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HECKE OPERATORS ON MIXED AUTOMORPHIC FORMS AND KUGA FIBER VARIETIES

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Abstract: We discuss Hecke operators acting on the space of mixed automorphic forms and on the cohomology of families of Abelian varieties, known as Kuga fiber varieties, parametrized by an arithmetic quotient of a symmetric space associated to a real reductive group. There is a correspondence between a certain class of mixed automorphic forms and the cohomology of a Kuga fiber variety. We prove that the two types of Hecke operator actions are compatible with respect to such a correspondence.

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1. Introduction

Mixed automorphic forms on Hermitian symmetric domains are automorphic forms associated to equivariant holomorphic maps, and they are related to certain families of Abelian varieties (cf. [6], [11]). The purpose of this paper is to discuss Hecke operators acting on the space of mixed automorphic forms on real reductive groups and the compatibility of their actions with Hecke operator actions on the cohomology of certain families of Abelian varieties, known as

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Kuga fiber varieties.

Let G_1 and G_2 be semisimple Lie groups of Hermitian type that occur as the sets of real points of algebraic groups defined over $\mathbb{Q}$, and let D_1 and D_2 be the associated Hermitian symmetric domains, respectively. We assume that there are a holomorphic map $\tau_0 : D_1 \to D_2$ and a homomorphism $\mu_0 : G_1 \to G_2$ of Lie groups such that τ_0 and μ_0 are equivariant, that is,

$$\tau_0(gz) = \mu_0(g)\tau_0(z)\,,$$

for all $z \in D_1$ and $g \in G_1$. Let $J_1 : G_1 \times D_1 \to GL(V_1)$ and $J_2 : G_2 \times D_2 \to GL(V_2)$ be automorphy factors for some finite-dimensional complex vector spaces V_1 and V_2, and let Γ_1 be an arithmetic subgroup of G_1. Then a holomorphic mixed automorphic form of type $(J_1, J_2, \mu_0, \tau_0)$ is a holomorphic map $f : D_1 \to V_1 \otimes V_2$ satisfying

$$f(\gamma z) = J_1(\gamma, z) \otimes J_2(\mu_0(\gamma), \tau_0(z))f(z)\,,$$

for all $\gamma \in \Gamma_1$ and $z \in D_1$.

Let Γ_0 be a torsion-free arithmetic subgroup of the symplectic group $Sp(n, \mathbb{R})$, and let $\mathcal{H}_n$ be the Siegel upper half space of degree n. Then it is well-known that the associated Siegel modular variety $X_0 = \Gamma_0\backslash\mathcal{H}_n$ parametrizes a family of polarized Abelian varieties. Such a family can be regarded as a fiber bundle $\pi_0 : Y_0 \to X_0$ over X_0 whose fibers are polarized Abelian varieties, and it plays an important role in number theory. For example, holomorphic forms of the highest degree on Y_0 can be identified with certain Siegel modular forms for Γ_0. Let G_1 and D_1 be as above, and let $G_2 = Sp(n, \mathbb{R})$ and $D_2 = \mathcal{H}_n$. Let Γ_1 be a torsion-free arithmetic subgroup of G_1 with $\mu_1(\Gamma_1) \subset \Gamma_0$. Then the fiber bundle $\pi_1 : Y_1 \to X_1$ over $X_1 = \Gamma_1\backslash D_1$ obtained by pulling the bundle $\pi_0 : Y_0 \to X_0$ back via the map $X_1 \to X_0$ induced by τ_0 is also a family of polarized Abelian varieties parametrized this time by the arithmetic quotient $X_1 = \Gamma_1\backslash D_1$ of the Hermitian symmetric domain D_1. The variety Y_1 is called a Kuga fiber variety, and holomorphic forms of the highest degree on Y_1 can be identified with holomorphic mixed automorphic forms of certain type (cf. [11]).

In this paper we discuss Hecke operators acting on the space of mixed automorphic forms. To consider Hecke operators we first extend the notion of mixed automorphic forms to the case of reductive groups. We then describe Hecke operators on the de Rham and Dolbeault cohomology of Kuga fiber varieties. Finally, we prove that such Hecke operator actions on the cohomology are compatible with the Hecke operator actions on mixed automorphic forms.

2. Mixed Automorphic Forms

In this section we discuss mixed automorphic forms on reductive groups, which generalize mixed automorphic forms on semisimple Lie groups discussed in [10], associated to homomorphisms of reductive groups. We also describe holomorphic mixed automorphic forms for groups of Hermitian type and show that they are mixed automorphic forms in the previous sense.

Let $\mathbb{G}$ (resp. $\mathbb{G}'$) be a reductive group defined over $\mathbb{Q}$, and set $G = \mathbb{G}(\mathbb{R})$ (resp. $G' = \mathbb{G}'(\mathbb{R})$). Let K (resp. K') be a maximal compact subgroup of G (resp. G'), and let $\sigma : K \to GL(V)$ (resp. $\sigma : K' \to GL(V')$) be its representation in a finite-dimensional complex vector space V (resp. V'). Let $\mu : G \to G'$ be a group homomorphism determined by a homomorphism $\mathbb{G} \to \mathbb{G}'$ of algebraic groups defined over $\mathbb{Q}$, and let Γ be an arithmetic subgroup of $\mathbb{G}(\mathbb{Q}) \subset G$. We assume that $\mu(K) \subset K'$, and denote by $\mathcal{Z}(\mathfrak{g})$ the center of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G.

Definition 1. A smooth function $\phi : G \to V \otimes V'$ is said to be a *mixed automorphic form for* Γ *of type* (σ, σ', μ) if it satisfies the following conditions:

(i) $\phi(\gamma g k) = (\sigma \otimes (\sigma' \circ \mu))(k)^{-1} \phi(g)$ for all $\gamma \in \Gamma$, $g \in G$ and $k \in K$.

(ii) ϕ is $\mathcal{Z}(\mathfrak{g})$-finite.

(iii) ϕ is slowly increasing, that is, there exist norms $\|\cdot\|$ on G and $|\cdot|$ on $V \otimes V'$, a constant C, and a positive integer n such that

$$|\phi(g)| \leq C \|g\|^n ,$$

for all $g \in G$. We shall denote by $\mathcal{A}(\Gamma, \sigma, \sigma', \mu)$ the complex vector space consisting of all mixed automorphic forms for Γ of type (σ, σ', μ).

If the groups are of Hermitian type, we can consider holomorphic mixed automorphic forms as follows. Let Z and Z' be the centers of G and G', respectively, and assume that $\mu(Z) \subset Z'$. Then the Riemannian symmetric spaces $D = G/ZK$ and $D' = G'/Z'K'$, and the homomorphism μ induces an equivariant holomorphic map $\tau : D \to D'$ of Hermitian symmetric domains satisfying

$$\tau(gz) = \mu(g)\tau(z) ,$$

for all $g \in G$ and $z \in D$.

Definition 2. Let $J : G \times D \to GL(V)$ and $J' : G' \times D' \to GL(V')$ be automorphy factors for G and G', respectively, and let $\Gamma \subset \mathbb{G}(\mathbb{Q})$ be an arithmetic subgroup.

(i) A holomorphic map $f : D \to V \otimes V'$ is said to be a *holomorphic mixed automorphic form for* Γ *of type* (J, J', μ, τ) if it satisfies

$$f(\gamma z) = (J(\gamma, z) \otimes J'(\mu(\gamma), \tau(z)))f(z)\,,$$

for all $z \in D$ and $\gamma \in \Gamma$. We denote by $\mathcal{M}(\Gamma, J, J', \mu, \tau)$ the space of holomorphic mixed automorphic forms for Γ of type (J, J', μ, τ).

(ii) A holomorphic map $f : D \to V$ is said to be a *holomorphic automorphic form for* Γ *of type* J if it satisfies

$$f(\gamma z) = J(\gamma, z)f(z)\,,$$

for all $z \in D$ and $\gamma \in \Gamma$.

We denote by $\mathcal{M}(\Gamma, J, J', \mu, \tau)$ the space of holomorphic mixed automorphic forms for Γ of type (J, J', μ, τ) and by $\mathcal{M}(\Gamma, J)$ the space of holomorphic automorphic forms for Γ of type J.

