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MULTITAPER MULTIVARIATE SPECTRAL ESTIMATORS OF TIME SERIES WITH DISTORTED OBSERVATIONS

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Abstract: The nonparametric estimation of some multitaper spectral measures for a strictly stationary multidimensional time series $\mathbf{X}(\mathbf{t}), \mathbf{t} = 0, \pm 1, ...,$is considered in the case, where only an amplitude modulated process $\mathbf{Y}(\mathbf{t}) = \mathbf{d}(\mathbf{t})\mathbf{X}(\mathbf{t})$ is observed. The stochastic process $\mathbf{d}(\mathbf{t})$ takes the value 1 when $\mathbf{X}(\mathbf{t})$ is observed and $\in$, non-zero constant, when $\mathbf{X}(\mathbf{t})$ is distorted. Statistical properties of the process $\mathbf{Y}(\mathbf{t})$ are obtained. Asymptotic statistical properties of these estimators are investigated.

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1. Introduction

We consider the problem of estimating some spectral measures for a strictly stationary multidimensional time series with some randomly distorted observations via multitapering. The distortion of data means replacing the original data by some multiples of the true values of data. Hinich and Weber [9] derived an estimator for distributed lag models with distorted observations via Hilbert transform method in the frequency domain. Walden [15, 16] has shown that

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multitapering technique gives an estimator with less variability for time series with complete observations.

Several authors have investigated such problems of spectral estimation. We can quote for instance: Brillinger [2] for untapered time series with complete observations and Brillinger [3] for tapered time series with complete observations. Parzen [11]; Bloomfield [1]; Dahlhaus [4]; Yajima & Nishino [17]; Ghazal [6] for untapered time series with some randomly missed observations. Ghazal et al [7] discussed a multidimensional stationary time series with missed observations in the frequency domain via tapering method. Teamah and Bakouch [13, 14] studied some spectral estimators on overlapped and non-overlapped segments via different tapers and discussed the asymptotic expressions of the means and covariances of these estimators for both continuous time and discrete time stationary real-valued processes.

The paper is organized as follows: In Section 2 we obtain some statistical properties of the amplitude modulated (observed) process $\mathbf{Y}(\mathbf{t}) = \mathbf{d}(\mathbf{t})\mathbf{X}(\mathbf{t})$, $\mathbf{t} = 0, \pm1, ...$, where $\mathbf{X}(\mathbf{t})$ is the original process and $\mathbf{d}(\mathbf{t}) = \{1$, if $\mathbf{X}(\mathbf{t})$ is observed; $\in$ (non-zero constant), if $\mathbf{X}(\mathbf{t})$ is distorted$\}$. The multitaper spectral density estimator of $\mathbf{X}(\mathbf{t})$ and its asymptotic properties are investigated in Section 3. Moreover, we conclude the confidence interval of the spectral density of $\mathbf{X}(\mathbf{t})$. The spectral distribution estimator of $\mathbf{X}(\mathbf{t})$ and its statistical properties are studied in Section 4. We finish the paper by some concluding remarks.

2. Model of Distorted Observations

Consider a real-valued strictly stationary time series $\mathbf{X}(\mathbf{t})$, $\mathbf{t} = 0, \pm1, ...$, with components $X_a(t), a = 1, 2, ..., r$, mean 0, autocovariance function $C_{\mathbf{XX}}(\tau) = \{C_{X_aX_b}(\tau) : \tau = 0, \pm1, ...\}_{a,b=1}^{r}$ and spectral density $f_{\mathbf{XX}}(\lambda) = \{f_{X_aX_b}(\lambda) : -\pi \le \lambda \le \pi\}_{a,b=1}^{r}$. To describe irregular observations in $\mathbf{X}(\mathbf{t}) = \{X_a(t) : t = 0, \pm1, ...\}_{a=1}^{r}$ we consider the amplitude modulated stochastic process $\mathbf{d}(\mathbf{t})$ $= \{d_a(t) : t = 0, \pm1, ...\}_{a=1}^{r}$ which is stochastically independent of $\mathbf{X}(\mathbf{t})$:

