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A NOTE CONCERNING A DISCRETE TWO DIMENSIONAL DIFFUSION PROBLEM AND RANDOM WALKS

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Abstract: We find the solution of the partial difference equation

$$\Delta_t v(x,y,t) = -r\nabla_x v(x,y,t) + l\Delta_x v(x,y,t) + u\Delta_y v(x,y,t) - d\nabla_y v(x,y,t),$$

with initial condition $v(x,y,0) = f(x,y)$, which corresponds to the discrete two dimensional diffusion equation, by using a Green's function determined through a random walk.

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1. Preliminaries

Diffusion is a physical process that can be modeled by partial difference equations. Partial difference equations that model diffusion arise naturally from the study of mathematical physics problems [4] and mathematical biology problems [5]. Many of these problems can be viewed as random walks [2, 8]. The derivation of the Green's function will require that the reader is familiar with basic combinatorial techniques related to random walks. Green's functions for partial differenceequations can be found in the literature. They have been studied by

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Van De Pol [10], de Boor et al [3] and Veit [9] among others, however their approach applies Fourier series arguments. For example, Van De Pol found a formula for the diagonal values of the Green's function but then left an infinite number of calculations to find the value of the Green's function elsewhere applying symmetry arguments. We will find the Green's function for the discrete two dimensional diffusion problem by considering a random walk, a very straightforward derivation resulting in a known formula for the Green's function. The use of the Green's function to solve discrete diffusion equations will require the reader has an understanding of Green's functions and their applications as well as a basic understanding of discrete calculus. Books on the subject include Kelley and Peterson [7], Elaydi [6] and Agarwal [1].

2. Introduction to a Discrete Two Dimensional Diffusion Equation

In this paper we will find the solution of the two dimensional discrete diffusion equation given by

$$\begin{aligned} v(x,y,t+1) = v(x,y,t) + r[v(x-1,y,t) - v(x,y,t)] \\ + l[v(x+1,y,t) - v(x,y,t)] + d[v(x,y+1,t) - v(x,y,t)] \\ + u[v(x,y-1,t) - v(x,y,t)]\,, \end{aligned}$$

with initial condition

$$v(x,y,0) = f(x,y)\,,$$

where x and y are integers $(x, y \in \mathbb{Z})$ and t is a non-negative integer $(t \in \mathbb{W})$. Thus by collecting terms, we want to find a solution of

$$\begin{aligned} v(x,y,t+1) = s\, v(x,y,t) + r\, v(x-1,y,t) + l\, v(x+1,y,t) \\ + d\, v(x,y+1,t) + u\, v(x,y-1,t)\,, \end{aligned}$$

with initial condition

$$v(x,y,0) = f(x,y)\,,$$

where $x, y \in \mathbb{Z}$ and $t \in \mathbb{W}$ with $s = 1 - r - l - d - u, r, l, d, u \in [0,1]$. The constants r (right), l (left), u (up) and d (down) correspond to the proportion of material (whatever v is modeling) moving from higher concentrations to lower concentrations (application of Fick's Law) in the respective directions.

Note, if for each fixed pair of integers w and z we can find a function P such that

$$\begin{aligned}P(x,y,w,z,t+1) &= s\,P(x,y,w,z,t)\\ &\quad + r\,P(x-1,y,w,z,t) + l\,P(x+1,y,w,z,t)\\ &\qquad + d\,P(x,y+1,w,z,t) + u\,P(x,y-1,w,z,t)\,,\end{aligned}$$

for all integers x and y and all non-negative integers t that satisfies the initial condition

$$P(x,y,w,z,0) = \delta_{x,w}\delta_{y,z}\,,$$

then the (unique) solution of our discrete two dimensional diffusion equation is

$$v(x,y,t) = \sum_{w=-\infty}^{\infty}\sum_{z=-\infty}^{\infty} P(x,y,w,z,t)f(w,z)\,,$$

and the sum converges since there are only a finite number of terms on the right hand side that are not zero. We find the function P, which we will call the Green's function for the discrete two dimensional diffusion problem, that satisfies these conditions by considering a random walk.

3. A Random Walk

Define a random walk as a movement in the two dimensional lattice $\mathbb{Z}^2$ with the property that, regardless of the current location and time, the walker moves Right with probability r, moves Left with probability l, moves Up with probability u, moves Down with probability d and Stays with probability s $(r+l+u+d+s=1)$ and at each time interval the walker must complete one of the five operations (which we will refer to as steps). Furthermore, for w and z fixed integers, we will assume that the walker is at (w,z) initially.

Let $P(x,y,w,z,t)$ be the probability that the walker started at (w,z) initially and is at (x,y) after t steps. At time zero, we know the location of the walker. That is, at time zero, the walker is at (w,z) (where w and z are fixed integers) with probability 1 (zero everywhere else). Hence,

$$P(x,y,w,z,0) = \delta_{x,w}\delta_{y,z}.$$

From conditional probability, the probability that the walker was at (w,z) initially and is at (x,y) at time $t+1$ corresponds to the following sum

$$\begin{aligned}P(x,y,w,z,t+1) &= s\,P(x,y,w,z,t) + r\,P(x-1,y,w,z,t)\\ &\quad + l\,P(x+1,y,w,z,t) + d\,P(x,y+1,w,z,t) + u\,P(x,y-1,w,z,t)\end{aligned}$$

since the walker must have been one step (Left, Right, Up, Down, or Stay) from (x, y) at time t and the corresponding coefficients are the probabilities of such steps. Thus the solution of this random walk is the Green's function for the two dimensional discrete diffusion problem as previously remarked. In the following theorem we find a formula for P, when $x \geq w$ and $y \geq z$.