Let the automorphy factors J and J' be as in Definition 2. Given an element $a \in G$ and a function $f : D \to V \otimes V'$, we set

$$(f \mid_{J,J'} [a])(z) = (J(a, z) \otimes J'(\mu(a), \tau(z)))^{-1} f(az)\,,$$

for all $z \in D$, and define the function $L(J, J')f : G \to V \otimes V'$ by

$$(L(J, J')f)(g) = (J(g, z_0) \otimes J'(\mu(g), \tau(z_0))^{-1} f(gz_0), \tag{1}$$

where $z_0 \in D$ denotes the base point of D, that is, the point corresponding to the coset ZK in $G/ZK = D$. Then we see that

$$f \mid_{J,J'} [aa'] = (f \mid_{J,J'} [a]) \mid_{J,J'} [a']\,,$$

for all $a, a' \in G$ and that, if f is a holomorphic automorphic form for Γ of type (J, J', μ, τ),

$$f \mid_{J,J'} [\gamma] = f\,,$$

for all $\gamma \in \Gamma$.

Lemma 3. *Let J and J' be the automorphy factors in Definition 2, and let $f : D \to V \otimes V'$ be a holomorphic map. Then we have*

$$(L(J, J')f)(ag) = (L(J, J')(f \mid_{J,J'} [a]))(g)\,,$$

for all $a, g \in G$.

Proof. Given a holomorphic map $f : D \to V \otimes V'$ and elements a, g of G, we see that

$$\begin{aligned}
&(L((J, J')f)(ag) \\
&\quad = (J(ag, z_0) \otimes J'(\mu(ag), \tau(z_0)))^{-1} f(agz_0) \\
&\quad = [(J(a, gz_0) \cdot J(g, z_0)) \\
&\qquad\qquad \otimes (J'(\mu(a), \mu(g)\tau(z_0)) \cdot J'(\mu(g), \tau(z_0)))]^{-1} f(agz_0) \\
&\quad = [(J(a, gz_0) \otimes J'(\mu(a), \tau(gz_0))) \\
&\qquad\qquad \cdot (J(g, z_0)) \otimes J'(\mu(g), \tau(z_0)))]^{-1} f(agz_0) \\
&\quad = [(J(g, z_0) \otimes J'(\mu(g), \tau(z_0)))^{-1} \\
&\qquad\qquad \cdot (J(a, gz_0) \otimes J'(\mu(a), \tau(gz_0)))^{-1}] f(agz_0) \\
&\quad = (J(g, z_0) \otimes J'(\mu(g), \tau(z_0)))^{-1} (f \mid_{J,J'} [a])(gz_0) \\
&\quad = (L(J, J')(f \mid_{J,J'} [a]))(g).
\end{aligned}$$

Hence the lemma follows. ∎

We consider semisimple Lie groups $\widetilde{G} \subset G$ and $\widetilde{G}' \subset G'$ such that their centers $Z \cap \widetilde{G}$ and $Z' \cap \widetilde{G}'$ are finite and

$$G = Z\widetilde{G}, \quad G' = Z'\widetilde{G}'. \tag{2}$$

If $\nu : Z \to \mathbb{C}^\times$ is a character of the center Z of G and if $j : \widetilde{G} \times D \to GL(V)$ is an automorphy factor for the semisimple Lie group $\widetilde{G}$, we define the associated map $j_\nu : G \times D \to GL(V)$ by

$$j_\nu(g, z) = |\nu(c)| j(g_1, z), \tag{3}$$

where $g = g_1 c \in G$ with $g_1 \in \widetilde{G}$ and $c \in Z$. Note that j_ν is well-defined because $Z \cap \widetilde{G}$ is finite. Given $g = g_1 c$, $g' = g'_1 c'$ with $g_1, g'_1 \in \widetilde{G}$ and $c, c' \in Z$, we have

$$\begin{aligned}
j_\nu(gg', z) &= |\nu(cc')| j(g_1 g'_1, z) \\
&= |\nu(c)| j(g_1, g'_1 z) \cdot |\nu(c')| j(g'_1, z) \\
&= j_\nu(g, g'_1 z) j_\nu(g', z),
\end{aligned}$$

for all $z \in D$. Since $g'z = g'_1 z$ for $z \in D$, we see that j_ν is an automorphy factor for the real reductive group G. Similarly, we define the automorphy factor $j'_{\nu'} : G' \times D' \to GL(V)$ associated to a character $\nu' : Z' \to \mathbb{C}^\times$ of $Z' \subset G'$ and an automorphy factor $j' : \widetilde{G}' \times D' \to GL(V')$ of $\widetilde{G}'$.

We shall now describe the construction of the canonical automorphy factor for a semisimple Lie group of $\mathcal{G}$ of Hermitian type (see e.g. [1], [14] for details). Let $\mathcal{K}$ be a maximal compact subgroup of $\mathcal{G}$, and let I be a $\mathcal{G}$-invariant complex structure on the associated Riemannian symmetric space $\mathcal{D} = \mathcal{G}/\mathcal{K}$. Then for each $z \in \mathcal{D}$ it determines a complex structure I_z on the tangent space $T_z(\mathcal{D})$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $\mathcal{G}$ and $\mathcal{K}$, respectively, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. If z_0 is the point in $\mathcal{D}$ with $\mathcal{K}z_0 = z_0$, then we can identify $\mathfrak{p}$ with the tangent space $T_{z_0}(\mathcal{D})$; hence we obtain a complex structure I_{z_0} on $\mathfrak{p}$. We set

$$\mathfrak{p}_\pm = \{X \in \mathfrak{p}_\mathbb{C} \mid I_{z_0}(X) = \pm iX\}, \tag{4}$$

and denote by $\mathcal{P}_+$, $\mathcal{P}_-$ the $\mathbb{C}$-subgroups of $\mathcal{G}_\mathbb{C}$ corresponding to $\mathfrak{p}_+$, $\mathfrak{p}_-$, respectively; here $(\cdot)_\mathbb{C}$ denotes the complexification. Then we have

$$\mathcal{P}_+ \cap \mathcal{K}_\mathbb{C}\mathcal{P}_- = \{1\}, \quad \mathcal{G} \subset \mathcal{P}_+\mathcal{K}_\mathbb{C}\mathcal{P}_-, \quad \mathcal{G} \cap \mathcal{K}_\mathbb{C}\mathcal{P}_- = \mathcal{K}$$

(cf. [14, Lemma II.4.2], [12]). If $g \in \mathcal{P}_+\mathcal{K}_\mathbb{C}\mathcal{P}_- \subset \mathcal{G}_\mathbb{C}$, we denote by $(g)_+ \in \mathcal{P}_+$, $(g)_0 \in \mathcal{K}_\mathbb{C}$ and $(g)_- \in \mathcal{P}_-$ the components of g such that

$$g = (g)_+ \cdot (g)_0 \cdot (g)_-.$$

Let $(\mathcal{G}_\mathbb{C} \times \mathfrak{p}_+)_*$ be the subset of $\mathcal{G}_\mathbb{C} \times \mathfrak{p}_+$ consisting of the elements (g, z) such that $g \cdot \exp z \in \mathcal{P}_+\mathcal{K}_\mathbb{C}P_-$, and set

$$\mathcal{J}(g, z) = (g \cdot \exp z)_0 \in \mathcal{K}_\mathbb{C} \tag{5}$$

for $(g, z) \in (\mathcal{G}_\mathbb{C} \times \mathfrak{p}_+)_*$. If we identify the Hermitian symmetric domain $\mathcal{D}$ with a subset of $\mathfrak{p}_+$ using the Harish-Chandra embedding $\mathcal{D} \hookrightarrow \mathfrak{p}_+$ (cf. [14, §II.4]), then we have

$$\mathcal{G} \times \mathcal{D} \subset (\mathcal{G}_\mathbb{C} \times \mathfrak{p}_+)_*.$$

Thus we obtain a map $\mathcal{J} : \mathcal{G} \times \mathcal{D} \to \mathcal{K}_\mathbb{C}$ which satisfies the condition

$$\mathcal{J}(g_1g_2, z) = \mathcal{J}(g_1, g_2z) \cdot \mathcal{J}(g_2, z)\,, \tag{6}$$

for $g_1, g_2 \in \mathcal{G}$ and $z \in \mathcal{D}$. The map $\mathcal{J}$ is called the *canonical automorphy factor* for $\mathcal{G}$.

