k p_v$$

$$\times \{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \lambda_v \mid p_v\}^{k-1} = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v \mid \lambda_v \mid p_v$$

$$= O(1)\sum_{v=1}^{m} P_v \mid \lambda_v \mid p_v \sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} = O(1)\sum_{v=1}^{m} p_v \mid \lambda_v \mid = O(1)\,,$$

as $m \to \infty$, by virtue of hypotheses of Theorem 1. Again

$$\sum_{n=2}^{m+1}(P_n/p_n)^{k-1} \mid T_{n,2} \mid^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid\mid s_v \mid^k P_v$$

$$\times \{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid P_v\}^{k-1} = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid P_v$$

$$= O(1)\sum_{v=1}^{m} P_v \mid \Delta\lambda_v \mid P_v \sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} = O(1)\sum_{v=1}^{m} P_v \mid \Delta\lambda_v \mid = O(1)\,,$$

as $m \to \infty$, by virtue of the hypotheses of Theorem 1. Again, similarly we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,3} \mid^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_n (P_{n-1})^k} \{ \sum_{v=1}^{n-1} P_v p_{v+1} \mid \lambda_{v+1} \mid\mid s_v \mid \}^k$$
$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \mid \lambda_v \mid p_v \times \{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \mid \lambda_v \mid p_v \}^{k-1}$$
$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \mid \lambda_v \mid p_v$$
$$= O(1) \sum_{v=1}^{m} P_v \mid \lambda_v \mid p_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O(1) \sum_{v=1}^{m} p_v \mid \lambda_v \mid = O(1) \, ,$$

as $m \to \infty$, by virtue of the hypotheses of Theorem 1. Finally, we have that

$$\sum_{n=1}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,4} \mid^k = \sum_{n=1}^{m} (P_n/p_n)^{k-1} \mid p_n s_n \lambda_n \mid^k$$
$$= O(1) \sum_{n=1}^{m} (P_n \mid \lambda_n \mid)^{k-1} p_n \mid \lambda_n \mid\mid s_n \mid^k$$
$$= O(1) \sum_{n=1}^{m} p_n \mid \lambda_n \mid = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the lemma and hypotheses of Theorem 1. Therefore, we get that

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} \mid T_{n,r} \mid^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

This completes the proof of Theorem 1. □

*Proof of Theorem 2.* Since the behavior of the Fourier series for a particular value of $x$, as far as convergence is concerned, depens on the behavior of the function in the immediate neighbourhood of this point only, Theorem 2 is an immediate consequence of Theorem 1.

If we take $p_n = 1$ for all values of $n$ in these theorems, then we get two results related to $| C, 1 |_k$ summability methods. □

## References

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