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ON THE GELBAUM-DE LAMADRID'S RESULT

R. Alencar[1], M.L. Lourenço[2 §]

[1]Instituto Tecnológico de Aeronautica
Praça Eduardo Gomes 50
CEP: 12228-900 São Jose dos Campos - SP, BRAZIL
e-mail: ralencar@ief.ita.br

[2]Departamento de Matemática
Universidade de São Paulo
CP 66281 - CEP: 05315-970
Rua do Matão, 1010-05508-090, São Paulo, BRAZIL
e-mail: mllouren@ime.usp.br

Abstract: The Gelbaum-DeLamadrid's result on tensor product bases is studied in view of a simple function that establish the square ordering. The extension of the function is applied to the k-fold product tensor bases.

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1. Introduction

If E and F are Banach spaces with Schauder bases (e_j) and $(f_k), (e_j, e_j^*)$ and (f_j, f_j^*) the associated biorthogonal systems, the natural projections on the first n coordenates are given by

$$P_n(x) = \sum_{j=1}^{n} e_j^*(x) \cdot e_j, \quad Q_n(x) = \sum_{l=1}^{n} f_l^*(x) \cdot f_l \,.$$

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§Correspondence author

It is well known in this case that the sequence $(e_j \otimes f_l)$ with the "square ordering" is a Schauder basis for both $E\widehat{\otimes}_\pi F$ and $E\widehat{\otimes}_\epsilon F$, the projective and injective tensor products. This result is due to Gelbaum–Gil DeLamadrid [2]. Most of the papers on the subject, do not indicate explicitly the square ordering and consequently do not exhibit properly the projections on the first coordenates of the product $E\widehat{\otimes}_\pi F$ (or $E\widehat{\otimes}_\epsilon F$). Also, each author has a different formula to represent the projections, defined as a combination of the tensor product operators $P_n \otimes Q_n$ in at least three branches.

In this study we introduce a simple function that establishes the square ordering on $\mathbb{N}^2$ from which is possible to exhibit precisely the projections on the first coordenates (defined with only two branches) on $E\widehat{\otimes}_\pi F$ (or $E\widehat{\otimes}_\epsilon F$). Since the function introduced has a natural extension to $\mathbb{N}^k$, the proof for the general case (finite many factors) is easily established.

2. Preliminaries

Let E be a Banach space and E^* its dual.

Definition 2.1. Let $(x_j)_{j=1}^\infty$ and $(x_j^*)_{j=1}^\infty$ be sequences in E and E^* respectively. If $x_j^*(x_k) = \delta_{jk}$ then (x_j, x_j^*) is called a biorthogonal system for E.

Definition 2.2. A sequence $(e_j)_{j=1}^\infty$ in E is called a Schauder basis of E if for every $x \in E$ there is a unique sequence of scalars $(\alpha_j)_{j=1}^\infty$, so that $x = \sum_{j=1}^{\infty} \alpha_j e_j$.

Some known facts about Schauder bases.

Let $(e_j)_{j=1}^\infty$ be a Schauder basis for E.

The projections $P_n : E \to E$ defined by $P_n(\sum_{j=1}^{\infty} \alpha_j e_j) = \sum_{j=1}^{n} \alpha_j e_j$ are bounded linear operators with $\sup_n \| P_n \| < \infty$. The number $c = \sup_n \| P_n \|$ is called the basis constant of $(e_j)_{j=1}^\infty$. For every integer n, the linear functional e_n^* on E defined by $e_n^*(\sum_{j=1}^{\infty} \alpha_j e_j) = \alpha_n$ is a bounded linear functional. These functionals e_j^* are characterized by the relation $e_j^*(e_k) = \delta_{jk}$ and therefore (e_j, e_j^*) is a biorthogonal system for E associated to the base $(e_j)_{j=1}^\infty$.

The next theorem is a characterization of bases.

Theorem 2.1. (see [3]) *Let (e_j, e_j^*) be a biorthogonal system for E. Then*

(e_j) is a basis of the closed linear span $\{e_j : j = 1.2..\}$ if and only if $1 \leq \sup_n \| P_n \| < \infty$.