Now we return to the case of semisimple Lie groups $\widetilde{G} \subset G$ and $\widetilde{G}' \subset G'$, and denote by $\widetilde{K} = K \cap \widetilde{G}$ and $\widetilde{K}' = K' \cap \widetilde{G}'$ the maximal compact subgroups of $\widetilde{G}$ and $\widetilde{G}'$ corresponding to $K \subset G$ and $K' \subset G'$, respectively. Let $\widetilde{J} : \widetilde{G} \times D \to \widetilde{K}_\mathbb{C}$ (resp. $\widetilde{J}' : \widetilde{G}' \times D' \to \widetilde{K}'_\mathbb{C}$) be the canonical automorphy factor for the semisimple

Lie group of Hermitian type $\widetilde{G}$ (resp. $\widetilde{G}'$), and let $\sigma : K \to GL(V)$ (resp. $\sigma' : K' \to GL(V'))$ be a finite-dimensional complex representation of K (resp. K') as before. If $\nu : Z \to \mathbb{C}^\times$ (resp. $\nu' : Z' \to \mathbb{C}^\times$) is a character of Z (resp. Z'), then σ (resp. σ') determines the map $\widetilde{\sigma} : \widetilde{K} \to GL(V)$ (resp. $\widetilde{\sigma}' : \widetilde{K}' \to GL(V'))$ given by

$$\sigma(cg_1) = |\nu(c)|\widetilde{\sigma}(g_1) \quad (\text{resp. } \sigma'(c'g_1') = |\nu'(c')|\widetilde{\sigma}'(g_1'))\,, \tag{7}$$

for $c \in \widetilde{K} = K \cap Z$, $g_1 \in \widetilde{K}$ (resp. $c \in \widetilde{K}' = K' \cap Z'$, $g_1' \in \widetilde{K}'$). Using the natural extensions

$$\widetilde{\sigma} : \widetilde{K}_{\mathbb{C}} \to GL(V), \quad \widetilde{\sigma}' : \widetilde{K}'_{\mathbb{C}} \to GL(V')$$

of the maps $\widetilde{\sigma}, \widetilde{\sigma}'$, we obtain the automorphy factors

$$\widetilde{J}_\sigma = \sigma \circ \widetilde{J} : \widetilde{G} \times D \to GL(V), \quad \widetilde{J}'_{\sigma'} = \sigma' \circ \widetilde{J}' : \widetilde{G}' \times D' \to GL(V'),$$

which then determine the associated automorphy factors $\widetilde{J}_{\sigma,\nu} : G \times D \to GL(V)$ and $\widetilde{J}'_{\sigma',\nu'} : G' \times D' \to GL(V')$ given as in (3).

Lemma 4. *Let $f : G \to V \otimes V'$ be a smooth function satisfying (i) and (ii) in Definition 1. Suppose that there is a character $\alpha : Z \to \mathbb{C}^\times$ such that*

$$f(cg) = \alpha(c)f(g)\,,$$

for all $c \in Z$ and $g \in G$. Then f is slowly increasing.

Proof. This follows from [3], Section 1.6. □

Theorem 5. *Let $\nu : Z \to \mathbb{C}^\times$ and $\nu' : Z' \to \mathbb{C}^\times$ be the characters of $Z \subset G$ and $Z' \subset G$, respectively, and let $f : D \to V \otimes V'$ be a holomorphic automorphic form for Γ of type $(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'}, \mu, \tau)$. Then the associated map*

$$L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'})f : G \to V \otimes V'$$

given by (1) is a mixed automorphic form for Γ of type (σ, σ', μ) in the sense of Definition 1.

Proof. Let $f : D \to V \otimes V'$ be a holomorphic automorphic form for Γ of type $(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'}, \mu, \tau)$, then we have

$$f \mid_{\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'}} [\gamma] = f\,,$$

for all $\gamma \in \Gamma$. Hence by Lemma 3 we see that

$$(L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'})f)(\gamma g) = (L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'})f)(g)\,,$$

for all $g \in G$ and $\gamma \in \Gamma$. On the other hand, for $k \in K$ and $g \in G$ we have

$$\begin{aligned}
&(L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'})f)(gk) \\
&\quad = (\widetilde{J}_{\sigma,\nu}(gk, z_0) \otimes \widetilde{J}'_{\sigma',\nu'}(\mu(gk), \tau(z_0)))^{-1} f(gkz_0) \\
&\quad = [(\widetilde{J}_{\sigma,\nu}(g, z_0) \cdot \widetilde{J}_{\sigma,\nu}(k, z_0)) \\
&\qquad\qquad \otimes (\widetilde{J}'_{\sigma',\nu'}(\mu(g), \tau(z_0)) \cdot \widetilde{J}'_{\sigma',\nu'}(\mu(k), \tau(z_0)))]^{-1} f(gz_0) \\
&\quad = (\widetilde{J}_{\sigma,\nu}(k, z_0) \otimes \widetilde{J}'_{\sigma',\nu'}(\mu(k), \tau(z_0)))^{-1} \\
&\qquad\qquad \times (\widetilde{J}_{\sigma,\nu}(g, z_0) \otimes \widetilde{J}'_{\sigma',\nu'}(\mu(g), \tau(z_0)))^{-1} f(gz_0) \\
&\quad = (\widetilde{J}_{\sigma,\nu}(k, z_0)^{-1} \otimes \widetilde{J}'_{\sigma',\nu'}(\mu(k), \tau(z_0)))^{-1} L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'}) f(g),
\end{aligned}$$

where $z_0 \in D$ is the fixed point of K. If z_0 is identified with its image under the Harish-Chandra embedding $D \hookrightarrow \mathfrak{p}_+$ considered above, then $\exp z_0$ corresponds to the identity element in $\widetilde{G}$. Thus by (5) we have

$$\widetilde{J}(k_1, z_0) = (k_1 \cdot \exp z_0)_0 = k_1\,,$$

for $k_1 \in \widetilde{K}$. Hence for $k = ck_1 \in K$ with $c \in Z$ and $k_1 \in \widetilde{K}$ we obtain

$$\widetilde{J}_{\sigma,\nu}(k, z_0) = \nu(c)\widetilde{\sigma}(\widetilde{J}(k_1, z_0)) = \nu(c)\widetilde{\sigma}(k_1).$$

Therefore by (7) we see that $\widetilde{J}_{\sigma,\nu}(k, z_0) = \sigma(k)$. Similarly, we have

$$\widetilde{J}'_{\sigma',\nu'}(\mu(k), \tau(z_0)) = (\sigma' \circ \mu)(k).$$

Hence it follows that $L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'})f$ satisfies the condition (i) in Definition 1. Using the arguments given in [2, p. 203], we see that $L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'})f$ also satisfies (ii) in Definition 1. To consider the growth condition we note that

$$\widetilde{J}_{\sigma,\nu}(c, z_0) = \nu(c)\widetilde{J}_{\sigma,\nu}(e, z_0) = \nu(c) \cdot 1_V,$$

$$\widetilde{J}'_{\sigma',\nu'}(\mu(c), \tau(z_0)) = \nu'(\mu(c))\widetilde{J}'_{\sigma',\nu'}(e', \tau(z_0)) = \nu'(\mu(c)) \cdot 1'_V\,,$$

for all $c \in Z$ and $g \in G$, where e, e' are the identity elements of G, G', respectively. Thus, for $c \in Z$ and $g \in G$, we obtain

$$
\begin{aligned}
&(L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'})f)(cg) \\
&\quad = (L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'})f)(gc) \\
&\quad = (\widetilde{J}_{\sigma,\nu}(gc, z_0) \otimes \widetilde{J}'_{\sigma',\nu'}(\mu(gc), \tau(z_0)))^{-1} f(gcz_0) \\
&\quad = (\widetilde{J}_{\sigma,\nu}(c, z_0) \otimes \widetilde{J}'_{\sigma',\nu'}(\mu(c), \tau(z_0)))^{-1} \\
&\qquad\qquad \times (\widetilde{J}_{\sigma,\nu}(g, z_0) \otimes \widetilde{J}'_{\sigma',\nu'}(\mu(g), \tau(z_0)))^{-1} f(cgz_0) \\
&\quad = (\nu(c) \cdot \nu'(\mu(c)))^{-1} \cdot (\widetilde{J}_{\sigma,\nu}(g, z_0) \otimes \widetilde{J}'_{\sigma',\nu'}(\mu(g), \tau(z_0)))^{-1} f(gz_0) \\
&\quad = \widehat{\nu}(c) L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'}) f(g),
\end{aligned}
$$

where $\widehat{\nu} : Z \to \mathbb{C}^\times$ is the character $c \mapsto (\nu(c) \cdot \nu'(\mu(c)))^{-1}$. Hence by Lemma 4 we see that $L(\widetilde{J}_{\sigma,\nu}, \widetilde{J}'_{\sigma',\nu'})f$ satisfies the condition (iii) in Definition 1, and therefore the proof of the theorem is complete. $\square$

3. Hecke Operators on Mixed Automorphic Forms

In this section we introduce Hecke operators acting on the space of mixed automorphic forms on reductive groups as well as Hecke operators acting on the space of holomorphic mixed automorphic forms. We show that such Hecke operator actions are compatible with respect to the correspondence described in Theorem 5.

