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ON THE INFINITE-DIMENSIONAL HOLOMORPHIC STRUCTURE ON TOPOLOGICAL BUNDLES WITH FIBERS ISOMORPHIC TO $\mathbf{C}^{(\mathbf{N})}$

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Abstract: Let (X, d) be a complete metrizable space and E a suitable topological vector bundle on X with fibers isomorphic to $\mathbf{C}^{(\mathbf{N})}$. Here we show how to give a continuous and injective map $j : X \to V$, V a complex Banach space, such that $j(X)$ is the zero-locus of a family of holomorphic functions on V and the bundle E has a holomorphic structure with respect to the complex analytic structure on X induced by j.

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1. Holomorphic Structures on Topological Bundles

The aim of this note is to give partial extensions of [1] (which in turn depends heavily from ideas and tools contained in [2], [4] and [5]) to the case of countably many complex topological vector bundles and to the case of vector bundles with infinite-dimensional fibers. Let X be a complete metrizable space and E a complex topological vector bundle with finite rank on X. Let $\mathcal{C}_X$ be the sheaf of all germs of continuous functions on X. By the theorem of Nagata

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and Smirnov any metric space is paracompact. Since X is paracompact, E is a pull-back of the tautological quotient bundle of a suitable Grassmannian ([3], Corollary 5.3 and Theorem 5.5). This implies the existence of positive integers n, m such that E is isomorphic to the Cokernel of a sheaf map $u : \mathcal{C}_X^{\oplus n} \to \mathcal{C}_X^{\oplus m}$. The map u is given by an $n \times m$ matrix of continuous function. We will say that u is a presentation of E. We will say that E is bounded (with respect to the metric d) if it has a presentation in which all entries of the matrix are bounded. In particular if X is compact, then every complex topological vector bundle is compact. First, we will outline a proof of the following result.

Theorem 1. *Let (X, d) be a complete metrizable topological space and $\{F_j\}$ a countable family of complex topological vector bundles on X, all except finitely many ones bounded. Then there is a continuous and injective map j of X into a complex Banach space V such that $j(X)$ is a closed analytic subset of V and all bundles F_j have a holomorphic structure with respect to the complex analytic structure on X induced by j.*

Then we will consider the case of topological vector bundles with infinite-dimensional fibers. We can allow only the following very particular Hausdorff complex topological vector space, the one with countable algebraic dimension, i.e. $\mathbf{C}^{(\mathbf{N})}$ with the so-called finite topoly, i.e. the inductive limit topology with respect to its finite-dimensional linear subspaces equipped with the usual euclidean topology. Let (X, d) be a complete metrizable topological space and E a complex topological vector bundle on X with fibers isomorphic to $\mathbf{C}^{(\mathbf{N})}$. We will say that E has a global presentation if there is countable sets I, J and a continuos linear map $u : \mathcal{C}_X^{(I)} \to \mathcal{C}_X^{(J)}$ such that $\mathrm{Coker}(u) \cong E$. A globally presented bundle E will be said of countable type if we can find such a presentation in which the matrix associated to u has only countably many non-zero entries. This is always the case if the set I is finite. If E has a presentation in which every entry of the matrix associated to u is bounded; this is always the case if X is compact. The proof of Theorem 1 gives verbatim the following result.

Theorem 2. *Let (X, d) be a complete metrizable topological space, $\{F_j\}$ a countable family of complex topological vector bundles on X, all except finitely many ones bounded, and $\{E_i\}$ a countable family of complex vector bundle on X with fibers isomorphic to $\mathbf{C}^{(\mathbf{N})}$. Assume that each E_i has a global presentation of countable type and bounded. Then there is a continuous and injective map j of X into a complex Banach space V such that $j(X)$ is a closed analytic subset of V and all bundles F_j and E_i have a holomorphic structure with respect to the complex analytic structure on X induced by j.*

Lemma 1. *Let (X, d) be a complete metric space and Lip(X) the algebra of all complex valued Lipschitz functions on X. Fix a countable set S, say $S = \{g_i\}_{i \geq 1}$, of continuous complex valued functions such that for $i \gg 0$, say $i \geq m$, the functions g_i's are bounded. Let A be the algebra of all complex valued functions on X generated by Lip$(X) \cup S$. Then there exist a complex Banach space V and a continuous injective map $u : X \to V$ such that $u(X)$ is the zero-locus of a family of holomorphic functions on V and for every $a \in A$ the continuous function $u^{-1*}(a) : u(X) \to \mathbf{C}$ is holomorphic.*

Proof. For $i \geq m$ set $\|g_i\|_\infty := \sup_{x \in X} |g_i(x)|$ and $f_i := g_i / \|g_i\|_\infty$. For $1 \leq i < m$ set $f_i := g_i$. For all $x, y \in X$ set $d''(x, y) := d(x, y) + \sum_{i=1} + \infty |f_i(x) - f_i(y)|/2^i$. It is easy to check that d'' is a complete metric equivalent to d. Use d'' instead of the metric d' used in the proof of [1], Lemma 1. □

Proof of Theorem 1. Just copy the proof of [1], Theorem 1, quoting Lemma 1 above instead of [1], Lemma 1. □

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