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ON SOME INEQUALITIES ON TIME SCALES

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Abstract: In this paper, we establish some integral inequalities on a time scale which generalize some results of Hayashi.

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1. Introduction

To unify the theory of continuous and discrete dynamic systems, in 1990, Hilger [4] proposed the study of dynamic systems on a time scale and developed necessary calculus for functions on a time scale (that is any closed subset of reals).

The main purpose of this paper is to establish the lower and upper bounds of

$$\left(\int_a^b h(t)f^2(t)\Delta t\right)\left(\int_a^b h(t)g^2(t)\Delta t\right),$$

for three given functions $h, f, g, \in C_{rd}([a,b], \mathbb{R}R)$ with $h(x) \geq 0$ on $[a,b]$. Some other integral inequalities are also given. These results generalized some results of Hayashi [3]. For other related results, we refer to Hardy, Littlewood and

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Pólya [2], Mintriović [6] and Mintriović, Pečarić and Fink [7].

We first briefly introduce the time scales calculus.

By a times scale $\mathbb{R}T$ we mean any closed subset of $\mathbb{R}R$ with order and topological structure in a canonical way. Since a time scale $\mathbb{R}T$ may or may not be connected, we need the concept of jump operators.

Definition. Let $t \in \mathbb{R}T$, where $\mathbb{R}T$ is a time scale, then two mappings

$$\sigma,\ \rho : \mathbb{R}T \to \mathbb{R}R\,,$$

satisfying

$$\sigma(t) = \inf\{s \in \mathbb{R}T | s > t\},\ \rho(t) = \sup\{s \in \mathbb{R}T | s > t\}$$

are called the jump operators.

If $\sigma(t) > t,\ t \in \mathbb{R}T$, we say t is *right-scattered.* If $\rho(t) < t,\ t \in \mathbb{R}T$, we say t is *left-scattered.* If $\sigma(t) = t,\ t \in \mathbb{R}T$, we say t is *right-dense.* If $\rho(t) = t,\ t \in \mathbb{R}T$, we say t is *left-dense.*

Definition. The mapping $f : \mathbb{R}T \to \mathbb{R}R$ is called *rd-continuous* if the following two conditions hold:

(a) f is continuous at each right-dense point or maximal point of $\mathbb{R}T$;

(b) $\lim_{s \to t^-} g(s) = g(t^-)$ exists for each left-dense point $t \in \mathbb{R}T$.

The set of all rd-continuous functions from $\mathbb{R}T$ to $\mathbb{R}R$ is denoted by $C_{rd}[\mathbb{R}T, \mathbb{R}R]$.

Let

$$\mathbb{R}T^k := \begin{cases} \mathbb{R}T - \{m\}, & \text{if } \mathbb{R}T \text{ has a left-scattered maximal point } m. \\ \mathbb{R}T, & \text{otherwise.} \end{cases}$$

Definition. Assume that $f : \mathbb{R}T \to \mathbb{R}R$ and $t \in \mathbb{R}T^k$, then we define $f^\Delta(t)$ to be the number (if it exists) with property that, for any given $\epsilon > 0$, there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \le \epsilon |\sigma(t) - s|\,,$$

for all $s \in U$. In this case $f^\Delta(t)$ is called the *delta*-derivative of $f(t)$ at t and f is *delta*-differentiable at t. If f is *delta*-differentiable at each $t \in \mathbb{R}T$, then f is called *delta*-differentiable on $\mathbb{R}T$.

Definition. The function $g : \mathbb{R}T \to \mathbb{R}R$ is called an antiderivative of $f : \mathbb{R}T \to \mathbb{R}R$ if $g^\Delta(t) = f(t)$, for all $t \in \mathbb{R}T^k$, and in this case, we define the integral of f by

$$\int_s^t f(u)\Delta u = g(t) - g(s)\,,$$

for all $s, t \in T$, and we say that f is integrable on $\mathbb{R}T$.

Throughout this paper, we suppose that:

(a) $\mathbb{R}R = (-\infty, \infty)$;

(b) an interval means the intersection of a real interval with the given time scale.