3. Tensor Product

Let X, Y and Z be vector spaces over the same field of scalars. $L(X, Z)$ denotes the spaces of all linear mappings from X to Z. $B(X, Y; Z)$ denotes the spaces of all bilinear mappings from $X \times Y$ to Z.

In case Z is the scalar field, the members of $B(X, Y; Z)$ are bilinear functionals and it is denoted by $B(X, Y)$.

For $x \in X$, $y \in Y$, $x \otimes y$ denotes the linear functional on $B(X, Y)$ defined by $(x \otimes y)(\varphi) = \varphi(x, y)$ $(\varphi \in B(X, Y))$.

Definition 3.1. The tensor product of X and Y, denoted by $X \otimes Y$, is the vector subspace of the dual of $B(X, Y)$ spanned by $\{x \otimes y : x \in X,\ y \in Y\}$; $X \otimes Y = [x \otimes y : x \in X,\ y \in Y]$. The basic properties of $x \otimes y$ (elementar tensor) are the following: For α in the scalar field and elements $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$:

(a) $(\alpha x) \otimes y = x \otimes (\alpha y) = \alpha(x \otimes y)$,

(b) $(x_1 + y_2) \otimes y = x_1 \otimes y + x_2 \otimes y$,

(c) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$.

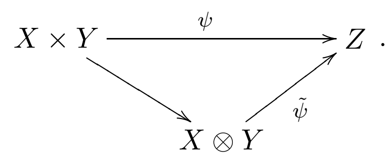
A typical member $u \in X \otimes Y$ can be represented in the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i, \quad x_i \in X, \quad y_i \in Y, \qquad 1 \leq i \leq n$$

(the scalars are absorbed either by x_i or y_i).

Remark 3.1. If $A \subset X, C \subset Y$ are linearly independent sets then the subset $A \otimes C$ of $X \otimes Y$ given by $A \otimes C = \{x \otimes y : x \in A, y \in C\}$ is a linearly independent set in $X \otimes Y$.

Theorem 3.1. (The Universal Mapping Property, see [1]) *For any vector spaces X, Y and Z there is an isomorphism between the space $L(X \otimes Y; Z)$, of linear mappings from $X \otimes Y$ to Z and the space $B(X, Y; Z)$, of bilinear* *mappings from $X \times Y$ to Z. That is, the following diagrama is commutative*



In case X and Y be normed spaces, two main norms are considered on $X \otimes Y$. Let $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$.

The *projective* π and *injective* ϵ norms are given by:

Definition 3.2. 1. The projective π norm is given by

$$\pi(u) = \inf\{\sum_{i=1}^{n} ||x_i|| \cdot ||y_i|| : u = \sum_{i=1}^{n} x_i \otimes y_i\},$$

where the infimum is taken over all finite representation $\sum_{i=1}^{n} x_i \otimes y_i$ of u.

2. The injective ϵ norm is given by

$$\epsilon(u) = \sup\{|\sum_{i=1}^{n} x^*(x_i) \cdot y^*(y_i)| : x^* \in X^*, ||x^*|| \leq 1, \\ y^* \in Y^*, ||y^*|| \leq 1\}.$$

The basic properties of π and ϵ norms are:

(a) $\pi(x \otimes y) = \epsilon(x \otimes y) = ||x|| \cdot ||y||$, for all $x \in X,\ y \in Y$.

(b) $\epsilon(u) \leq \pi(u)$, for all $u \in X \otimes Y$.

The normed spaces $(X \otimes Y, \pi)$ and $(X \otimes Y, \epsilon)$ are denoted by $X \otimes_\pi Y$ and $X \otimes_\epsilon Y$ respectively, their completions are denoted by $X \widehat{\otimes}_\pi Y$ and $X \widehat{\otimes}_\epsilon Y$.