Let $\mathbb{G}$ be as in Section 2, and let $\Gamma \subset \mathbb{G}(\mathbb{Q})$ be an arithmetic subgroup. We denote by $\mathcal{H}(\mathbb{G}(\mathbb{Q}), \Gamma)$ the space of all functions $h : \mathbb{G}(\mathbb{Q}) \to \mathbb{C}$ supported in a finite union of double cosets $\Gamma a \Gamma$ with $a \in \mathbb{G}(\mathbb{Q})$ such that

$$h(\gamma_1 g \gamma_2) = h(g)\,,$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $g \in \mathbb{G}(\mathbb{Q})$. Thus, if $h \in \mathcal{H}(\mathbb{G}(\mathbb{Q}), \Gamma)$ has support in $\bigcup_{i=1}^{\nu} \Gamma x_i \Gamma$ for some positive integer ν and $x_1, \ldots, x_\nu \in \mathbb{G}(\mathbb{Q})$, then $h = \sum_{i=1}^{\nu} c_i \chi_{x_i}$ with $c_i = h(x_i)$, where χ_a for $a \in \mathbb{G}(\mathbb{Q})$ denotes the characteristic function of $\Gamma a \Gamma$ on $\mathbb{G}(\mathbb{Q})$ given by

$$\chi_a(x) = \begin{cases} 1 & \text{if } x \in \Gamma a \Gamma, \\ 0 & \text{if } x \notin \Gamma a \Gamma. \end{cases}$$

Given $a \in \mathbb{G}(\mathbb{Q})$, if $\{\gamma_1, \dots, \gamma_m\}$ is a complete set of coset representatives of $(\Gamma \cap a^{-1}\Gamma a)\backslash\Gamma$, the double coset $\Gamma a \Gamma$ has a decomposition of the form

$$\Gamma a \Gamma = \coprod_{i=1}^{m} \Gamma a \gamma_i .$$

If h_1 and h_2 are elements of $\mathcal{H}(\mathbb{G}(\mathbb{Q}), \Gamma)$, then we denote by $h_1 * h_2 : \mathbb{G}(\mathbb{Q}) \to \mathbb{C}$ their convolution given by

$$h_1 * h_2(x) = \int_{\mathbb{G}(\mathbb{Q})} h_1(xy^{-1}) h_2(y) dy \,,$$

for all $x \in \mathbb{G}(\mathbb{Q})$.

Lemma 6. *Let* $\Gamma a \Gamma = \coprod_{i=1}^{m} \Gamma a_i$ *and* $\Gamma a \Gamma = \coprod_{j=1}^{m} \Gamma b_j$ *be decompositions of the double cosets* $\Gamma a \Gamma$ *and* $\Gamma b \Gamma$*, respectively, with* $a, b, a_i, b_j \in \mathbb{G}(\mathbb{Q})$*. Then we have*

$$\chi_a * \chi_b = \sum_{c \in \mathbb{G}(\mathbb{Q})} \lambda(a, b; c) \chi_c, \tag{8}$$

where $\lambda(a, b; c)$ *is the number of pairs* (i, j) *for which* $\Gamma a_i b_j = \Gamma c$*.*

Proof. See e.g. [7]. □

By extending linearly the operation in (8) for characteristic functions to arbitrary elements of $\mathcal{H}(\mathbb{G}(\mathbb{Q}), \Gamma)$, we see that $\mathcal{H}(\mathbb{G}(\mathbb{Q}), \Gamma)$ has the structure of an associative algebra.

Definition 7. The associative algebra $\mathcal{H}(\mathbb{G}(\mathbb{Q}), \Gamma)$ is called the *Hecke algebra* for Γ.

Let $\mathcal{A}(\Gamma, \sigma, \sigma', \mu)$ be the space of automorphic forms for Γ of type (σ, σ', μ) in the sense of Definition 1. Given an element ϕ of $\mathcal{A}(\Gamma, \sigma, \sigma', \mu)$ and a characteristic function χ_a for $a \in \mathbb{G}(\mathbb{Q})$, we set

$$(\chi_a \cdot \phi)(g) = \sum_{i=1}^{m} \phi(a_i g) \,, \tag{9}$$

for all $g \in G$ if

$$\Gamma a \Gamma = \coprod_{i=1}^{m} \Gamma a_i . \tag{10}$$

By extending this operation linearly, we obtain an action of the Hecke algebra $\mathcal{H}(\mathbb{G}(\mathbb{Q}),\Gamma)$ on the space $\mathcal{A}(\Gamma,\sigma,\sigma',\mu)$. We also see that $\mathcal{H}(\mathbb{G}(\mathbb{Q}),\Gamma)$ acts on the space

$$\mathcal{M}(\Gamma, j_\nu, j'_{\nu'}, \mu, \tau)$$

of holomorphic mixed automorphic forms for Γ of type $(j_\nu, j'_{\nu'}, \mu, \tau)$ by extending the action of χ_a for $a \in \mathbb{G}(\mathbb{Q})$ given by

$$(\chi_a \cdot f)(z) = \sum_{i=1}^{m} (f \mid_{j_\nu, j'_{\nu'}} [a_i])(z)\,,$$

for $z \in D$ and $f \in \mathcal{M}(\Gamma, j_\nu, j'_{\nu'}, \mu, \tau)$.

Proposition 8. *The actions of the Hecke algebra $\mathcal{H}(\mathbb{G}(\mathbb{Q}),\Gamma)$ on the spaces*

$$\mathcal{M}(\Gamma, j_\nu, j'_{\nu'}, \mu, \tau), \quad \mathcal{A}(\Gamma,\sigma,\sigma',\mu)$$

described above are compatible with respect to the correspondence described in Theorem 5, that is,

$$L(j_\nu, j'_{\nu'})(\Xi \cdot f) = \Xi \cdot (L(j_\nu, j'_{\nu'}) f)\,,$$

for all $f \in \mathcal{M}(\Gamma, j_\nu, j'_{\nu'}, \mu, \tau)$ and $\Xi \in \mathcal{H}(\mathbb{G}(\mathbb{Q}),\Gamma)$, where $L(j_\nu, j'_{\nu'})$ is as in (1).

Proof. Given $f \in \mathcal{M}(\Gamma, j_\nu, j'_{\nu'}, \mu, \tau)$, it suffices to show that

$$L(j_\nu, j'_{\nu'})(\chi_a \cdot f) = \chi_a \cdot (L(j_\nu, j'_{\nu'}) f),$$

for all $a \in \mathbb{G}(\mathbb{Q})$. Let the double coset $\Gamma a \Gamma$ with $a \in \mathbb{G}(\mathbb{Q})$ have a decomposition as in (10). Using (9) and Lemma 3, we have

$$\begin{aligned}
\chi_a \cdot (L(j_\nu, j'_{\nu'}) f)(g) &= \sum_{i=1}^{m} (L(j_\nu, j'_{\nu'}) f)(a_i g) \\
&= \sum_{i=1}^{m} (L(j_\nu, j'_{\nu'}) f \mid_{j_\nu, j'_{\nu'}} [a_i])(g) \\
&= \Big(L(j_\nu, j'_{\nu'}) \Big(\sum_{i=1}^{m} f \mid_{j_\nu, j'_{\nu'}} [a_i] \Big) \Big)(g) \\
&= (L(j_\nu, j'_{\nu'})(\chi_a \cdot f))(g)\,,
\end{aligned}$$

for all $g \in G$; hence the proposition follows. ∎

4. Kuga Fiber Varieties

A Kuga fiber variety is a family of Abelian varieties parametrized by an arithmetic quotient of a Hermitian symmetric domain (cf. [7], [9], [14]). In this section we describe the construction of Kuga fiber varieties associated to symplectic representations of reductive groups and establish connections between holomorphic mixed automorphic forms and holomorphic forms on such Kuga fiber varieties.