$$d_a(t) = \left\{ \begin{array}{ll} 1, & \text{if } X_a(t) \text{ is observed;} \\ \in_a, & \text{otherwise} \end{array} \right\}, \tag{1}$$

where $\in_a$ is a constant. Let $d_a(t)$ be independent and identically distributed random variables with $P(d_a(t) = 1) = p_a$, $P(d_a(t) = \in_a) = q_a$, where $p_a + q_a = 1$. Clearly, the $\ell^{\underline{th}}$ moment of $d_a(t)$ is given by

$$E\{d_a^{\ell}(t)\} = p_a, + \in_a^{\ell} q_a, \quad \ell = 1, 2, \dots. \tag{2}$$

We set

$$\gamma_a = p_a + \in_a q_a. \tag{3}$$

The process $d_a(t)$ defined by (1) is giving both the case of randomly missed observations (Dahlhaus [4]) for $\in_a = 0$ and the case of distorted observations for $\in_a \neq 0$. Our interest is the case of distorted observations. Hence,the observed time series can be represented as

$$Y_a(t) = d_a(t)X_a(t), \quad t = 1, 2, ..., N; \quad a = 1, 2, ..., r\,, \tag{4}$$

which is an amplitude modulated series. From this, distortion means $Y_a(t)$ is constructed by replacing the original time series $X_a(t)$ by some multiples of the true values of $X_a(t)$.That is, we cannot observe $X_a(t)$ directly but we are only able to observe $Y_a(t)$. The series in equation (4) was discussed in the case of missed observations by Parzen [11], Bloomfield [1], and Yajima and Nishino [17] for a univariate time series. Also, Dahlhaus [4] and Ghazal [6] discussed this case for multidimensional untapered time series.

2.1. Statistical Properties of $Y_a(t)$

Suppose that the moments of $X_a(t)$, $a = 1, 2, ..., r$, exist and the joint cumulant of $X_{a_1}(t_1 + \tau)$, ..., $X_{a_{n-1}}(t_{n-1} + \tau), X_{a_n}(\tau)$ is given by

$$C_{X_{a_1} \cdots X_{a_n}}(t_1, ..., t_{n-1}) = \mathrm{Cum}\{X_{a_1}(t_1 + \tau), ..., X_{a_{n-1}}(t_{n-1} + \tau),\ X_{a_n}(\tau)\},$$

where $a_1, ..., a_n = 1, 2, ..., r$; $t_1, ..., t_{n-1}, \tau = 0, \pm 1, ...,$ $n = 2, 3, ...$ We put $\mathbf{X}(\mathbf{t})$ such that

$$\Sigma_{t_1,...,t_{n-1}=-\infty}^{\infty} \left| t_s C_{X_{a_1} \cdots X_{a_n}}(t_1, ..., t_{n-1}) \right| < \infty, \tag{5}$$

where $a_1, ..., a_n = 1, 2, ..., r$; $s = 1, 2, ..., n-1$; $n = 2, 3, ...$. If $\mathbf{X}(\mathbf{t})$ satisfies equation (5), we define its cumulant spectral densities by

$$\begin{aligned} &f_{X_{a_1} \cdots X_{a_n}}(\lambda_1, ..., \lambda_{n-1}) \\ &\quad = (2\pi)^{1-n} \Sigma_{t_1,...,t_{n-1}=-\infty}^{\infty} C_{X_{a_1} \cdots X_{a_n}}(t_1, ..., t_{n-1}) \exp(-i \Sigma_{s=1}^{n-1} \lambda_s t_s), \\ &\qquad\qquad -\pi \le \lambda_1, ..., \lambda_{n-1} \le \pi; \quad n = 2, 3, ...; \quad i = \sqrt{-1}, \end{aligned}$$

where $f_{X_{a_1} \cdots X_{a_n}}(\lambda_1, ..., \lambda_{n-1})$ is bounded and uniformly continuous.