Theorem 3.2. (see [1]) *Let E, X, F and Y be Banach spaces, $S : E \longrightarrow X$ and $T : F \longrightarrow Y$ bounded linear operators. Then, there are bounded linear operators $S \otimes_\pi T : E\widehat{\otimes}_\pi F \longrightarrow X\widehat{\otimes}_\pi Y, S \otimes_\epsilon T : E\widehat{\otimes}_\epsilon F \longrightarrow X\widehat{\otimes}_\epsilon Y$ so that $S \otimes_\pi T(x \otimes y) = S(x) \otimes T(y), S \otimes_\epsilon T(x \otimes y) = S(x) \otimes T(y)$ for all $x \in E, y \in F$. Furthermore, $||S \otimes_\pi T|| = ||S \otimes_\epsilon T|| = ||S|| \cdot ||T||$.*

4. The Square Ordering on $\mathbb{N}^k$.

The square enumeration for $\mathbb{N}^2$ follows the diagram:

$$
\begin{array}{ccccccccccc}
 & & & & & \downarrow & & \downarrow & & \downarrow \\
 & 1 & & 2 & & 3 & & 4 & & 5 \\
1 & (1,1) & \rightarrow & (1,2) & & (1,3) & & (1,4) & & \vdots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \vdots \\
2 & (2,1) & \leftarrow & (2,2) & & (2,3) & & (2,4) & & \vdots \\
 & & & & & \downarrow & & \downarrow & & \vdots \\
3 & (3,1) & \leftarrow & (3,2) & \leftarrow & (3,3) & & (3,4) & & \vdots \\
 & & & & & & & \downarrow & & \vdots \\
4 & (4,1) & \leftarrow & (4,2) & \leftarrow & (4,3) & \leftarrow & (4,4) & & \vdots \\
 & & & & & & & & & \vdots \\
5 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots
\end{array}
$$

The initial terms of the list in the square ordering are:

$$(1,1),\ (1,2),\ (2,2),\ (2,1),\ (1,3),\ (2,3),\ (3,3),\ (3,2),\ (3,1),\ (1,4),\ \dots$$

The next proposition, whose proof is elementar, gives the position of the pair (i,j) in the square ordering.

Proposition 4.1. *The function $sq_2 : \mathbb{N}^2 \to \mathbb{N}$ defined by*

$$sq_2(i,j) = \begin{cases} i^2 - j + 1 & \text{if } i \geq j\,, \\ j^2 - 2j + i + 1 & \text{if } i < j\,, \end{cases}$$

establishes a square ordering for $\mathbb{N}^2$.

Remark 4.1. It can be shown directly that sq_2 is an 1-1 onto function.

For $n \in \mathbb{N}$ and m a non-negative integer,

$$sq_2^{-1}(n) = \begin{cases} (n - m^2, m+1) & \text{if } m^2 < n \leq m^2 + m + 1 \\ (m+1, (m+1)^2 - n + 1) & \text{if } m^2 + m + 1 \leq n \leq (m+1)^2. \end{cases}$$

The square ordering sq_k on $\mathbb{N}^k$ can be defined by induction.

If $(i_1, i_2, i_3) \in \mathbb{N}^3$, we define sq_3 by

$$sq_3(i_1, i_2, i_3) = sq_2(sq_2(i_1, i_2), i_3).$$

Therefore, supposing sq_{k-1} is defined on $\mathbb{N}^{k-1}$, we define sq_k on $\mathbb{N}^k$ by

$$sq_k(i_1, i_2, \dots, i_k) = sq_2(sq_{k-1}(i_1, \dots, i_{k-1}), i_k).$$

Example 4.1. $sq_2(5,12) = 126,\ sq_3(5,12,20) = 15857.$

5. Tensor Product Basis

Let E and F be Banach spaces with Schauder bases $(e_k)_{k=1}^{\infty}$ and $(f_l)_{l=1}^{\infty}$ respectively, $(e_k, e_k^*)_{k=1}^{\infty}$ and $(f_l, f_l^*)_{l=1}^{\infty}$ the associated biorthogonal systems.

The corresponding projections on the first n-coordenate are given by

$$P_n(x) = \sum_{k=1}^{n} e_k^*(x)e_k,\ \sup_n ||P_n|| = a\,,$$

$$Q_n(y) = \sum_{l=1}^{n} f_l^*(x)f_l,\ \sup_n ||Q_n|| = b\,.$$

Remark 5.1. We mean P_0 and Q_0 the null operator.