Theorem 1. *If $x \geq w$, $y \geq z$, and $t \geq (x - w) + (y - z)$ then*

$$P(x, y, w, z, t) = \sum_{k=0}^{\left\lfloor\frac{t-(x-w)-(y-z)}{2}\right\rfloor} \sum_{j=0}^{\left\lfloor\frac{t-(x-w)-(y-z)-2k}{2}\right\rfloor} H(x, y, w, z, t, j, k)\,,$$

where

$$H(x, y, w, z, t, j, k) = \frac{t!r^{x-w+k}l^k u^{y-z+j} d^j s^{t-(x-w)-(y-z)-2k-2j}}{(x-w+k)!k!(y-z+j)!j!(t-(x-w)-(y-z)-2k-2j)!}.$$

Proof. To start at (w, z) and end up at (x, y) after t steps the walker must have taken at least $x - w$ Rights and $y - z$ Ups. Thus there are

$$t - (x - w) - (y - z)$$

additional steps that must be taken. The walker can take k additional Rights, where k is between 0 and

$$\left\lfloor \frac{t - (x - w) - (y - z)}{2} \right\rfloor,$$

since for each additional Right the walker must take a Left (additional Rights are paired with Lefts) to end up with the horizontal component x. Thus leaving

$$t - (x - w) - (y - z) - 2k$$

steps to be taken. With the

$$t - (x - w) - (y - z) - 2k$$

steps to be taken the walker can take j additional Ups, where j is between 0 and

$$\left\lfloor \frac{t - (x - w) - (y - z) - 2k}{2} \right\rfloor,$$

since for each additional Up the walker must take a Down (additional Ups are paired with Downs) to end up with a vertical component of y. Thus leaving

$$t - (x - w) - (y - z) - 2k - 2j$$

steps remaining which must all then be Stays. The number of different walks consisting of $(x-w)+k$ Rights, k Lefts, $(y-z)+j$ Ups, j Downs and $t-(x-z)-(y-z)+2k+2j$ Stays (permutations of like objects) is

$$\frac{t!}{(x-w+k)!k!(y-z+j)!j!(t-(x-w)-(y-z)-2k-2j)!},$$

each with probability

$$r^{x-w+k}l^k u^{y-z+j} d^j s^{t-(x-w)-(y-z)-2k-2j}.$$

Therefore,

$$P(x,y,w,z,t)=\sum_{k=0}^{\lfloor\frac{t-(x-w)-(y-z)}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{t-(x-w)-(y-z)-2k}{2}\rfloor} H(x,y,w,z,t,j,k)\,. \qquad \square$$

If $t<(x-w)+(y-z)$ then $P(x,y,w,z,t)=0$ since there are fewer steps taken, t, then the minimum number of required steps, $(x-w)+(y-z)$, to get from the initial point (w,z) to the point (x,y). The remaining cases are proven in the same manner. For non-negative integers a,b,τ with $\tau\geq a+b$ define the functions H_1, H_2, H_3 and H_4 by

$$H_1(a,b,\tau)=\sum_{k=0}^{\lfloor\frac{\tau-a-b}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{\tau-a-b-2k}{2}\rfloor}\frac{\tau!r^{a+k}l^k u^{b+j} d^j s^{\tau-a-b-2k-2j}}{(a+k)!k!(b+j)!j!(\tau-a-b-2k-2j)!},$$

$$H_2(a,b,\tau)=\sum_{k=0}^{\lfloor\frac{\tau-a-b}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{\tau-a-b-2k}{2}\rfloor}\frac{\tau!r^{k}l^{a+k} u^{b+j} d^j s^{\tau-a-b-2k-2j}}{(a+k)!k!(b+j)!j!(\tau-a-b-2k-2j)!},$$

$$H_3(a,b,\tau)=\sum_{k=0}^{\lfloor\frac{\tau-a-b}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{\tau-a-b-2k}{2}\rfloor}\frac{\tau!r^{k}l^{a+k} u^{j} d^{b+j} s^{\tau-a-b-2k-2j}}{(a+k)!k!(b+j)!j!(\tau-a-b-2k-2j)!},$$

and

$$H_4(a,b,\tau)=\sum_{k=0}^{\lfloor\frac{\tau-a-b}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{\tau-a-b-2k}{2}\rfloor}\frac{\tau!r^{a+k}l^{k} u^{j} d^{b+j} s^{\tau-a-b-2k-2j}}{(a+k)!k!(b+j)!j!(\tau-a-b-2k-2j)!},$$

then for $x, y, w, z, t \in \mathbb{Z}$ with $t \geq 0$

$$P(x,y,w,z,t) = \begin{cases} H_1(x-w, y-z, t) & \text{if } x \geq w,\ y \geq z \text{ and } t \geq |x-w| + |y-z|, \\ H_2(w-x, y-z, t) & \text{if } w \geq x,\ y \geq z \text{ and } t \geq |x-w| + |y-z|, \\ H_3(w-x, z-y, t) & \text{if } w \geq x,\ z \geq y \text{ and } t \geq |x-w| + |y-z|, \\ H_4(x-w, z-y, t) & \text{if } x \geq w,\ z \geq y \text{ and } t \geq |x-w| + |y-z|, \\ 0 & \text{if } t < |x-w| + |y-z|. \end{cases}$$

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