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TRAJECTORIES WITH UNBOUNDED CONSUMPTION FOR A MODEL WITH DISCRETE INNOVATIONS

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Abstract: In this paper we consider a model of economic dynamics with discrete innovations and establish the existence of trajectories with unbounded consumption.

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1. Introduction

We consider the model with discrete innovations introduced in Makarov [1] and studied in Zaslavski [2-9]. In this model the state of the economy is determined by a set of operating technologies, a collection of funds corresponding to these technologies, and a set of known, but as yet not implemented technologies. To introduce a new technology, expenditures of the already utilized types of funds are required. As a result of these expenditures, the new technology at the next instant of time will be introduced into action with a certain initial reserve of new funds.

We consider a single-product economy which deals with two production factors: labor L and funds K. The time is assumed to be discrete and the amount of labor is constant and equal to unity. The state of the economy is determined by a set of operating technologies, a collection of funds corresponding to these

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technologies and a set of known, but as yet not implemented technologies.

A technology is a pair (f, v), where f is a production function of two variables K, L and $v \in [0, 1)$. Possessing at time t funds K and labor resources L, the economy utilizing the technology (f, v) will produce during a unit time interval, a product in the amount of $f(K, L)$. Moreover, at time $t + 1$ the economy will still have in its possession the used old funds in the amount of vK.

To introduce a new technology, expenditures of the already utilized types of funds are required. As a result of these expenditures, the new technology at the next instant of time will be introduced into action with a certain intial reserve of the new funds.

Let $I = \{0, 1, \dots\}$ and $\{(f^i, v^i) : \ i \in I\}$ be the set of all technologies which can be utilized in the production process. At time $t \in I$ the state of the economy is given in the form

$$(I_0^t, I_n^t, (K_t^i, C_t^i)(i \in I_0^t)),$$

where I_0^t is a finite set of numbers (indices) of technologies introduced by the time t, I_n^t is the set of numbers of technologies which are available in principle but not introduced, and $K_t^i, C_t^i \geq 0$ are the funds and consumption of the i-th type available at time t which correspond to the technology with the number i. We will assume that at time t the following information is available

$$(K^i), \ (s^{ij}) \ (i \in I_n^t, \ j \in I_n^t \cup I_0^t),$$

where $s^{ij} \geq 0$ is the expenditure of the j-th funds required for introduction of the i-th technology and $K^i > 0$ is the initial amount of the i-th fund which is obtained at the initial time of utilization of the i-th technology. At time $t + 1$ the economy may pass over to the state

$$(I_0^{t+1}, I_n^{t+1}, (K_{t+1}^i, C_{t+1}^i) \ (i \in I_0^{t+1})) \, ,$$

for which

$$I_0^t \subset I_0^{t+1} \subset I_0^t \cup I_n^t,$$

$$I_0^{t+1} \setminus I_0^t \subset \{i \in I_n^t : \ s^{ij} = 0 \text{ for all } j \in I_n^t\} \, ,$$

and the numbers $L_t^i \geq 0$, $i \in I_0^t$ are determined such that

$$\sum_{i \in I_0^t} L_t^i \leq 1,$$

$$K_{t+1}^j \geq v^j K_t^j \, , \text{ for all } j \in I_0^t,$$

$$K^i_{t+1} = K^i,\ C^i_{t+1} = 0,\ i \in I_0^{t+1} \setminus I_0^t,$$

$$K^j_{t+1} - v^j K^j_t + C^j_{t+1} + \sum_{i \in I_0^{t+1} \setminus I_0^t} s^{ij} \le f^j(K^j_t, L^j_t)\,,\ \text{for all } j \in I_0^t$$

(we assume here that the result of a summation over an empty set equals zero). Note that in the model under consideration the newly procuced product is used for consumption and expenditures related to an introduction of new technologies. Sometimes the state of the economy at time t will be written in the form

$$(I_0^t, I_n^t, (K^i_t, C^i_t, L^i_t)\ (i \in I_0^t))\,,$$

where (L^i_t) $(i \in I_0^t)$ is the distribution of labor resources at time t. We do it in the case when some information about this distribution is required. When describing a trajectory of the model we would also include into its description the corresponding sequence of distributions of labor resources, most often only in the case when some information about these resources is required. However, in any case a definite sequence of distributions of labor resources is always associated with a trajectory of the model.