Lemma 5.1. *Let H be the subset of $E \otimes F$ given by $H = \{e_k \otimes f_l,\ k,l = 1,2,\cdots\}$ and $[H]$ the linear span of H. Then $[H]$ is a dense subset in $E \otimes F$ either for π or ϵ norms on $E \otimes F$.*

Proof. Let us notice that for any $k,m \in \mathbb{N}$ the tensor product operator $P_k \otimes Q_m : E \otimes F \longrightarrow E \otimes F$ takes its values in $[H]$, let $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$,

$$\begin{aligned} P_k \otimes Q_m(u) &= \sum_{i=1}^{n} P_k \otimes Q_m(x_i \otimes y_i) = \sum_{i=1}^{n} P_k(x_i) \otimes Q_m(y_i) \\ &= \sum_{i=1}^{n}(\sum_{j=1}^{k} e_j^*(x_i)e_j) \otimes (\sum_{l=1}^{m} f_l^*(y_i)f_l) = \sum_{i=1}^{n}\sum_{j,l=1}^{k,m} e_j^*(x_i)f_l^*(y_i)e_j \otimes f_l \\ &= \sum_{j,l=1}^{k,m}(\sum_{i=1}^{n} e_j^*(x_i)f_l^*(y_i))e_j \otimes f_l \in [H]. \end{aligned}$$

Now, for every $x \in E, y \in F$ we have the identity,

$$x \otimes y - P_k \otimes Q_m(x \otimes y) = x \otimes (y - Q_m(y)) + (x - P_k(x)) \otimes Q_m(y).$$

It follows, by using the triangular inequality and the basic properties of π and ϵ norms, that

$$\pi(x \otimes y - P_k \otimes Q_m(x \otimes y)) \leq ||x|| \cdot ||y - Q_m(y)|| + ||x - P_k(x)|| \cdot ||Q_m(y)||.$$

Therefore, for a given $\delta > 0$ and k, m sufficiently large we get,

$$\pi(x \otimes y - P_k \otimes Q_m(x \otimes y)) \leq ||x|| \frac{\delta}{2||x||} + \frac{\delta}{2b||y||} ||Q_m(y)|| \leq \delta.$$

Clearly, the same is true for ϵ norm.

Since u is represented by a finite sum of $x \otimes y$, the lemma is proved. □

Theorem 5.2. *Let $(e_j)_{j=1}^{\infty}$, $(f_l)_{l=1}^{\infty}$ be Schauder bases E and F respectively, P_k, Q_k the projections on the first k-coordenate associated with the biorthogonal systems, $(e_j, e_j^*)_{j=1}^{\infty}$ and $(f_l, f_l^*)_{l=1}^{\infty}$. Let $H = \{e_j \otimes f_l,\ j, l = 1, 2, \cdots\}$ and define $u_n = e_j \otimes f_l, u_n^* = e_j^* \otimes f_l^*$, where $n = sq_2(j, l)$. Then*

(i) $(u_n, u_n^*)_{n=1}^{\infty}$ is a biorthogonal system for $E\widehat{\otimes}_\pi F$ and $E\widehat{\otimes}_\epsilon F$.

(ii) If the operator $R_n : E\widehat{\otimes}_\pi F \longrightarrow E\widehat{\otimes}_\pi F$ is defined by

$$R_n = P_{n-m^2} \otimes_\pi Q_{m+1} + P_m \otimes_\pi Q_m - P_{n-m^2} \otimes_\pi Q_m, \quad \text{if } m^2 < n \leq m^2 + m + 1\,,$$

or

$$R_n = P_{m+1} \otimes_\pi Q_{m+1} - P_{m+1} \otimes_\pi Q_{(m+1)^2-n} + P_m \otimes_\pi Q_{(m+1)^2-n}\,, \quad \text{if } m^2 + m + 1 < n \leq (m+1)^2$$

(the form of R_n is unchanged when the π norm is replaced by the ϵ norm) then

$$R_n(z) = \sum_{i=1}^{n} u_i^*(z) u_i, \ \ \forall z \in E\widehat{\otimes}_\pi F \ \ (\forall z \in E\widehat{\otimes}_\epsilon F) \text{ and all } n \in \mathbb{N}.$$

Notice that $||R_n|| \geq 1$.