Let V be a real vector space of dimension $2n$ defined over $\mathbb{Q}$, and let β be a nondegenerate alternating bilinear form on V also defined over $\mathbb{Q}$. We denote by $\mathbb{G}Sp(V,\beta)$ the linear algebraic group of similitudes of the bilinear form β on V, and consider the homomorphism $\mu : G \to G'$ described in Section 2 for $G' = \mathbb{G}'(\mathbb{R})$ with $\mathbb{G}' = \mathbb{G}Sp(V,\beta)$. Let $\widetilde{G}' = Sp(V_{\mathbb{R}},\beta)$, and let $\widetilde{G} \subset G$ be the semisimple Lie group of Hermitian type with finite center satisfying (2). We denote by $\widetilde{\mu} : \widetilde{G} \to \widetilde{G}' = Sp(V_{\mathbb{R}},\beta)$ the homomorphism induced by μ. If K and K' are maximal compact subgroups of G and G', respectively, as before, we set

$$D = G/ZK = \widetilde{G}/\widetilde{K}, \quad \mathcal{H} = G'/Z'K' = \widetilde{G}'/\widetilde{K}'.$$

Thus the groups G, $\widetilde{G}$ act on D naturally, and G', $\widetilde{G}'$ act on $\mathcal{H}$. Note that $\mathcal{H}$ is isomorphic to the Siegel upper half space of degree n. Let $\Gamma \subset \mathbb{G}(\mathbb{Q})$ be a torsion-free arithmetic subgroup. Then the locally symmetric space $X = \Gamma\backslash D = \widetilde{\Gamma}\backslash D$ has a natural structure of a complex quasi-projective variety (cf. [1]). Since it is assumed that $\mu(K) \subset K'$, the homomorphism μ induces an equivariant holomorphic map $\tau : D \to \mathcal{H}$ satisfying

$$\tau(gz) = \mu(g)\tau(z)$$

for all $g \in G$ and $z \in D$. Let $G \ltimes_\mu V$ be the semidirect product of G and V with respect to the action of G on V given by $v \mapsto \mu(g)v$ for $v \in V$ and $g \in G$. Thus $G \ltimes_\mu V$ consists of the elements (g,v) in $G \times V$, and its multiplication operation is given by

$$(g,v) \cdot (g',v') = (gg', \mu(g)v' + v)\,,$$

for $g, g' \in G$ and $v, v' \in V$. The group $G \ltimes_\mu V$ acts on $D \times V$ by

$$(g,v) \cdot (z,v') = (gz, \mu(g)v' + v)\,, \tag{11}$$

for all $g \in G$, $z \in D$ and $v, v' \in V$.

Note that $\mathcal{H}$ can be regarded as the space of all complex structures I on V such that the bilinear form $(v,v') \mapsto \beta(v, Iv')$ on V is symmetric and positive definite. Thus we see that each element $z \in D$ determines a complex vector space $(V, I_{\tau(z)})$, where $I_{\tau(z)}$ is the complex structure on V corresponding to $\tau(z) \in \mathcal{H}$. Let z_0 be a fixed element of D, and let I_0 be the complex structure on V corresponding to the element $\tau(z_0)$ of $\mathcal{H}$. Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V, and denote by V_+ and V_- the subspaces of $V_{\mathbb{C}}$ defined by

$$V_{\pm} = \{v \in V_{\mathbb{C}} \mid I_0 v = \pm iv\},$$

so that we have

$$V_{\mathbb{C}} = V_+ \oplus V_-, \quad V_+ = \overline{V}_-.$$

Then each element v in $(V, I_{\tau(z)})$ determines an element

$$\xi(z,v) = v_z = v_+ - \tau(z)v_- = v_+ - I_{\tau(z)}v_- \tag{12}$$

of the subspace V_+ of $V_{\mathbb{C}}$, where the elements $v_{\pm}$ denote the $V_{\pm}$-components of $v \in V \subset V_{\mathbb{C}} = V_+ \oplus V_-$. We set

$$W = \coprod_{z \in D} (V, I_{\tau(z)}),$$

the disjoint union of the vector spaces $(V, I_{\tau(z)})$ with complex structure $I_{\tau(z)}$ for the elements $z \in D$. Then the map

$$W \to D \times V_+, \quad (z,v) \mapsto (z, \xi(z,v)) \tag{13}$$

determines a bijection $W \cong D \times V_+$ and a $\mathbb{C}$-linear isomorphism $(V, I_{\tau(z)}) \cong \{z\} \times V_+$. Thus the natural projection map $W \to D$ has the structure of a holomorphic vector bundle with fiber V_+. Now $G \ltimes_\mu V$ acts on W by

$$(g,v) \cdot (z,w) = (gz, \mu(g)(w+v)), \tag{14}$$

for $(g,v) \in G \ltimes_\mu V$ and $(z,w) \in W$, that is, $z \in D$ and $w \in (V, I_{\tau(z)})$ (cf. [14, p. 199]). Using the isomorphism between W and $D \times V_+$ given in (13), we see that $G \ltimes_\mu V$ acts on $D \times V_+$ by

$$\begin{aligned}(g,v) \cdot (z,w) &= (gz, \xi(gz, \mu(g)w) + \xi(gz, \mu(g)v)) \\ &= (gz, \xi(gz, \mu(g)u) + (\mu(g)v)_{gz}),\end{aligned} \tag{15}$$

for $(g,v) \in G \ltimes_\mu V$ and $(z,w) \in D \times V_+$ with $w = \xi(z,u)$ and $u \in (V, I_{\tau(z)})$.

We extend the alternating bilinear form β on V to the $\mathbb{C}$-bilinear form $\beta : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ on $V_{\mathbb{C}}$. Then the bilinear form $(v, v') \mapsto \beta(v, Iv')$ on V_+ is both symmetric and alternating, and hence we see that $\beta\,|_{V_+ \times V_+} = 0$. Similarly, we have $\beta\,|_{V_-\times V_-}=0$. Let $\{u_1,\dots,u_{2n}\}$ be a symplectic basis of $V_{\mathbb{C}}$ for $i\beta$ such that $\{u_1,\dots,u_n\}$ is a basis of V_+ and

$$i\beta(u_j,u_k)=0=i\beta(u_{n+j},u_{n+k})\,,$$

$$i\beta(u_j u_{n+j})=0=i\beta(u_{n+k},u_k)\,,$$

for $1\le j,k\le n$.

Lemma 9. *Let (g,v) be an element of $G\ltimes_\mu V$ such that*

$$\widetilde{\mu}(\widetilde{g})=\begin{pmatrix}A_\mu & B_\mu\\ C_\mu & D_\mu\end{pmatrix}\in Sp(V_{\mathbb{C}},\beta)$$

with respect to the basis $\{u_1,\dots,u_{2n}\}$ described above, where $\widetilde{g}\in\widetilde{G}$ is the image of g under the map $G\to\widetilde{G}$. Then the action in (15) *can be written in the form*

$$(g,v)\cdot(z,w)=(gz,{}^t(C_\mu\tau(z)+D_\mu)^{-1}w+(\mu(g)v)_{gz})\,,$$

for all $(z,w)\in D\times V_+$.

Proof. This can be proved by slightly modifying the proof of Corollary 7.4 in [14, Chapter II]. □

Let L be a lattice in V such that $\mu(\Gamma)L\subset L$ and $\beta(L,L)\subset\mathbb{Z}$. The action of $G\ltimes_\mu V$ on W in (14) induces an action of $\Gamma\ltimes_\mu L$ on W. We set

$$Y=\Gamma\ltimes_\mu L\backslash W.$$

Then the natural projection map $W\to D$ determines the structure of a fiber bundle on the induced map $\pi:Y\to X$ whose fiber over $\Gamma z\in X$ with $z\in D$ is the quotient $(V,I_{\tau(z)})/L$ of the complex vector space $(V,I_{\tau(z)})$ by the lattice L. The complex torus $(V,I_{\tau(z)})/L$ is in fact an Abelian variety because the alternating bilinear form β can be used as a Riemann form. Thus we obtain a fiber bundle $\pi:Y\to X$ whose fibers are Abelian varieties of the form $(V,I_{\tau(z)})/L$. The total space Y of such a fiber bundle is called a *Kuga fiber variety.*

Let $J_H:\widetilde{G}\times D\to K_{\mathbb{C}}$ and $J_V:\widetilde{G}'\times\mathcal{H}\to K'_{\mathbb{C}}$ be the canonical automorphy factors for the semisimple Lie groups $\widetilde{G}$ and $\widetilde{G}'=Sp(V_{\mathbb{R}},\beta)$, respectively, discussed in Section 2. We identify the complexification $\widetilde{G}'_{\mathbb{C}}$ of $\widetilde{G}'$ with $Sp(n,\mathbb{C})$

using the symplectic basis $\{u_1, \dots, u_{2n}\}$ described above, and regard $\mathcal{H}$ as the Siegel upper half space of degree n. Then it is known that

$$J_V(h, \zeta) = (C\zeta + D)^t\,,$$

for all $\zeta \in \mathcal{H}$ and

$$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{C}) = \widetilde{G}'_{\mathbb{C}}.$$