Properties of $X_a(t)$ and $d_a(t)$ yield $E\{Y_a(t)\} = 0$. Also, the covariance of $Y_{a_1}(t_1)$ and $Y_{a_2}(t_2)$ is

$$C_{Y_{a_1} Y_{a_2}}(\tau) = \gamma_{a_1} \gamma_{a_2} C_{X_{a_1} X_{a_2}}(\tau), \quad \tau = |t_1 - t_2\ , \tag{6}$$

where γ_a is defined by (3). Moreover, $\text{Var}\{Y_a(t)\} = (p_a + \in_a^2 \ q_a)$ $\text{Var}\{X_a(t)\}$. From Brillinger [3] we get

$$\text{Cum}\{Y_{a_1}(t_1), ..., Y_{a_n}(t_n)\}$$

$$= (-1)^{m-1}(m-1)!\Sigma_\upsilon E[\Pi_{j\in\upsilon_1} d_{a_j}(t_j)X_{a_j}(t_j)]...E[\Pi_{j\in\upsilon_m} d_{a_j}(t_j)X_{a_j}(t_j)],$$

where the summation extends over all partitions $\upsilon = \cup_{b=1}^{m}\upsilon_b$, $m = 1, 2, ..., n$. Consequently

$$C_{Y_{a_1}...Y_{a_n}}(t_1, ..., t_{n-1}) = \Pi_{s=1}^{n}\gamma_{a_s} C_{X_{a_1}...X_{a_n}}(t_1, ..., t_{n-1}), \tag{7}$$

formula (5) implies

$$\Sigma_{t_1,...,t_{n-1}=-\infty}^{\infty} \left| t_s C_{Y_{a_1}...Y_{a_n}}(t_1, ..., t_{n-1}) \right| < \infty. \tag{8}$$

From equations (7) and (8), we deduce that $\mathbf{Y(t)} = \{Y_a(t) : t = 0, \pm1, ...\}_{a=1}^{r}$ is a strictly stationary r-vector valued time series and all of whose moments exist. Also, we can define the n-th spectrum of $Y_{a_1}, ..., Y_{a_n}$ by

$$f_{Y_{a_1}...Y_{a_n}}(\lambda_1, ..., \lambda_{n-1}) = (2\pi)^{1-n}\Pi_{s=1}^{n}\gamma_{a_s}\Sigma_{t_1,...,t_{n-1}=-\infty}^{\infty}$$

$$C_{X_{a_1}...X_{a_n}}(t_1, ..., t_{n-1}) \exp(-i\Sigma_{s=1}^{n-1}\lambda_s t_s), \tag{9}$$

which is bounded and uniformly continuous. If $p_{a_s} \to 1$, $a_s = 1, 2, ..., r$, then equation (9) gives $f_{X_{a_1}...X_{a_n}}(\lambda_1, ..., \lambda_{n-1})$, this case was considered by Brillinger [2]. Also,

$$f_{Y_{a_1}...Y_{a_n}}(\lambda_1, ..., \lambda_{n-1}) = \Pi_{s=1}^{n}\gamma_{a_s} f_{X_{a_1}...X_{a_n}}(\lambda_1, ..., \lambda_{n-1}).$$