Proof. (i) Since is easely seen that $B(E, F)^* \subset B(E^*, F^*)$ and $E \otimes F \subset B(E, F)^*$, $X^* \otimes Y^* \subset B(E^*, F^*)^*$ it follows that, $x^* \otimes y^*$ is a linear functional on $E \otimes F$, acting as $x^* \otimes y^*(x \otimes y) = x^*(x)y^*(y)$.

Furthermore, for $u = \sum\limits_{i=1}^{n} x_i \otimes y_i \in E \otimes F$, we have

$$\begin{aligned} |x^* \otimes y^*(u)| &= \Big| \sum_{i=1}^{n} x_i^* \otimes y_i^*(x_i \otimes y_i) \Big| \\ &= \Big| \sum_{i=1}^{n} x_i^*(x_i) \cdot y_i^*(y_i) \Big| \leq ||x^*|| \cdot ||y^*|| \sum_{i=1}^{n} ||x_i|| \cdot ||y_i||\,, \end{aligned}$$

therefore $|x^* \otimes y^*(u)| \leq ||x^*|| \cdot ||y^*|| \cdot \pi(u)$. Also,

$$\begin{aligned}|x^* \otimes y^*(u)| &= |||x^*|| \cdot ||y^*|| \sum_{i=1}^{n} \frac{x^*}{||x^*||}(x_i) \cdot \frac{y^*}{||y^*||}(y_i)| \\ &\leq ||x^*|| \cdot ||y^*|| \sup\{|\sum_{i=1}^{n} s^*(x_i)t^*(y_i)| : s^* \in B_{E^*}, t^* \in B_{F^*}\} \\ &= ||x^*|| \cdot ||y^*|| \epsilon(u)\,.\end{aligned}$$

Therefore, $x^* \otimes y^* \in (E\widehat{\otimes}_\pi F)^*$ and $x^* \otimes y^* \in (E\widehat{\otimes}_\epsilon F)^*$.

Now, for any $n, k \in \mathbb{N}$ let $(j_1, l_1) = sq_2^{-1}(n)$ and $(j_2, l_2) = sq_2^{-1}(k)$, then

$$\begin{aligned}u_k^*(u_n) &= e_{j_2}^* \otimes f_{l_2}^*(e_{j_1} \otimes f_{l_1}) = e_{j_2}^*(e_{j_1}) \cdot f_{l_2}^*(f_{l_1}) \\ &= \begin{cases} 1 & \text{if and only if } \ j_2 = j_1 \text{ and } l_2 = l_1\,, \\ 0 & \text{otherwise}\,. \end{cases}\end{aligned}$$

Therefore, $u_k^*(u_n) = \delta_{k,n}$ (since sq_2 is a 1-1 onto function).

(ii) In view of Lemma 5.1, it is enough to check that

$$R_n(h) = \sum_{i=1}^{n} u_i^*(h)u_i \ \text{ for } h \in [H].$$

Let $n \in \mathbb{N}$ be fixed.

Claim.

$$R_n(e_j \otimes f_l) = \begin{cases} e_j \otimes f_l & \text{if and only if } \ 1 \leq sq_2(j,l) \leq n\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The proof of the claim is splited in the following cases: (I) $sq_2(j,l) = r < n$, (II) $sq_2(j,l) = r = n$ and (III) $sq_2(j,l) = r > n$. We choose the case $sq_2(j,l) = r < n$ to prove completly. the others can be proved with minors modification.

I. Suppose $sq_2(j,l) = r < n$.

1. If $j \geq l$, from Proposition 4.1 there is $m_r \in \mathbb{N}$, $m_r \geq 0$ such that $m_r^2 + m_r + 1 < r \leq (m_r + 1)^2$ and $j = m_r + 1$, $l = (m_r + 1)^2 - r + 1$.

(a) Suppose $m^2 < n \leq m^2 + m + 1$ (so, $n - m^2 \leq m + 1$). From the inequalities

$$\left.\begin{array}{l} m_r^2 + m_r + 1 < r \leq (m_r + 1)^2 \\ m^2 < n \leq m^2 + m + 1 \end{array}\right\} \overset{r<n}{\Longrightarrow} m_r^2 + m_r + 1 < m^2 + m + 1$$

so, $m_r < m$ and therefore $m_r + 1 \leq m$.