For further concerning the time scale, we refer to [1, 4, 5]. Throughout this paper, we let the interval $[a, b] := [a, b] \cap \mathbb{R}T$, where $\mathbb{R}T$ is a time scale and $a, b \in \mathbb{R}T$ with $a < b$.

2. Main Results

First, we state the well-known Cauchy inequality on time scales $\mathbb{R}T$ as follows:

Theorem 1. *Let $h, f, g \in C_{rd}([a, b], \mathbb{R}R)$ with $h(x) \geq 0$ on $[a, b]$. Then*

$$\left(\int_a^b h(x)f(x)g(x)\Delta x\right)^2 \leq \left(\int_a^b h(x)f^2(x)\Delta x\right)\left(\int_a^b h(x)g^2(x)\Delta x\right).$$

Theorem 2. *Let $h, f, g \in C_{rd}([a, b], \mathbb{R}R)$ with $h(x) \geq 0$ on $[a, b]$. If*

$$g(x)[g(x) - f(x)] \geq 0 \quad \text{and} \quad f(x)[g(x) - f(x)] \geq 0 \tag{C_0}$$

on $[a, b]$, then

$$2\int_a^b h(x)f(x)g(x)\Delta x \int_a^b h(x)g^2(x)\Delta x - \left(\int_a^b h(x)f(x)g(x)\Delta x\right)^2 \geq \int_a^b h(x)f^2(x)\Delta x \int_a^b h(x)g^2(x)\Delta x. \tag{R_1}$$

Proof. It follows from condition (C_0) that

$$2\int_a^b h(x)f(x)g(x)\Delta x \int_a^b h(x)g^2(x)\Delta x - \left(\int_a^b h(x)f(x)g(x)\Delta x\right)^2 - \int_a^b h(x)f^2(x)\Delta x \int_a^b h(x)g^2(x)\Delta x$$

$$
\begin{aligned}
&= \int_a^b h(x)f(x)g(x)\Delta x \int_a^b h(x)g^2(x)\Delta x - \Big(\int_a^b h(x)f(x)g(x)\Delta x\Big)^2 \\
&\quad + \int_a^b h(x)f(x)g(x)\Delta x \int_a^b h(x)g^2(x)\Delta x \\
&\quad - \int_a^b h(x)f^2(x)\Delta x \int_a^b h(x)g^2(x)\Delta x \\
&= \int_a^b h(x)f(x)g(x)\Delta x \Big[\int_a^b h(x)g(x)\Big(g(x)-f(x)\Big)\Delta x\Big] \\
&\quad + \int_a^b h(x)g^2(x)\Delta x \Big[\int_a^b h(x)f(x)\Big(g(x)-f(x)\Big)\Delta x\Big] \geq 0.
\end{aligned}
$$

This completes the proof of (R_1). □

Remark 1. It follows from Theorems 1 and 2 that

$$
\begin{aligned}
2\int_a^b h(x)f(x)g(x)\Delta x \int_a^b h(x)g^2(x)\Delta x - \Big(\int_a^b h(x)f(x)g(x)\Delta x\Big)^2 \\
\geq \int_a^b h(x)f^2(x)\Delta x \int_a^b h(x)g^2(x)\Delta x \\
\geq \Big(\int_a^b h(x)f(x)g(x)\Delta x\Big)^2, \quad (R_2)
\end{aligned}
$$

if $g(x) \geq f(x) \geq 0$ or $g(x) \leq f(x) \leq 0$ on $[a,b]$.

Corollary 3. *Let $h, f, g \in C_{rd}([a,b], \mathbb{R}R)$ with $h(x) \geq 0$ on $[a,b]$. If*

$$g(x)[g(x)-f(x)] \leq 0 \quad \text{and} \quad f(x)[g(x)-f(x)] \leq 0$$

on $[a,b]$, then the reverse inequality in (R_1) holds.

