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INTERPOLATION FOR DISCRETE SUBSETS OF DOMAINS OF $\mathbf{C}^{\mathbf{N}}$

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Abstract: Let $U \subseteq \mathbf{C}^{\mathbf{N}}$ be a connected open subset and S a discrete and infinite subset of U. Here we give some conditions on S which force the failure the interpolation problem for S with respect to the holomorphic functions on U.

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1. Interpolation

Let X be a complex space (even infinite-dimensional) and $S \subset X$ a discrete subset. We will see S as a reduced complex space. Let $W(S) \subseteq H^0(S, \mathcal{O}_S)$ be the image of the restriction map $H^0(X, \mathcal{O}_X) \to H^0(S, \mathcal{O}_S)$. We will say that $W(S)$ is the space of all solutions for the interpolation problem determined by S. Here we will prove the following result.

Theorem 1. *Let $U \subseteq \mathbf{C}^{\mathbf{N}}$ be a connected open subset and S a discrete and infinite subset of U. Let $\pi_n : \mathbf{C}^{\mathbf{N}} \to \mathbf{C}^n$, $n \geq 1$, be the projection on the first n coordinates.*

(a) *If $\pi_n(S)$ is finite for all integers $n \geq 1$, then $W(S)$ has countable dimension and uncountable codimension in $H^0(S, \mathcal{O}_S)$.*

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(b) *If there is an integer $x \geq 1$ such that $\pi_x(S)$ is infinite, but for all integers $n \geq 1$ the set $\pi_n(S)$ is not discrete in $\pi_n(U)$, then $W(S)$ has uncountable dimension and uncountable codimension in $H^0(S, \mathcal{O}_S)$.*

(c) *Assume that U is a domain of holomorphy in $\mathbf{C}^{\mathbf{N}}$ and that there is an integer $n \geq 1$ such that $\pi_n|S : S \to \pi_n(S)$ is injective and $\pi_n(S)$ is discrete in $\pi_n(U)$. Then $W(S) = H^0(S, \mathcal{O}_S)$.*

Proof. Since U is connected, for every $f \in H^0(U, \mathcal{O}_U)$ there is an integer $n \geq 1$ such that f depends only from the first n coordinates of $\mathbf{C}^{\mathbf{N}}$ ([1], Theorem 1). For every integer $n \geq 1$ let $W(S)_n$ be the linear subspace of $W(S)$ formed by the images of the holomorphic functions on U depending only from the first n variables. If $\pi_n(S)$ is finite, then $W(S)_n$ is a finite-dimensional $\mathbf{C}$-vector space. Hence in case (a) $W(S)$ has countable algebraic dimension. Furthermore, $W(S)$ is finite-dimensional if and only if there is n_0 such that $W(S)_n = W(S)_{n_0}$ for every $n \geq n_0$. Since the holomorphic functions on any open subset of any Hausdorff topological vector space separates distinct points, it is immediate that $W(S)$ is a finite-dimensional $\mathbf{C}$-vector space if and only if S is finite, concluding the proof of part (a). Now we will check part (b). Fix an integer $n \geq 1$. By assumption there is $P_n \in \mathbf{C}^n \backslash \pi_n(U)$ and a sequence $\{x_{m,n}\}_{m \geq 1}$, $x_{m,n} \in U$ such that $\lim_{m \to +\infty} x_{m,n} = P_n$. Since the Banach space c_0 has uncountable codimension in ℓ^∞ we easily see that $W(S)_n$ has uncountable codimension in $H^0(S, \mathcal{O}_S)$. Since $W(S) = \bigcup_{n \geq 1} W(S)_n$, we obtain part (b). Now assume that U is a domain of holomorphy. By [1], Theorem 3, there is an integer $s \geq 1$ and a domain of holomorphy A of $\mathbf{C}^s$ such that $U = A \times \mathbf{C}^{\mathbf{N} \backslash \{1, \dots, s\}}$. Assume the existence of at least one integer $m \geq s$ such that $\pi_m|S$ is injective and $\pi_m(S)$ is discrete in $\pi_m(U)$. Hence $\pi_m(S)$ is closed submanifold of $\pi_n(U)$. Since $\pi_m(U) = A \times \mathbf{C}^{m-s}$, $\pi_n(U)$ is open and it is a Stein manifold. By Theorem B of Cartan-Serre the restriction map $H^0(\pi_m(U), \mathcal{O}_{\pi_m(U)}) \to H^0(\pi_m(S), \mathcal{O}_{\pi_m(S)})$ is surjective, concluding part (c). $\square$

Definition 1. Let U be an open subset of a complex topological vector space and $P \in U$. A (non-reduced) closed analytic subset Z of U will be called a connected zero-dimensional subscheme of U (or of V) supported by P or with $\{P\}$ as support if it is finite-dimensional and $Z_{red} = \{P\}$. This is equivalent to require the existence of a finite-dimensional affine subspace M of V such that $P \in M$, Z is a closed analytic subspace of $M \cap U$ (or of M) and $Z_{red} = \{P\}$. Let $S \subset U$ a discrete subset. For each $P \in S$ fix a connected zero-dimensional subscheme A_P of U and set $A := \bigcup_{P \in S} A_P$. A is a closed analytic subspace of U and $A_{red} = S$. We will say that A is a zero-dimensional subscheme of U

and that S is the support of A. Let $W(A) \subseteq H^0(S, \mathcal{O}_A)$ be the image of the restriction map $H^0(U, \mathcal{O}_U) \to H^0(S, \mathcal{O}_A)$.

The proof of Theorem 1 gives verbatim the following result.

Theorem 2. *Let $U \subseteq \mathbf{C}^{\mathbf{N}}$ be a connected open subset and S a discrete and infinite subset of U. Let Z be a zero-dimensional subscheme of U with S as support.*

(a) *If $\pi_n(S)$ is finite for all integers $n \geq 1$, then $W(Z)$ has countable dimension and uncountable codimension in $H^0(Z, \mathcal{O}_Z)$.*

(b) *If there is an integer $x \geq 1$ such that $\pi_x(S)$ is infinite, but for all integers $n \geq 1$ the set $\pi_n(S)$ is not discrete in $\pi_n(U)$, then $W(Z)$ has uncountable dimension and uncountable codimension in $H^0(Z, \mathcal{O}_Z)$.*

(c) *Assume that U is a domain of holomorphy in $\mathbf{C}^{\mathbf{N}}$ and that there is an integer $n \geq 1$ such that $\pi_n|S : S \to \pi_n(S)$ is injective and $\pi_n(S)$ is discrete in $\pi_n(U)$. Then $W(Z) = H^0(Z, \mathcal{O}_Z)$.*

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References

[1] A. Hirschowitz, Remarques sur les ouverts d'holomorphie d'un produit dénombrable de droits, *Ann. Inst. Fourier (Grenoble)*, **19**, No. 1 (1969), 219–229.