For such n, R_n has the form

$$R_n = P_{n-m^2} \otimes Q_{m+1} + P_m \otimes Q_m - P_{n-m^2} \otimes Q_m \,.$$

Since $l \leq j \Longrightarrow (m_r+1)^2 - r + 1 \leq m_r + 1 \leq m$, for
$z = e_{m_r+1} \otimes f_{(m_r+1)^2-r+1}$
we get,

$$\begin{array}{rcl} P_{n-m^2} \otimes Q_{m+1}(z) & = & P_{n-m^2}(e_{m_r+1}) \otimes f_{(m_r+1)^2-r+1}\,, \\ P_m \otimes Q_m(z) & = & z\,, \\ P_{n-m^2} \otimes Q_m(z) & = & P_{n-m^2}(e_{m_r+1}) \otimes f_{(m_r+1)^2-r+1}, \quad \text{therefore }\,, \\ R_n(z) & = & z\,. \end{array}$$

(b) Suppose $m^2 + m + 1 < n \leq (m+1)^2$. From the inequalities

$$\left.\begin{array}{l} m_r^2 + m_r + 1 < r < (m_r+1)^2 \\ m^2 + m + 1 < n \leq (m+1)^2 \end{array}\right\} \overset{r<n}{\Longrightarrow} \; m_r \leq m$$

in fact, if $m_r > m \Longrightarrow m+1 \leq m_r \Longrightarrow (m+1)^2 \leq m_r^2 < m_r^2 + m_r + 1$ contradiction.

For such n, R_n has the form,

$$R_n = P_{m+1} \otimes Q_{m+1} - P_{m+1} \otimes Q_{(m+1)^2-n} + P_m \otimes Q_{(m+1)^2-n}$$

Let us see the case $m_r = m$ ($\Longrightarrow (m+1)^2 - n < (m_r+1)^2 - r$)
for $z = e_{m_r+1} \otimes f_{(m_r+1)^2-r+1}$, we get

$$\begin{array}{l} P_{m+1} \otimes Q_{m+1}(z) =_{m_r+1} \otimes f_{(m_r+1)^2-r+1} \\ \qquad\qquad\qquad\qquad \text{(as before } (m_r+1)^2 - r + 1 \leq m_r + 1)\,, \\ P_{m+1} \otimes Q_{(m+1)^2-n}(z) = e_{m_r+1} \otimes 0 = 0\,, \\ \qquad\qquad\qquad\qquad \text{since } (m+1)^2 - n < (m_r+1)^2 - r + 1\,, \\ P_{m+1} \otimes Q_{(m+1)^2-n}(z) = 0 \otimes 0 = 0\,, \\ \qquad\qquad\qquad\qquad\qquad \text{since } m < m_r + 1\,, \quad \text{therefore} \\ R_n(z) = z\,. \end{array}$$

Let us see the case $m_r < m$ ($\Longrightarrow m_r + 1 \leq m$)

$$\begin{array}{rcl} P_{m+1} \otimes Q_{m+1}(z) & = & e_{m_r+1} \otimes f_{(m_r+1)^2-r+1}\,, \\ P_{m+1} \otimes Q_{(m+1)^2-n}(z) & = & e_{m_r+1} \otimes Q_{(m+1)^2-n}(f_{(m_r+1)^2-r+1})\,, \\ P_{m+1} \otimes Q_{(m+1)^2-n}(z) & = & e_{m_r+1} \otimes Q_{(m+1)^2-n}(f_{(m_r+1)^2-r+1})\,, \\ \text{therefore} \quad R_n(z) & = & z\,. \end{array}$$

(2) $sq_2(j,l) = r < n$

If $j < l$ $m_r^2 < r \leq m_r^2 + m_r + 1$, $j = r - m_r^2 < m_r + 1 = l$, for such r we have $z = e_{r-m_r^2} \otimes f_{m_r+1}$

(a) Suppose $m^2 < n \leq m^2 + m + 1 \Longrightarrow$

$$R_n = P_{n-m^2} \otimes Q_{m+1} + P_m \otimes Q_m - P_{n-m^2} \otimes Q_m .$$

We have the inequalities

$$\left.\begin{array}{ll} r - m_r^2 < m_r + 1, & m_r^2 < r \leq m_r^2 + m_r + 1 \\ n - m^2 \leq m + 1, & m^2 < n \leq m^2 + m + 1 \end{array}\right\} \overset{r<n}{\Longrightarrow} m_r \leq m .$$

In fact, if $m < m_r$ ($\Longrightarrow m+1 \leq m_r$) so $n \leq m^2+m+1 \leq (m+1)^2 \leq m_r^2 < r$; contradiction.