Corollary 4. *Let $h_i, a_i, b_i \in \mathbb{R}R$ with $h_i \geq 0$ for each $i = 1, 2, \cdots, n$. If*

$$
\begin{cases} b_i[b_i-a_i] \geq 0, \\ a_i[b_i-a_i] \geq 0, \end{cases} \quad \left(\text{or} \quad \begin{cases} b_i[b_i-a_i] \leq 0, \\ a_i[b_i-a_i] \leq 0 \end{cases}\right),
$$

for each $i = 1, 2, \cdots, n$, then

$$
2\Big(\sum_{i=1}^n h_i a_i b_i\Big)\Big(\sum_{i=1}^n h_i b_i^2\Big) - \Big(\sum_{i=1}^n h_i a_i b_i\Big)^2
$$

$$\geq (\text{or} \leq) \quad \left(\sum_{i=1}^{n} h_i a_i^2\right)\left(\sum_{i=1}^{n} h_i b_i^2\right).$$

Theorem 5. *Let $h, f_1, f_2, g_1, g_2 \in C_{rd}([a,b], \mathbb{R}R)$ with $h(x) \geq 0$ on $[a,b]$. If one of the following conditions holds:*

(C_1) *$f_1(x) \geq g_1(x) \geq 0$ and $f_2(x) \geq g_2(x) \geq 0$ on $[a,b]$;*

(C_2) *$f_1(x) \geq g_1(x) \geq 0$ and $f_2(x) \leq g_2(x) \leq 0$ on $[a,b]$;*

(C_3) *$f_1(x) \leq g_1(x) \leq 0$ and $f_2(x) \geq g_2(x) \geq 0$ on $[a,b]$;*

(C_4) *$f_1(x) \leq g_1(x) \leq 0$ and $f_2(x) \leq g_2(x) \leq 0$ on $[a,b]$,*

then

$$\begin{aligned}\left(\int_a^b h(x)f_1(x)g_2(x)\Delta x + \int_a^b h(x)f_2(x)g_1(x)\Delta x\right)& \\ \times \int_a^b h(x)f_1(x)f_2(x)\Delta x& \\ - \left(\int_a^b h(x)f_1(x)g_2(x)\Delta x\right)\left(\int_a^b h(x)f_2(x)g_1(x)\Delta x\right)& \\ \geq \left(\int_a^b h(x)f_1(x)f_2(x)\Delta x\right)\left(\int_a^b h(x)g_1(x)g_2(x)\Delta x\right).& \quad (R_3)\end{aligned}$$

Proof. Let $x, y \in [a,b]$. It follows from conditions (C_i), $(i = 1,2,3,4)$ that

$$\begin{aligned}h(x)h(y)f_1(x)g_2(x)f_2(y)[f_1(y) - g_1(y)]& \\ + h(x)h(y)f_1(x)f_2(x)g_1(y)[f_2(y) - g_2(y)] \geq 0.&\end{aligned}$$

Integrating the above inequality with respect to y from a to b and then integrating it with respect to x from a to b, we see that

$$\begin{aligned}\int_a^b h(x)f_1(x)g_2(x)\Delta x \int_a^b h(y)f_1(y)f_2(y)\Delta y - \int_a^b h(x)f_1(x)g_2(x)\Delta x& \\ \times \int_a^b h(x)f_2(y)g_1(y)\Delta y + \int_a^b h(x)f_1(x)f_2(x)\Delta x \int_a^b h(y)f_2(y)g_1(y)\Delta y& \\ - \int_a^b h(x)f_1(x)f_2(x)\Delta x \int_a^b h(y)g_1(y)g_2(y)\Delta y \geq 0.&\end{aligned}$$

Hence

$$\left(\int_a^b h(x)f_1(x)g_2(x)\Delta x + \int_a^b h(x)f_2(x)g_1(x)\Delta x\right)\int_a^b h(x)f_1(x)f_2(x)\Delta x$$
$$- \int_a^b h(x)f_1(x)g_2(x)\Delta x \int_a^b h(y)f_2(y)g_1(y)\Delta y$$
$$\geq \int_a^b h(x)f_1(x)f_2(x)\Delta x \int_a^b h(x)g_1(x)g_2(x)\Delta x.$$

This completes our proof. □

Remark 2. Let $f_1 = f_2 = g$ and $g_1 = g_2 = f$. Then (R_3) reduces to (R_1).