We define the automorphy factor $j_V : G' \times \mathcal{H} \to \mathbb{C}^\times$ by

$$j_V(g', \zeta) = \det[J_V(\widetilde{g}', \zeta)]\,,$$

for all $(g', \zeta) \in G' \times \mathcal{H}$, where $\widetilde{g}' \in \widetilde{G}'$ is the image of g' under the natural projection $G' \to \widetilde{G}'$. Let $\mathrm{Ad} : \widetilde{G} \to GL(\widetilde{\mathfrak{g}})$ be the adjoint representation of $\widetilde{G}$. Thus we have $\mathrm{Ad}(\widetilde{g}) = d\eta_{\widetilde{g}}$ for each $\widetilde{g} \in \widetilde{G}$, where $\eta_{\widetilde{g}} : \widetilde{G} \to \widetilde{G}$ is the homomorphism given by $\eta_{\widetilde{g}}(\widetilde{h}) = \widetilde{g}\widetilde{h}\widetilde{g}^{-1}$, for all $\widetilde{h} \in \widetilde{G}$. We extend Ad to the representation $\mathrm{Ad} : \widetilde{G}_{\mathbb{C}} \to GL(\widetilde{\mathfrak{g}}_{\mathbb{C}})$ of the complexification $\widetilde{G}_{\mathbb{C}}$ of $\widetilde{G}$. Then we see that

$$\mathrm{Ad}(\widetilde{k}) \cdot \widetilde{\mathfrak{p}}_+ \subset \widetilde{\mathfrak{p}}_+\,,$$

for all $\widetilde{k} \in \widetilde{K}_{\mathbb{C}}$, where $\widetilde{\mathfrak{p}}_+ \subset \widetilde{\mathfrak{g}}$ is given as in (4) for $\widetilde{G}$ and $\widetilde{K}$. We denote by $\mathrm{Ad}_{\widetilde{\mathfrak{p}}_+} : \widetilde{K}_{\mathbb{C}} \to GL(\widetilde{\mathfrak{p}}_+)$ the representation of $\widetilde{K}_{\mathbb{C}}$ in $\widetilde{\mathfrak{p}}_+$ given by

$$\mathrm{Ad}_{\widetilde{\mathfrak{p}}_+}(\widetilde{k}) = \mathrm{Ad}(\widetilde{k}) \mid_{\widetilde{\mathfrak{p}}_+}\,,$$

for $\widetilde{k} \in \widetilde{K}_{\mathbb{C}}$, and define the automorphy factor $j_H : G \times D \to \mathbb{C}^\times$ by

$$j_H(g, z) = \det[\mathrm{Ad}_{\mathfrak{p}_+}(J_H(\widetilde{g}, z))]\,,$$

for $(g, z) \in G \times D$, where $\widetilde{g} \in \widetilde{G}$ denotes the image of $g \in G$ as before.

Theorem 10. *Let Y^m be the fiber product of m copies of the Kuga fiber variety Y over X described above, and let $k = \dim_{\mathbb{C}} D$. Then its Dolbeault cohomology space $H^{k+mn,0}(Y^m)$ of type $(k + mn, 0)$ is canonically isomorphic to the space of mixed automorphic forms on D of type $(j_H^{-1}, j_V^m, \mu, \tau)$.*

Proof. Note that the Dolbeault cohomology space $H^{k+mn,0}(Y^m)$ is canonically isomorphic to the space $H^0(Y^m, \Omega^{k+mn})$ of sections of Ω^{k+mn}, where Ω^{k+mn} is the sheaf of holomorphic $(k+mn)$-forms on Y^m (cf. [4, p. 45]). Using the construction of Y above, we see that

$$Y^m \cong \Gamma \ltimes_\mu L^m \backslash D \times V_+^m,$$

and therefore each fiber of Y^m is a complex torus of complex dimension mn of the form $(V_+/L)^m$. Let $\xi^{(j)} = (\xi_1^{(j)}, \ldots, \xi_n^{(j)})$ be a coordinate system for the complex vector space V_+ for $1 \leq j \leq n$, and let $z = (z_1, \ldots, z_k)$ be the global coordinate system for D regarded as a bounded symmetric domain in $\mathbb{C}^k$. Let Φ be a holomorphic $(k+mn)$-form on Y^m, which we consider as a $(\Gamma \ltimes_\mu L^m)$-invariant form on $D \times V_+^m$. Then there is a holomorphic function $f_\Phi(z, \xi)$ on $D \times V_+^m$ such that

$$\Phi(z, \xi) = f_\Phi(z, \xi) dz \wedge d\xi^{(1)} \wedge \cdots \wedge d\xi^{(m)},$$

where $z = (z_1, \ldots, z_k) \in D$, $\xi = (\xi^{(1)}, \ldots, \xi^{(m)}) \in V_+^m$ and

$$\xi^{(j)} = (\xi_1^{(j)}, \ldots, \xi_n^{(j)}) \in V_+, \quad d\xi^{(j)} = d\xi_1^{(j)} \wedge \cdots \wedge d\xi_n^{(j)},$$

for $1 \leq j \leq m$. Given an element $z_0 \in D$, the restriction of the form Φ to the fiber $Y_{z_0}^n$ over z_0 is the holomorphic mn-form

$$\Phi(z_0, \xi) = f_\Phi(z_0, \xi) d\xi^{(1)} \wedge \cdots \wedge d\xi^{(m)}$$

on V_+^m, where $\xi \mapsto f_\Phi(z_0, \xi)$ is a holomorphic function on $Y_{z_0}^n \cong (V_+/L)^m$. Since any holomorphic function on a compact complex manifold is constant, we see that f_Φ is a function of z only. Thus we have

$$\Phi(z, \xi) = \widehat{f}_\Phi(z) dz \wedge d\xi^{(1)} \wedge \cdots \wedge d\xi^{(m)},$$

where $\widehat{f}_\Phi$ is a holomorphic function on D. In order to use the condition that Φ is invariant under the action of $\Gamma \ltimes_\mu L^m$, consider an element

$$(\gamma, l) = (\gamma, l^{(1)}, \ldots, l^{(m)}) \in \Gamma \ltimes_\mu L^m.$$

Then its action on dz is given by

$$dz \circ (\gamma, l) = j_H(\gamma, z) dz,$$

since $z \mapsto j_H(\gamma, z)$ is the Jacobian map for the transformation $z \mapsto \gamma z$ of D into itself as stated above. On the other hand, by Lemma 9 the action of $\Gamma \ltimes_\mu L^m$ on

$$d\xi^{(j)} = (d\xi_1^{(j)}, \ldots, d\xi_n^{(j)}) \in V_+$$

is given by

$$\begin{aligned} d\xi^{(j)} \circ (\gamma, l) &= \bigwedge_{i=1}^n d[{}^t(C_\mu \tau(z) + D_\mu)^{-1} \xi_i^{(j)} + (\widetilde{\mu}(\widetilde{\gamma}) l^{(j)})_{\gamma z}] \qquad (16) \\ &= \det(C_\mu \tau(z) + D_\mu)^{-1} \bigwedge_{i=1}^n d\xi_i^{(j)} \\ &= j_V(\mu(\gamma), \tau(z))^{-1} d\xi^{(j)}, \end{aligned}$$

for $1 \le j \le m$, where

$$\widetilde{\mu}(\widetilde{\gamma}) = \begin{pmatrix} A_\mu & B_\mu \\ C_\mu & D_\mu \end{pmatrix} \in Sp(n, \mathbb{C}) = \widetilde{G}'_{\mathbb{C}}.$$

Thus we obtain

$$\Phi \circ (\gamma, l) = \widehat{f}_\Phi(\gamma z) j_H(\gamma, z) j_V(\mu(\gamma), \tau(z))^{-m} dz \wedge d\xi^{(1)} \wedge \cdots \wedge d\xi^{(m)};$$

hence it follows that

$$\widehat{f}_\Phi(\gamma z) = j_H(\gamma, z)^{-1} j_V(\mu(\gamma), \tau(z))^m \widehat{f}_\Phi(z)\,,$$

for all $\gamma \in \Gamma$ and $z \in D$. On the other hand, given a mixed automorphic form f on D of type $(j_H^{-1}, j_V^m, \mu, \tau)$, we define the $(k+mn)$-form Φ_f on Y^m by

$$\Phi_f(z, \xi) = f(z) dz \wedge d\xi^{(1)} \wedge \cdots \wedge d\xi^{(m)}.$$

Then for $(\gamma, l) = (\gamma, l^{(1)}, \ldots, l^{(m)}) \in \Gamma \ltimes_\mu L^m$ we have

$$\begin{aligned}
&(\Phi_f \circ (\gamma, l))(z, \xi) \\
&\quad = f(\gamma z) j_H(\gamma, z) j_V(\mu(\gamma), \tau(z))^{-m} dz \wedge d\xi^{(1)} \wedge \cdots \wedge d\xi^{(m)} \\
&\quad = f(z) dz \wedge d\xi^{(1)} \wedge \cdots \wedge d\xi^{(m)} = \Phi_f(z, \xi).
\end{aligned}$$

Thus Φ_f is a $(\Gamma \ltimes_\mu L^m)$-invariant $(k+mn)$-form on $D \times V_+^m$, and therefore the map $f \mapsto \Phi_f$ gives an isomorphism between the space of mixed automorphic forms on D of type $(j_H^{-1}, j_V^m, \mu, \tau)$ and the space $H^0(Y^m, \Omega^{k+mn})$, and hence the theorem follows. □

5. Hecke Operators on Cohomology

Hecke operators acting on various cohomology spaces have been investigated in numerous papers over the years (see e.g. [5], [8], [13], [15]). Let Y^m be the fiber product of m copies of the Kuga fiber variety Y over the arithmetic quotient $X = \Gamma \backslash D$ of the Hermitian symmetric domain D constructed in Section 4. In this section, we introduce Hecke operators acting on the cohomology of Y^m and show that their actions are compatible with Hecke operator actions on holomorphic mixed automorphic forms.