Denote by R^l_+ the cone of elements of the Euclidean space R^l with non-negative coordinates. Below all the technologies under consideration (f, v) will assume to be such that $f : R^2_+ \to R_+$ be a continuous, superlinear (superadditive, positively homogeneous) function,

$$f(0,1) = f(1,0) = 0,$$

$$f(x,1) < f(\lambda x, 1) < \lambda f(x,1),\ \text{for each } \lambda > 1 \text{ and each } x > 0,$$

and there exists $X \in R_+$ such that $f(1, X) > 1 - v$.

Let (f, v) be a technology. It is easy to see that there exists a unique number $x(f, v) > 0$ such that

$$f(x(f,v), 1) = (1-v)x(f,v).$$

For $x_0 > 0$ the inequality $f(x_0, 1) > (1-v)x_0$ holds if and only if $x_0 < x(f,v)$ and the sequence

$$x_t = v x_{t-1} + f(x_{t-1}, 1),\ t = 1, 2, \ldots$$

converges to $x(f,v)$ as $t \to \infty$. Evidently $x(f,v)$ is a characteristic of the technology (f,v) which evaluates its production capabilities.

The technology (f, v) is associated with a dynamic model of the economy whose trajectory is a sequence (K_t, C_t), $t = 0, 1, \dots,$ where $K_t, C_t \geq 0$ are the funds and consumption available at time t which satisfy

$$K_{t+1} - vK_t \geq 0$$

and

$$K_{t+1} - vK_t + C_{t+1} \leq f(K_t, 1)\,,$$

for all $t = 0, 1, \dots$

It is easy to see that for any model trajectory (K_t, C_t), $t = 0, 1, \dots$ we have

$$\limsup_{t\to\infty}(vK_t + f(K_t, 1)) \leq x(f, v)$$

and

$$\limsup_{t\to\infty} K_t \leq x(f, v).$$

Moreover, for any initial state of the model (K_0, C_0) with $K_0 > 0$ there exists a trajectory (K_t, C_t), $t = 0, 1, \dots$ such that $K_t \to x(f, v)$ as $t \to \infty$.

Let $X = (K_t, C_t)(t \in I)$ be a model trajectory. Set

$$w(X) = \limsup_{T\to\infty} T^{-1} \sum_{t=0}^{T-1} C_t.$$

Evidently $w(X) \leq x(f, v)$. Set

$$w(f, v) = \sup\{w(X) : \; X \text{ is a model trajectory}\}.$$

The number $w(f, v)$ is a characteristic of the technology (f, v) which evaluates its consumption capabilities.

It is easy to verify that the following result is true.

Proposition 1.1. *There exists a number $h(f, v) \in (0, 1)$ such that*

$$\lim_{x\to\infty}(h(f, v)f(1, x)) \in (1 - v, \infty]$$

and

$$(1 - h(f, v))f(x(h(f, v)f, v), 1) = w(f, v) > 0.$$

2. Existence of Trajectories with Unbounded Consumption

Consider the model with discrete innovations introduced in Section 1. We assume that

$$s^{ij} = 0\,, \text{ for all } i, j \in I \text{ satisfying } j \geq i,$$

and that for any state of the model $(I_0^t, I_n^t, (K_t^i, C_t^i))(i \in I_0^t)$ at any instant of time t the relation

$$\sup\{i : \ i \in I_0^t\} + 1 \in I_n^t$$

holds.

In Zaslavski [2] we studied the existence of model trajectories with unbounded consumption and established the following result.

Theorem 2.1. *Let* $\sup\{w(f^i, v^i) : \ i \in I\} = \infty$ *and*

$$\sup_{p \in I} \left\{ \sum_{i \in I} (s^{pi} (x(f^i, v^i)(1 - v^i))^{-1} \right\} < 1. \tag{2.1}$$

Assume that $(I_0^0, I_n^0, (K_0^i, C_0^i)(I \in I_0^0))$ *be an initial state of the economy such that* $K_0^i > 0,\ i \in I_0^0$, *and*

$$s^{pj} = 0 \text{ for any } p > p(0) := \sup\{i : \ i \in I_0^0\}$$

$$\text{and any } j \in \{0, \ldots, p(0)\} \setminus I_0^0.$$

Then there exists a model trajectory

$$(I_0^t, I_n^t, (K_t^i, C_i^t)(I \in I_0^t))(t \in I)\,,$$

such that

$$0 < \sup\{C_t^i : \ i \in I_0^t\} \to \infty \text{ as } t \to \infty.$$

Note that inequality (2.1) means that the expenditures of the i-th funds required for introduction of the technology (f^p, v^p) should be small compared to the value

$$x(f^i, v^i)(1 - v^i) = f^i(x(f^i, v^i), 1)\,,$$

which characterizes the production capabilities of the technology (f^i, v^i).