Let us see the case $m_r = m (\Longrightarrow r - m_r^2 < n - m^2)$

$P_{n-m^2} \otimes Q_{m+1}(z) = P_{n-m^2}(e_{r-{m_r}^2}) \otimes Q_{m+1}(f_{m_r+1}) = e_{r-{m_r}^2} \otimes f_{m_r+1}$
$P_m \otimes Q_m(z) = P_m(e_{r-{m_r}^2}) \otimes Q_m(f_{m_r+1}) = e_{r-{m_r}^2} \otimes 0 = 0$
since $r - m_r^2 < m_r + 1 = m + 1 \Longrightarrow r - m_r^2 \leq m$ and $m < m_r + 1$.
$P_{n-m^2} \otimes Q_m(z) = e_{r-{m_r}^2} \otimes 0 = 0$, therefore
$R_n(z) = z$.

If $m_r < m (\Longrightarrow m_r + 1 \leq m)$

$$P_{n-m^2} \otimes Q_{m+1}(z) = P_{n-m^2}(e_{r-{m_r}^2}) \otimes Q_{m+1}(f_{m_r+1}) = P_{n-m^2}(e_{r-{m_r}^2}) \otimes f_{m_r+1} ,$$

$$P_m \otimes Q_m(z) = P_m(e_{r-{m_r}^2}) \otimes Q_m(f_{m_r+1}) = e_{r-{m_r}^2} \otimes f_{m_r+1} ,$$

since $r - m_r^2 < m_r + 1 \leq m$.

$$\begin{array}{rcl} P_{n-m^2} \otimes Q_m(z) & = & P_{n-m^2}(e_{r-{m_r}^2}) \otimes f_{m_r+1}, \text{ therefore} \\ R_n(z) & = & z. \end{array}$$

(b) Suppose $m^2 + m + 1 < n \leq (m+1)^2$ $((m+1)^2 - n + 1 \leq m + 1)$ for such n, $R_n = P_{m+1} \otimes Q_{m+1} - P_{m+1} \otimes Q_{(m+1)^2-n} + P_m \otimes Q_{(m+1)^2-n}$
$z = e_{r-m_r^2} \otimes f_{m_r+1}$ (as before).

We have the inequalities:

$$\left.\begin{array}{ll} r - m_r^2 < m_r + 1 & m_r^2 < r \leq m_r^2 + m_r + 1 \\ (m+1)^2 - n + 1 \leq m + 1 & m^2 + m + 1 < n \leq (m+1)^2 \end{array}\right\} \overset{r<n}{\Longrightarrow}$$

$$m_r^2 + m_r + 1 \leq m^2 + m + 1 \quad \therefore \ m_r \leq m\,.$$

If $m = m_r$,

$$P_{m+1} \otimes Q_{m+1}(z) = P_{m+1}(e_{r-m_r^2}) \otimes Q_{m+1}(f_{m_r+1}) = e_{r-m_r^2} \otimes f_{m_r+1}\,,$$

since $r - m_r^2 < m_r + 1 = m + 1$.

$$P_{m+1} \otimes Q_{(m+1)^2-n}(z) = e_{r-m_r^2} \otimes 0 = 0\,,$$

since $(m+1)^2 - n \leq m < m_r + 1$.

$$P_m \otimes Q_{(m+1)^2-n}(z) = P_m(e_{r-m_r^2}) \otimes 0 = 0, \text{ therefore } \ R_n(z) = z\,.$$