2.2. Cumulants of the Tapered Fourier Transform of $Y_a(t)$

Let $h_a^{(N;k)}(t) = h_a^{(k)}(\frac{t}{N})$, $k = 1, 2, ..., K$; $a = 1, 2, ..., r$, be bounded functions of bounded variations and equal zero outside the interval $[1, N]$. The function $h_a^{(N;k)}(t)$ is called a taper or data window. K is the number of tapers that will be used to form the multitaper spectrum estimator. Also, we put

$$H_{a_1...a_n}^{(N;k)}(\lambda) = \Sigma_{\tau=1}^{N}[\Pi_{s=1}^{n} h_{a_s}^{(N;k_s)}(\tau)] \exp(-i\lambda\tau)\,, \tag{10}$$

where k is a vector: $k = (k_1, ..., k_n)$, $k_s = 1, 2, ..., K$; $s = 1, 2, ..., n$; $-\pi \le \lambda \le \pi$. At $\lambda = 0$, we may write (Brillinger [3])

$$H_{a_1...a_n}^{(N;k)}(0) = N H_{a_1...a_n}^{(k)}(0)\,, \tag{11}$$

where $H^{(k)}_{a_1 \ldots a_n}(0) = \int_0^1 [\Pi_{s=1}^n h^{(k_s)}_{a_s}(u)]du$. From the properties of $h^{(N;k)}_a(t)$, $a = 1, 2, ..., r$, we can get

$$\left| \Sigma_{\tau=1}^N [\Pi_{s=1}^{n-1} h^{(N;k_s)}_{a_s}(\tau + u_s)] \; h^{(N;k_n)}_{a_n}(\tau) \exp(-i\lambda\tau) - H^{(N;k)}_{a_1 \ldots a_n}(\lambda) \right|$$

$$\leq A[\Sigma_{s=1}^{n-1} |u_s|], \quad A \text{ is a constant}, \tag{12}$$

for $a_1, ..., a_n = 1, 2, ..., r$; $k_s = 1, 2, ..., K$. Also,

$$\left| H^{(N;k)}_{a_1 \ldots a_n}(\lambda) \right| \leq B / |\sin(\lambda/2)|\, , \quad \lambda \neq 0; \tag{13}$$

B is a constant.

Given a stretch of data $X_a(t)$, $t = 1, 2, ..., N$, of the series $\mathbf{X(t)}$ with some randomly distorted observations, then the tapered Fourier transform of $Y_a(t)$ is given by

$$J^{(N;k)}_{Y_a}(\lambda) = \Sigma_{t=1}^N h^{(N;k)}_a(t) Y_a(t) \exp(-i\lambda t), \quad -\pi \leq \lambda \leq \pi, \tag{14}$$

where $Y_a(t)$ is defined by (4). Since $E\{Y_a(t)\} = 0$, then the mean value of $J^{(N;k)}_{Y_a}(\lambda)$ is given by

$$E\{J^{(N;k)}_{Y_a}(\lambda)\} = 0, \quad k = 1, 2, ..., K. \tag{15}$$

The following lemma gives the n-th cumulant of $J^{(N;k)}_{Y_a}(\lambda)$:

Lemma 2.1. *From Properties of* $Y_a(t)$, $h^{(N;k)}_a(t)$ and equations (8), (12) and (13) we get

$$\text{Cum}\{J^{(N;k_1)}_{Y_{a_1}}(\lambda_1), ..., J^{(N;k_n)}_{Y_{a_n}}(\lambda_n)\}$$

$$= (2\pi)^{n-1} \Pi_{s=1}^n \gamma_{a_s} H^{(N;k)}_{a_1 \cdots a_n}(\Sigma_{s=1}^n \lambda_s) f_{X_{a_1} \ldots X_{a_n}}(\lambda_1, ..., \lambda_{n-1}) + O(1)\, , \tag{16}$$

where $H^{(N;k)}_{a_1 \cdots a_n}(\cdot)$ *is defined by (10),* $a_s = 1, 2, ..., r$; $s = 1, 2, ..., n$, *and the error term* $O(\cdot)$ *is uniform in* $\lambda_1, ..., \lambda_n$ *as* $N \to \infty$.