From now on we consider a particular case of the model such that for each $i, j \in I$ the inequality $s^{ij} > 0$ holds if and only if $i = j + 1$. This assumption means that for introduction of the (i+1)-th technology we only need expenditures of the i-th fund.

In this case inequality (2.1) is equivalent to the following inequality:

$$\sup_{i\in I}\left\{s^{(i+1)i}(x(f^i,v^i)(1-v^i))^{-1}\right\}<1\,. \tag{2.2}$$

In this paper we prove the following result and establish the existence of a trajectory with unbounded consumption under an assumption which is weaker than (2.2).

Theorem 2.2. *Let*

$$s^{(i+1)i}<x(f^i,v^i)(1-v^i)=f^i(x(f^i,v^i),1)\,, \tag{2.3}$$

for each $i\in I$ *and*

$$w(f^i,v^i)(1-s^{(i+1)i}(f^i(x(f^i,v^i),1))^{-1})\to\infty \text{ as } i\to\infty. \tag{2.4}$$

Assume that $(I_0^0,I_n^0,(K_0^i,C_0^i)(I\in I_0^0))$ *is an initial state of the economy,*

$$p=\sup\{I_0^0\},\ K_0^p>0.$$

Then there exists a model trajectory

$$(I_0^t,I_n^t,(K_t^i,C_t^i)(i\in I_0^t))(t\in I)\,,$$

such that

$$0<\sup\{C_t^i:\ i\in I_0^t\}\to\infty\,,\ \text{as }\ \to\infty.$$

Note that the unboundedness of the set $\{w(f^i,v^i):\ i\in I\}$ and inequality (2.2) imply (2.4).

3. Proof of Theorem 2.2

We may assume without loss of generality that for any instant of time t and any state of the model

$$(\tilde I_0^t,\tilde I_n^t,(\tilde K_t^i,\tilde C_t^i)(i\in\tilde I_0^t))\,,$$

the following relation holds:

$$\tilde I_n^t=\{\max\{i:\ i\in\tilde I_0^t\}+1\}. \tag{3.1}$$

In view of this assumption we omit below the notation I_n^t in discribing the state of the model.

We may assume without loss of generality that

$$I_0^0 = \{p\}. \tag{3.2}$$

For $i \in I$ we set

$$w(i) = w(f^i, v^i),\ h(i) = h(f^i, v^i), \tag{3.3}$$
$$x(i) = x(f^i, v^i),\ \Lambda(i) = x(h(f^i, v^i)f^i, v^i).$$

We preface the proof of Theorem 2.2 by the following auxiliary result.

Lemma 3.1. *Let $(I_0^s, I_n^s, (K_s^i, C_s^i)(i \in I_0^s))$ be a state of the economy at time $s \in I$ and*

$$q = \max\{i :\ i \in I_0^s\}. \tag{3.4}$$

Assume that

$$K_s^q > 4^{-1}\Lambda(q)(1 - s^{(q+1)q}(f^q(x(f^q, v^q), 1))^{-1}) \tag{3.5}$$
$$= 4^{-1}\Lambda(q)(1 - s^{(q+1)q}(f^q(x(q), 1))^{-1}).$$

Then there exists a model trajectory

$$(I_0^t, I_n^t, (K_t^i, C_t^i)(i \in I_0^t))(t = s, \dots, \tau),$$

with an integer $\tau > s$ such that

$$I^\tau = I^s \cup \{q + 1\},$$
$$C_t^q \geq 32^{-1} w(q)(1 - s^{(q+1)q}(f^q(x(q), 1))^{-1}), \tag{3.6}$$
$$t = 1 + s, \dots, \tau,$$
$$K_\tau^{q+1} > 4^{-1}\Lambda(q + 1)(1 - s^{(q+2)(q+1)}(f^{q+1}(x(q + 1), 1))^{-1}). \tag{3.7}$$

Proof. Set

$$a_s = 4^{-1}\Lambda(q)(1 - s^{(q+1)q}(f^q(x(q), 1))^{-1}), \tag{3.8}$$
$$b_s = K_s^q - a_s. \tag{3.9}$$