If $m_r < m$,

$$P_{m+1} \otimes Q_{m+1}(z) = P_{m+1}(e_{r-m_r^2}) \otimes Q_{m+1}(f_{m_r+1}) = e_{r-m_r^2} \otimes f_{m_r+1}$$

since $r - m_r^2 < m_r + 1 < m + 1$.

$$P_{m+1} \otimes Q_{(m+1)^2-n}(z) = e_{r-m_r^2} \otimes Q_{(m+1)^2-n}(f_{m_r+1})\,,$$

$$P_m \otimes Q_{(m+1)^2-n}(z) = e_{r-m_r^2} \otimes Q_{(m+1)^2-n}(f_{m_r+1})\,,$$

since $r - m_r^2 < m_r + 1 \leq m$.
Therefore $R_n(z) = z$. □

Theorem 5.3. *Let $u_n = e_j \otimes f_l$, where $n = sq_2(j,l)$. Then $(u_n)_{n=1}^{\infty}$ is a Schauder basis either for $E\widehat{\otimes}_\pi F$ or $E\widehat{\otimes}_\epsilon F$ and the operators R_n. are the associated projections on the first n-coordenate.*

Proof. From the previous theorem, we have that (u_n, u_n^*) is a biorthogonal system either for $E\widehat{\otimes}_\pi F$ or $E\widehat{\otimes}_\epsilon F$ and $R_n(z) = \sum_{i=1}^{n} u_i^*(z)u_i$ for all $z \in E\widehat{\otimes}_\pi F$ or $E\widehat{\otimes}_\epsilon F$. Since $||R_n|| \leq a\cdot b + a\cdot b + a\cdot b = 3\,a\cdot b$ either for $m^2 < n < m^2+m+1$ or $m^2+m+1 < n \leq (m+1)^2$, by appling the theorem 2.1 it follows that (u_n) is a Schauder basis for $\overline{[H]}^{\pi} = E\widehat{\otimes}_\pi F$ or $\overline{[H]}^{\epsilon} = E\widehat{\otimes}_\epsilon F$. Consequently R_n is the projection on the n first coordenate. □

Corollary 5.4. *Let $X_1, \ldots, X_k$ be Banach spaces with Schauder bases $(e_{n_1}^1), \ldots, (e_{n_k}^k)$ respectively. Then the sequence $(e_{n_1}^1 \cdots e_{n_k}^k)$ with the k-square ordering is a Schauder basis for both*

$$X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_k \quad \text{and} \quad X_1 \widehat{\otimes}_\epsilon \cdots \widehat{\otimes}_\epsilon X_k.$$

Proof. Induction on k. For $2 \leq r \leq k$ let us denote

$$X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_r \quad \text{or} \quad X_1 \widehat{\otimes}_\epsilon \cdots \widehat{\otimes}_\epsilon X_r, \qquad \text{by } E^r.$$

Let (f_n^2) be the Schauder basis (given by Theorem 5.3) for E^2. Therefore, $f_n^2 = e_{i_1}^1 \otimes e_{i_2}^2$ for $n = sq_2(i_1, i_2)$. By Theorem 5.3, the sequence $(f_n^2 \otimes e_{n_3}^3)$ with the square ordering is a Schauder basis for E^3.

Let (f_m^3) be the Schauder basis of E^3, therefore, $f_m^3 = f_n^2 \otimes e_{i_3}^3$ for $m = sq_2(n, i_3)$. Since $n = sq_2(i_1, i_2)$ it follows that $m = sq_2(sq_2(i_1, i_2), i_3) = sq_3(i_1, i_2, i_3)$.

Suppose now by induction that the sequence $(f_t^{k-1})_{t=1}^{\infty}$ is a Schauder basis for E^{k-1}, where $t = sq_{k-1}(n_1, \ldots, n_{k-1})$. By Theorem 5.3 it follows that $(f_t^{k-1} \otimes e_{n_k}^k)$ is a Schauder basis for E^k in the square ordering.

If $f_s^k = f_t^{k-1} \otimes e_{n_k}^k$ then $s = sq_2(t, n_k) = sq_2(sq_{k-1}(n_1, \cdots, n_{k-1}), n_k)$ $= sq_k(n_1, \cdots, n_k)$. □

Remark 5.2. The π and ϵ norms can be mixed in the tensor product without changing the basis.

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