Setting $\in_a = 0$ and $K = 1$ in formula (16) gives $\text{Cum}\{J^{(N;k_1)}_{Y_{a_1}}(\lambda_1), ..., J^{(N;k_n)}_{Y_{a_n}}(\lambda_n)\}$ for the case of missed observations (Ghazal et al [7]). If $p_{a_s} \to 1$, the case of time series with complete observations, then equation (16) gives $\text{Cum}\{J^{(N;k_1)}_{X_{a_1}}(\lambda_1), ..., J^{(N;k_n)}_{X_{a_n}}(\lambda_n)\}$, $k_s = 1, 2, ..., K$; $s = 1, 2, ..., n$, this cumulant was obtained by Brillinger [3] for $K = 1$.

2.3. Asymptotic Distribution of $J_{Y_a}^{(N;k)}(\lambda)$

The following theorem will give the asymptotic distribution of $J_{Y_a}^{(N;k)}(\lambda)$.

Theorem 2.1. *$J_{Y_a}^{(N;k)}(\lambda)$, $a = 1, 2, ..., r$; $-\pi \leq \lambda \leq \pi$, are asymptotically independent at the different frequencies and distributed asymptotically as*

$$J_{Y_a}^{(N;k)}(\lambda) \sim \left\{ \begin{array}{ll} N_r^C(0, 2\pi\gamma_a\gamma_b H_{ab}^{(N;k)}(0) f_{X_aX_b}(\lambda)), & \lambda \neq 0, \ \pm\pi; \\ N_r(0, 2\pi\gamma_a\gamma_b H_{ab}^{(N;k)}(0) f_{X_aX_b}(\lambda)), & \lambda = 0, \ \pm\pi \end{array} \right\},$$

where $N_r^C(0, 2\pi\gamma_a\gamma_b H_{ab}^{(N;k)}(0) f_{X_aX_b}(\lambda))$, $a, b = 1, 2, ..., r$; $k = 1, 2, ..., K$; $\gamma_a = p_a + \in_a q_a$, denotes the r-dimensional complex normal distribution with mean zero and covariance matrix $2\pi\gamma_a\gamma_b H_{ab}^{(N;k)}(0) f_{X_aX_b}(\lambda)$ and $N_r(0, 2\pi\gamma_a\gamma_b \, H_{ab}^{(N;k)}(0) f_{X_aX_b}(\lambda))$ denotes the real equivalent.

Proof. We have investigated that $E\{J_{Y_a}^{(N;k)}(\lambda)\} = 0$ by formula (15). Making use of equations (11), (13) and (16), we get

$$\lim_{N\to\infty} N^{-1} \operatorname{Cov}\{J_{Y_a}^{(N;k_1)}(\pm\lambda_1), J_{Y_b}^{(N;k_2)}(\pm\lambda_2)\}$$

$$= \left\{ \begin{array}{ll} 2\pi\gamma_a\gamma_b H_{ab}^{(k)}(0) f_{X_aX_b}(\pm\lambda_1), & \text{if } \pm\lambda_1 = \pm\lambda_2; \\ 0, & \text{if } \pm\lambda_1 \neq \pm\lambda_2. \end{array} \right\}.$$

That is, $J_{Y_a}^{(N;k)}(\lambda)$, $a = 1, 2, ..., r$, are asymptotically independent at the different frequencies. Moreover, we can show that

$$\lim_{N\to\infty} N^{-n/2} \operatorname{Cum}\{J_{Y_{a_1}}^{(N;k_1)}(\pm\lambda_1), ..., J_{Y_{a_n}}^{(N;k_n)}(\pm\lambda_n)\} = 0,$$

$n \geq 3$, for $\Sigma_{s=1}^n(\pm\lambda_s) = 0$ and $\Sigma_{s=1}^n(\pm\lambda_s) \neq 0$. We observe that the cumulants of the variates and the conjugates of these variates tend to the cumulants of the normal distribution, which completes the proof. □

When $p_{a_s} \to 1$ in Theorem 2.1, then we get the distribution of $J_{X_a}^{(N;k)}(\lambda)$ which was discussed by Ghazal et al [8] for $K = 1$. Also, putting $\in_a = 0$ and $K = 1$ give the distribution of $J_{Y_a}^{(N;k)}(\lambda)$ in the case of missed observations (Ghazal et al [7]).