(3.5), (3.8) and (3.9) imply that $b_s > 0$. Set

$$l_1 = s^{(q+1)q}(f^q(x(q), 1))^{-1},\ l_2 = (1 - l_1)/2. \tag{3.10}$$

By induction we define a sequence $\{b_j\}_{j=s}^{\infty}$ of real numbers as follows:

$$b_{j+1} = v^q b_j + f^q(b_j, l_1 + l_2) \tag{3.11}$$

$$= (l_1 + l_2)[v^q b_j/(l_1 + l_2) + f^q(b_j/(l_1 + l_2), 1)].$$

Define a sequence of real numbers $\{a_j\}_{j=s}^{\infty}$ as follows:

$$a_{j+1} = v^q a_j + h(q) f^q(a_j, l_2) \tag{3.12}$$

$$= l_2[v^q a_j/l_2 + h(q) f^q(a_j/l_2, 1)]\,,$$

for all integers $j \geq s$. In view of (3.8) and (3.10)

$$a_s/l_2 \geq 8^{-1}\Lambda(q).$$

When combined with (3.12), (3.3) and the monotonicity of f^q this inequality implies that

$$a_j/l_2 \geq 8^{-1}\Lambda(q) \text{ for all integers } j \geq s. \tag{3.13}$$

By (3.13), (3.3) and Proposition 1.1 for all integers $j \geq s$

$$(1 - h(q)) f^q(a_j, l_2) = l_2(1 - h(q)) f^q(a_j/l_2, 1) \tag{3.14}$$

$$\geq 8^{-1} l_2 (1 - h(q)) f^q(\Lambda(q), 1) = 8^{-1} l_2 w(q).$$

It is not difficult to see that

$$\lim_{j \to \infty} b_j = (l_1 + l_2) x(q), \tag{3.15}$$

$$\lim_{j \to \infty} f^q(b_j, l_1 + l_2) = (l_1 + l_2) f^q(x(q), 1).$$

It follows from (3.15) and (3.10) that there is an integer $\tau_0 > s + 2$ such that

$$f^q(b_{\tau_0 - 1}, l_1 + l_2) \geq s^{(q+1)q}. \tag{3.16}$$

For $t = 1 + s, \ldots, \tau_0 - 1$ we set

$$I_0^t = I_0^s,\ L_{t-1}^q = 1,\ L_{t-1}^i = 0,\ i \in I_0^s \setminus \{q\}, \tag{3.17}$$

$$K_t^i = v^i K_{t-1}^i,\ C_t^i = 0,\ i \in I_0^s \setminus \{q\},$$

$$K_t^q = a_t + b_t,\ C_t^q = (1 - h(q)) f^q(a_{t-1}, l_2).$$

It is not difficult to see that the states of the trajectory are well defined for $j = s, \dots, \tau_0 - 1$. (3.17) and (3.14) imply that

$$C_t^q \geq 8^{-1} l_2 w(q),\ t = s+1, \dots, \tau_0 - 1. \tag{3.18}$$

Set

$$\begin{gathered} L^i_{\tau_0 - 1} = 0,\ i \in I_0^s \setminus \{q\},\ L^q_{\tau_0-1} = 1, \\ I_0^{\tau_0} = I_0^s \cup \{q+1\}, \\ K^{q+1}_{\tau_0} = K^{q+1},\ C^{q+1}_{\tau_0} = 0, \\ K^i_{\tau_0} = v^i K^i_{\tau_0 - 1},\ C^i_{\tau_0} = 0,\ i \in I_0^s \setminus \{q\}, \\ C^q_{\tau_0} = (1 - h(q)) f^q(a_{\tau_0 - 1}, l_2), \\ K^q_{\tau_0} = a_{\tau_0} + b_{\tau_0} - s^{(q+1)q}. \end{gathered} \tag{3.19}$$

It follows from (3.19), (3.17), (3.12), (3.11) and (3.16) that the state of the economy at time τ_0 is well defined. (3.19) and (3.14) imply that

$$C^q_{\tau_0} \geq 8^{-1} l_2 w(q). \tag{3.20}$$

By (3.19), (3.12), (3.16), (3.11) and (3.13),

$$K^q_{\tau_0} \geq a_{\tau_0} \geq 8^{-1} l_2 \Lambda(q). \tag{3.21}$$

For any integer $t \geq \tau_0$ we set

$$\begin{gathered} I_0^{t+1} = I_0^{\tau_0},\ L^i_t = 0,\ i \in I_0^{\tau_0} \setminus \{q, q+1\}, \\ L^q_t = l_2,\ L^{q+1}_t = 1 - l_2 = (1 + l_1)/2, \\ C^i_{t+1} = 0,\ K^i_{t+1} = v^i K^i_t,\ i \in I_0^{\tau_0} \setminus \{q, q+2\}, \\ K^q_{t+1} = v^q K^q_t + h(q) f^q(K^q_t, l_2), \\ C^q_{t+1} = (1 - h(q)) f^q(K^q_t, l_2), \\ C^{q+1}_{t+1} = 0,\ K^{q+1}_{t+1} = v^{q+1} K^{q+1}_t + f^{q+1}(K^{q+1}_t, 1 - l_2). \end{gathered} \tag{3.22}$$