For each nonnegative integer p we denote by $\mathcal{E}^p(Y^m)$ the space of differential p-forms on Y^m. Using the notations in Section 4, the space Y^m is isomorphic to the quotient $\Gamma \ltimes_\mu L^m \backslash D \times V_+^m$; hence we see that $\mathcal{E}^p(Y^m)$ can be identified with the space $\mathcal{E}^p(D \times V_+^m)^{\Gamma \ltimes_\mu L^m}$ of $(\Gamma \ltimes_\mu L^m)$-invariant p-forms on $D \times V_+^m$. Here the action of an element $(\gamma, l) \in \Gamma \ltimes_\mu L^m$ on a p-form $\omega \in \mathcal{E}^p(D \times V_+^m)$ on $D \times V_+^m$ is given by

$$\begin{aligned} ((\gamma, l) \cdot \omega)(z, \xi) &= \omega((\gamma, l) \cdot (z, \xi)) \\ &= \omega(\gamma z, J^{\mu,\tau}(\gamma, z)\xi^{(1)} + (\mu(\gamma)\ell^{(1)})_{\gamma z}, \dots, \\ &\qquad J^{\mu,\tau}(\gamma, z)\xi^{(m)} + (\mu(\gamma)\ell^{(m)})_{\gamma z}), \end{aligned} \tag{17}$$

for $\gamma \in \Gamma$, $z \in D$, $l = (l^{(1)}, \dots, l^{(m)})$ and $\xi = (\xi^{(1)}, \dots, \xi^{(m)}) \in V_+^m$, where $(\mu(\gamma)\ell^{(j)})_{\gamma z}$ with $1 \leq j \leq m$ is given by (12) and

$$J^{\mu,\tau}(\gamma, z) = {}^t(C_\mu \tau(z) + D_\mu)^{-1}, \quad \widetilde{\mu}(\widetilde{\gamma}) = \begin{pmatrix} A_\mu & B_\mu \\ C_\mu & D_\mu \end{pmatrix} \in Sp(V_\mathbb{C}, \beta)$$

(see Lemma 9). Given nonnegative integers r and s, let $\mathcal{E}^{r,s}(Y^m)$ (resp. $\mathcal{E}^{r,s}(D \times V_+^m)$) be the space of differential forms on Y^m (resp. $D \times V_+^m$) of type (r, s). Then $\mathcal{E}^{r,s}(Y^m)$ can be regarded as the space $\mathcal{E}^{r,s}(D \times V_+^m)^{\Gamma \ltimes_\mu L^m}$ of $(\Gamma \ltimes_\mu L^m)$-invariant forms on $D \times V_+^m$ of type (r, s), and we have

$$\mathcal{E}^p(Y^m) = \bigoplus_{r+s=p} \mathcal{E}^{r,s}(Y^m), \quad \mathcal{E}^p(D \times V_+^m) = \bigoplus_{r+s=p} \mathcal{E}^{r,s}(D \times V_+^m).$$

If $\mathbb{G}$ is the reductive group with $G = \mathbb{G}(\mathbb{R})$ and if $L \subset V_+$ is the lattice associated to Y as before, we set

$$\mathbb{G}(\mathbb{Q})_{\mu,L} = \{a \in \mathbb{G}(\mathbb{Q}) \mid \mu(a)L \subset L\}.$$

Since $\mu(\Gamma)L \subset L$, we see that $\Gamma \subset \mathbb{G}(\mathbb{Q})_{\mu,L}$. We denote by

$$\mathcal{H}(\mathbb{G}(\mathbb{Q})_{\mu,L}, \Gamma)$$

the subalgebra of the Hecke algebra $\mathcal{H}(\mathbb{G}(\mathbb{Q}), \Gamma)$ described in Section 3 consisting of the functions $h : \mathbb{G}(\mathbb{Q}) \to \mathbb{C}$ supported in a finite union of double cosets $\Gamma a \Gamma$ with $a \in \mathbb{G}(\mathbb{Q})_{\mu,L}$ such that

$$h(\gamma_1 g \gamma_2) = h(g),$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $g \in \mathbb{G}(\mathbb{Q})_{\mu,L}$.

Definition 11. The algebra $\mathcal{H}(\mathbb{G}(\mathbb{Q})_{\mu,L}, \Gamma)$ is called the *Hecke algebra associated to* μ, L *and* Γ.

Let $a \in \mathbb{G}(\mathbb{Q})_{\mu,L}$ with $\Gamma a \Gamma = \coprod_{i=1}^{q} \Gamma a_i$, and let

$$\chi_a \in \mathcal{H}(\mathbb{G}(\mathbb{Q})_{\mu,L}, \Gamma)$$

be the characteristic function of $\Gamma a \Gamma$. Given a p-form $\omega \in \mathcal{E}^p(D \times V_+^m)$, we set

$$\begin{aligned}(\chi_a \cdot \omega)(z,\xi) &= \sum_{i=1}^{q} \omega((a_i, 0)\cdot(z,\xi)) \qquad (18)\\ &= \sum_{i=1}^{q} \omega(a_i z, J^{\mu,\tau}(a_i,z)\xi^{(1)}, \dots, J^{\mu,\tau}(a_i,z)\xi^{(m)}),\end{aligned}$$

for all $(z,\xi) \in D \times V_+^m$ with $\xi = (\xi^{(1)}, \dots, \xi^{(m)})$.

Lemma 12. *If $\omega \in \mathcal{E}^p(D \times V_+^m)$ is a $(\Gamma \ltimes_\mu L^m)$-invariant p-form on $D \times V_+^m$, then $\chi_a \cdot \omega$ is also a $(\Gamma \ltimes_\mu L^m)$-invariant p-form on $D \times V_+^m$.*

Proof. Given elements $(\gamma, l) \in \Gamma \ltimes_\mu L^m$ and $(z,\xi) \in D \times V_+^m$ with

$$l = (l^{(1)}, \dots, l^{(m)}), \quad \xi = (\xi^{(1)}, \dots, \xi^{(m)}),$$

using (17) and (18), we have

$$\begin{aligned}&(\chi_a \cdot \omega)((\gamma,l)\cdot(z,\xi))\\ &\quad= (\chi_a \cdot \omega)(\gamma z, J^{\mu,\tau}(\gamma,z)\xi^{(1)} + (\mu(\gamma)\ell^{(1)})_{\gamma z}, \dots,\\ &\qquad\qquad J^{\mu,\tau}(\gamma,z)\xi^{(m)} + (\mu(\gamma)\ell^{(m)})_{\gamma z})\\ &\quad= \sum_{i=1}^{q} \omega(a_i\gamma z, J^{\mu,\tau}(a_i,\gamma z)(J^{\mu,\tau}(\gamma,z)\xi^{(1)} + (\mu(\gamma)\ell^{(1)})_{\gamma z}), \dots,\\ &\qquad\qquad J^{\mu,\tau}(a_i,\gamma z)(J^{\mu,\tau}(\gamma,z)\xi^{(m)} + (\mu(\gamma)\ell^{(m)})_{\gamma z}))\\ &\quad= \sum_{i=1}^{q} \omega(a_i\gamma z, J^{\mu,\tau}(a_i\gamma,z)\xi^{(1)} + J^{\mu,\tau}(a_i,\gamma z)(\mu(\gamma)\ell^{(1)})_{\gamma z}, \dots,\\ &\qquad\qquad J^{\mu,\tau}(a_i\gamma,z)\xi^{(m)} + J^{\mu,\tau}(a_i,\gamma z)(\mu(\gamma)\ell^{(m)})_{\gamma z}),\end{aligned}$$

where we also used the fact that $J^{\mu,\tau}: G \times D \to \mathbb{C}^\times$ is an automorphy factor, that is, it satisfies

$$J^{\mu,\tau}(\delta\delta', z) = J^{\mu,\tau}(\delta, \delta' z)\cdot J^{\mu,\tau}(\delta', z),$$

for all $\delta, \delta' \in G$ and $z \in D$. However, for $1 \le j \le m$ we have

$$J^{\mu,\tau}(a_i,\gamma z)(\mu(\gamma)\ell^{(m)})_{\gamma z} = (\mu(\gamma)\ell^{(m)})_{a_i\gamma z}$$