Corollary 6. *Let $h_i, a_i, b_i, p_i, q_i \in \mathbb{R}R$ with $h_i \geq 0$ for each $i = 1, 2, \cdots, n$. If one of the following conditions holds:*

(C_1^*) $a_i \geq b_i \geq 0$ *and* $p_i \geq q_i \geq 0$ *for each* $i = 1, 2, \cdots, n$;

(C_2^*) $a_i \geq b_i \geq 0$ *and* $p_i \leq q_i \leq 0$ *for each* $i = 1, 2, \cdots, n$;

(C_3^*) $a_i \leq b_i \leq 0$ *and* $p_i \geq q_i \geq 0$ *for each* $i = 1, 2, \cdots, n$;

(C_4^*) $a_i \leq b_i \leq 0$ *and* $p_i \leq q_i \leq 0$ *for each* $i = 1, 2, \cdots, n$,

then

$$\left(\sum_{i=1}^n h_i a_i q_i + \sum_{i=1}^n h_i b_i p_i\right)\sum_{i=1}^n h_i a_i p_i - \left(\sum_{i=1}^n h_i a_i q_i\right)\left(\sum_{i=1}^n h_i b_i p_i\right)$$
$$\geq \left(\sum_{i=1}^n h_i a_i p_i\right)\left(\sum_{i=1}^n h_i b_i q_i\right).$$

Theorem 7. *Let $h, f, g \in C_{rd}([a, b], [0, \infty))$ with $f(x) \geq 1$ and $g(x) \geq 1$ on $[a, b]$. Then*

$$\left(\int_a^b h(x)f(x)\Delta x + \int_a^b h(x)g(x)\Delta x\right)\left(\int_a^b h(x)f(x)g(x)\Delta x\right)$$
$$- \int_a^b h(x)f(x)\Delta x \int_a^b h(x)g(x)\Delta x$$
$$\geq \int_a^b h(x)\Delta x \int_a^b h(x)f(x)g(x)\Delta x. \quad (R_4)$$

Proof. It follows from assumptions that

$$\Big(h(x)f(x)+h(x)g(x)\Big)h(y)f(y)g(y) - h(x)f(x)h(y)g(y) - h(x)h(y)f(y)g(y)$$
$$= h(x)h(y)f(x)g(y)\Big[f(y)-1\Big] + h(x)h(y)f(y)g(y)\Big[g(x)-1\Big] \geq 0.$$

Hence

$$\int_a^b\int_a^b \Big(h(x)f(x)+h(x)g(x)\Big)h(y)f(y)g(y)\Delta x\Delta y - \int_a^b h(x)f(x)\Delta x \times \int_a^b h(y)g(y)\Delta y - \int_a^b h(x)\Delta x \int_a^b h(y)f(y)g(y)\Delta y \geq 0.$$

This completes the proof of our Theorem. □

Remark 3. If f and g are replaced by fg and fg, respectively, then (R_4) reduces to

$$2\int_a^b h(x)f(x)g(x)\Delta x \int_a^b h(x)f^2(x)g^2(x)\Delta x - \left(\int_a^b h(x)f(x)g(x)\Delta x\right)^2 \geq \int_a^b h(x)\Delta x \int_a^b h(x)f^2(x)g^2(x)\Delta x.$$

Corollary 8. *(a) $\mathbb{R}T=\mathbb{R}R$. Let $h,f,g\in C([a,b],[0,\infty))$ with $f(x)\geq 1$ and $g(x)\geq 1$ on $[a,b]$. Then*

$$\left(\int_a^b h(x)f(x)dx + \int_a^b h(x)g(x)dx\right)\left(\int_a^b h(x)f(x)g(x)dx\right) - \int_a^b h(x)f(x)dx\int_a^b h(x)g(x)dx$$
$$\int_a^b h(x)dx\int_a^b h(x)f(x)g(x)dx.$$

(b) $\mathbb{R}T=\mathbb{R}Z$. Let $h_i\geq 0$, $a_i\geq 1$ and $b_i\geq 1$ for each $i=1,2,\cdots,n$. Then

$$\left(\sum_{i=1}^n h_ia_i + \sum_{i=1}^n h_ib_i\right)\left(\sum_{i=1}^n h_ia_ib_i\right) - \left(\sum_{i=1}^n h_ia_i\right)\left(\sum_{i=1}^n h_ib_i\right) \geq \left(\sum_{i=1}^n h_i\right)\left(\sum_{i=1}^n h_ia_ib_i\right).$$

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