In view of (3.22), (3.21) and (3.3) for any integer $t \geq \tau_0 + 1$

$$K^q_t \geq 8^{-1} l_2 \Lambda(q),\ C^q_t \geq 8^{-1} l_2 w(q). \tag{3.23}$$

It follows from (3.19) and (3.22) that

$$\lim_{t \to \infty} (1 - l_2)^{-1} K^{q+1}_t = x(q+1).$$

Therefore there is an integer $\tau > \tau_0 + 4$, for which

$$K_\tau^{q+1} > (1 - l_2)\Lambda(q+1) = ((1 + l_1)/2)\Lambda(q+1). \tag{3.24}$$

Consider the trajectory of the model $(I_0^t, (K_t^i, C_t^i)(i \in I_0^t))$ $(t = s, \dots, \tau)$. (3.6) follows from (3.10), (3.17)-(3.20), (3.22) and (3.23). In order to complete the proof of the lemma it is sufficient to prove (3.7). The inequality (3.24) implies that

$$K_\tau^{q+1} > 2^{-1}\Lambda(q+1)$$
$$> 4^{-1}\Lambda(q+1)(1 - s^{(q+2)(q+1)}(f^{q+1}(x(q+1), 1))^{-1}).$$

Thus (3.7) is true. Lemma 3.1 is proved. □

Proof of Theorem 2.2. Set $a_0 = K_0^p$,

$$a_{t+1} = v^p a_t + h(p) f^p(a_t, 1). \tag{3.25}$$

Clearly $\lim_{t\to\infty} a_t = \Lambda(p)$. There exists a natural number $t(0)$ such that

$$a_{t(0)} > 2^{-1}\Lambda(p). \tag{3.26}$$

For $t = 0, \dots, t(0)$ set

$$I_0^t = I_0^0 = \{p\},\ L_{t-1}^p = 1,\ K_t = a_t,$$
$$C_t = (1 - h(p)) f^p(a_{t-1}, 1).$$

It is easy to see that that the states of the trajectory are well defined at times from zero to $t(0)$ inclusively. By (3.26)

$$K_{t(0)}^p > 2^{-1}\Lambda(p).$$

Further construction of the trajectory is by induction.

Assume that $k \in I$ and we have defined $t(q) \in I,\ q = 0, \dots, k$ and the model trajectory

$$(I_0^t, (K_t^i, C_t^i)(i \in I_0^t))(t = 0, \dots, t(k)),$$

such that $t(0) < \dots < t(k)$, for each $q = 0, \dots, k$

$$I_0^{t(q)} = \{p, \dots, p+q\}, \tag{3.27}$$

$$K_{t(q)}^{p+q} > 4^{-1}(\Lambda(p+q))(1 - s^{(p+q+1)(p+q)}(f^{p+q}(x(p+q), 1))^{-1}),$$

and that for each integer q satisfying $0 \le q \le k-1$

$$C_t^{p+q} \ge 32^{-1} w(p+q)(1 - s^{(p+q+1)(p+q)}(f^{p+q}(x(p+q),1))^{-1}), \tag{3.28}$$

$$t = t(q)+1, \dots, t(q+1)$$

(note that for $k=0$ this assumption holds).

We apply Lemma 3.1 with $s = t(k)$ and obtain that there exists a model trajectory

$$(I_0^t, I_n^t, (K_t^i, C_t^i)(i \in I_0^t)), \quad (t = t(k), \dots, t(k+1)),$$

with $t(k+1) > t(k)$ such that

$$I_0^{t(k+1)} = \{p, \dots, p+k+1\},$$

$$C_t^{p+k} \ge 32^{-1} w(p+k)(1 - s^{(p+k+1)(p+k)}(f^{p+k}(x(p+k),1))^{-1}),$$

$$t = t(k)+1, \dots, t(k+1),$$

$$K_{t(k+1)}^{p+k+1} > 4^{-1}\Lambda(p+k+1)(1 - s^{(p+k+2)(p+k+1)}(f^{p+k+1}(x(p+k+1),1))^{-1}).$$

Thus (3.27) holds for $q = k+1$ and (3.28) holds for $q = k$. By induction we have constructed a strictly increasing sequence of integers $\{t(q)\}_{q=0}^{\infty} \subset I$ and the model trajectory $(I_0^t, (K_t^i, C_t^i)(i \in I_0^t))(t \in I)$ such that (3.27) and (3.28) hold for each $q \in I$. Theorem 2.2 now follows from (3.28) and (2.4). □

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