(see [14, Chaper 4]); hence we obtain

$$\begin{aligned}(\chi_a \cdot \omega)((\gamma, l) \cdot (z,\xi)) \\ = \sum_{i=1}^{q} \omega(\gamma a_i z, J^{\mu,\tau}(a_i\gamma, z)\xi^{(1)} + (\mu(\gamma)\ell^{(1)})_{a_i\gamma z}, \ldots, \\ J^{\mu,\tau}(a_i\gamma, z)\xi^{(m)} + (\mu(\gamma)\ell^{(m)})_{a_i\gamma z}).\end{aligned} \tag{19}$$

On the other hand, using (17) and (18) again, we have

$$\begin{aligned}(\gamma, l) \cdot ((\chi_a \cdot \omega)(z,\xi)) \\ = \sum_{i=1}^{q} \omega(\gamma a_i z, J^{\mu,\tau}(\gamma, z)(J^{\mu,\tau}(a_i, \gamma z)\xi^{(1)}) + (\mu(\gamma)\ell^{(1)})_{\gamma a_i z}, \ldots, \\ J^{\mu,\tau}(\gamma, z)(J^{\mu,\tau}(a_i, \gamma z)\xi^{(m)}) + (\mu(\gamma)\ell^{(m)})_{\gamma a_i z}) \\ = \sum_{i=1}^{q} \omega(\gamma a_i z, J^{\mu,\tau}(\gamma a_i, z)\xi^{(1)}) + (\mu(\gamma)\ell^{(1)})_{\gamma a_i z}, \ldots, \\ J^{\mu,\tau}(\gamma a_i, z)\xi^{(m)} + (\mu(\gamma)\ell^{(m)})_{\gamma a_i z}).\end{aligned} \tag{20}$$

Note, however, that each of the sets $\{a_i\gamma \mid 1 \le i \le m\}$ and $\{\gamma a_i \mid 1 \le i \le m\}$ is a permutation of $\{a_i \mid 1 \le i \le m\}$. Thus, comparing (19) and (20), we see that

$$(\chi_a \cdot \omega)((\gamma, l) \cdot (z,\xi)) = (\gamma, l) \cdot ((\chi_a \cdot \omega)(z,\xi)),$$

and hence the lemma follows. □

Lemma 13. *The differential operators $d : \mathcal{E}^p(D \times V_+^m) \to \mathcal{E}^{p+1}(D \times V_+^m)$ and $\overline{\partial} : \mathcal{E}^{r,s}(D \times V_+^m) \to \mathcal{E}^{r,s+1}(D \times V_+^m)$ commutes with each $\chi_a \in \mathcal{H}(\mathbb{G}(\mathbb{Q})_{\mu,L}, \Gamma)$, that is,*

$$d \circ \chi_a = \chi_a \circ d, \quad \overline{\partial} \circ \chi_a = \chi_a \circ \overline{\partial},$$

for all $a \in \mathbb{G}(\mathbb{Q})_{\mu,L}$.

Proof. Let $\Gamma a \Gamma = \coprod_{i=1}^{q} \Gamma a_i$, and define for each $i \in \{1, \ldots, \ell\}$ the map $\eta_{a_i} : D \times V_+^m \to D \times V_+^m$ by

$$\eta_{a_i}(z, v) = (a_i, 0) \cdot (z, v) = (a_i z, \mu(a_i)v),$$

for all $(z, v) \in D \times V_+^m$. Then the pullback map $\eta_{a_i}^* : \omega \mapsto \omega \circ \eta_{a_i}$ on differential forms on $D \times V_+^m$ satisfies

$$\eta_{a_i}^* \circ d = d \circ \eta_{a_i}^*, \quad \eta_{a_i}^* \circ \overline{\partial} = \overline{\partial} \circ \eta_{a_i}^*$$

because a_i is an element of the real reductive group G, and we have

$$\chi_a(\omega) = \sum_{i=1}^{q} \eta_{a_i}^*(\omega)\,,$$

for each differential form ω on $D \times V_+^m$. Thus for $\omega \in \mathcal{E}^p(D \times V_+^m)$ we obtain

$$\begin{aligned}(d \circ \chi_a)(\omega) &= d\Big(\sum_{i=1}^{q} \eta_{a_i}^*(\omega)\Big) = \sum_{i=1}^{q} (d \circ \eta_{a_i}^*)(\omega) \\ &= \sum_{i=1}^{q} (\eta_{a_i}^* \circ d)(\omega) = (\chi_a \circ d)(\omega).\end{aligned}$$

Similarly, we have $(\overline{\partial} \circ \chi_a)(\omega) = (\chi_a \circ \overline{\partial})(\omega)$ for all $\omega \in \mathcal{E}^p(D \times V_+^m)$, and therefore the proof of the lemma is complete. □

By Lemma 13 we see that the Hecke algebra $\mathcal{H}(\mathbb{G}(\mathbb{Q})_{\mu,L}, \Gamma)$ associated to μ, L and Γ operates on the de Rham cohomology space $H^p(Y^m)$ as well as on the Dolbeault cohomology space $H^{r,s}(Y^m)$ of the m-fold fiber power Y^m of the Kuga fiber variety Y.

Theorem 14. *Let $\varepsilon : \mathcal{M}(\Gamma, j_H^{-1}, j_V^m, \mu, \tau) \to H^{k+mn,0}(Y^m)$ be the canonical isomorphism given in Theorem 10. Then the action of the Hecke algebra $\mathcal{H}(\mathbb{G}(\mathbb{Q})_{\mu,L}, \Gamma)$ on the space $\mathcal{M}(\Gamma, j_H^{-1}, j_V^m, \mu, \tau)$ of mixed automorphic forms is compatible with its action on the Dolbeault cohomology space $H^{k+mn,0}(Y^m)$ via the isomorphism ε.*

Proof. Let $f \in \mathcal{M}(\Gamma, j_H^{-1}, j_V^m, \mu, \tau)$, and let $a \in \mathbb{G}(\mathbb{Q})_{\mu,L}$ with $\Gamma a \Gamma = \coprod_{i=1}^q \Gamma a_i$. It suffices to show that

$$\chi_a \cdot (\varepsilon(f)) = \varepsilon(\chi_a \cdot f)\,,$$

for all $f \in \mathcal{M}(\Gamma, j_H^{-1}, j_V^m, \mu, \tau)$. Given $(z, \xi) \in D \times V_+^m$ with $\xi = (\xi^{(1)}, \dots, \xi^{(m)})$ and $\xi^{(j)} = (\xi_1^{(j)}, \dots, \xi_n^{(j)})$ for $1 \le j \le m$, we have

$$\begin{aligned}(\chi_a \cdot \varepsilon(f))(z, \xi) &= \sum_{i=1}^{q} (\varepsilon(f))((a_i, 0) \cdot (z, \xi)) \\ &= \sum_{i=1}^{q} f(a_i z) d(z \cdot (a_i, 0)) \wedge d(\xi^{(1)} \cdot (a_i, 0)) \wedge \cdots \\ &\qquad \wedge d(\xi^{(m)} \cdot (a_i, 0)).\end{aligned}$$

However, as in (16), we have

$$d(\xi^{(j)} \cdot (a_i, 0)) = j_V(\mu(a_i), \tau(z))^{-1} d\xi^{(j)},$$

for $1 \leq i \leq \ell$ and $1 \leq j \leq m$. On the other hand, for each $g \in G$ the map $j_H(g, \cdot)$ is the Jacobian map for the transformation $z \mapsto gz$, and therefore we see that

$$d(z \cdot (a_i, 0)) = d(a_i z) = j_H(a_i, z) dz.$$

Thus we have

$$\begin{aligned}
&(\chi_a \cdot \varepsilon(f))(z, \xi) \\
&\quad = \sum_{i=1}^{q} f(a_i z) j_H(a_i, z) j_V(\mu(a_i), \tau(z))^{-m} dz \wedge d\xi^{(1)} \wedge \cdots \wedge d\xi^{(m)} \\
&\quad = (\chi_a \cdot f)(z) dz \wedge d\xi^{(1)} \wedge \cdots \wedge d\xi^{(m)} \\
&\quad = \varepsilon(\chi_a \cdot f)(z, \xi),
\end{aligned}$$

and hence the theorem follows. □

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