3. Statistical Properties of the Multitaper Spectrum Estimator of $X_a(t)$

We will give the spectral density estimator of $X_a(t)$ by

$$f_{X_aX_b}^{(N)}(\lambda) = \frac{1}{\gamma_a\gamma_b} f_{Y_aY_b}^{(N)}(\lambda), \quad a, b = 1, 2, ..., r, \tag{17}$$

where $f_{Y_aY_b}^{(N)}(\lambda) = \frac{1}{K}\Sigma_{k=1}^{K} I_{Y_aY_b}^{(N;k)}(\lambda)$ is the multitaper spectrum estimator (Walden [15, 16]) of $Y_a(t)$ and the asymptotic distribution of $J_{Y_a}^{(N;k)}(\lambda)$ suggests the statistic

$$I_{Y_aY_b}^{(N;k)}(\lambda) = (2\pi H_{ab}^{(N;k)}(0))^{-1} J_{Y_a}^{(N;k)}(\lambda) J_{Y_b}^{*(N;k)}(\lambda), \quad -\pi \le \lambda \le \pi, \tag{18}$$

$J_{Y_a}^{(N;k)}(\lambda)$ is defined by (14). The asterisk denotes the complex conjugate.

3.1. Expected Value of $f_{X_aX_b}^{(N)}(\lambda)$

Making use of equations (16), (17) and (18) we get

$$E\{f_{X_aX_b}^{(N)}(\lambda)\} = f_{X_aX_b}(\lambda) + O(K^{-1}N^{-1}), \quad a, b = 1, 2, ..., r. \tag{19}$$

It is clear that

$$E\{f_{X_aX_b}^{(N)}(\lambda)\} \to f_{X_aX_b}(\lambda),$$

as $N \to \infty$ or $K \to \infty$, that is, $f_{X_aX_b}^{(N)}(\lambda)$ is an asymptotically unbiased estimator of $f_{X_aX_b}(\lambda)$.

3.2. Covariance of $f_{X_aX_b}^{(N)}(\lambda)$

From Brillinger [3] we obtain

$$\begin{aligned}
&\mathrm{Cov}\,\{J_{Y_{a_1}}^{(N;k_1)}(\lambda_1) J_{Y_{b_1}}^{(N;k_1)}(-\lambda_1), J_{Y_{a_2}}^{(N;k_2)}(\lambda_2) J_{Y_{b_2}}^{(N;k_2)}(-\lambda_2)\} \\
&= \mathrm{Cum}\{J_{Y_{a_1}}^{(N;k_1)}(\lambda_1), J_{Y_{b_1}}^{(N;k_1)}(-\lambda_1), J_{Y_{a_2}}^{(N;k_2)}(-\lambda_2), J_{Y_{b_2}}^{(N;k_2)}(\lambda_2)\} \\
&+ \mathrm{Cum}\{J_{Y_{a_1}}^{(N;k_1)}(\lambda_1), J_{Y_{a_2}}^{(N;k_2)}(-\lambda_2)\}\mathrm{Cum}\{J_{Y_{b_1}}^{(N;k_1)}(-\lambda_1), J_{Y_{b_2}}^{(N;k_2)}(\lambda_2)\} \\
&+ \mathrm{Cum}\{J_{Y_{a_1}}^{(N;k_1)}(\lambda_1), J_{Y_{b_2}}^{(N;k_2)}(\lambda_2)\}\mathrm{Cum}\{J_{Y_{b_1}}^{(N;k_1)}(-\lambda_1), J_{Y_{a_2}}^{(N;k_2)}(-\lambda_2)\}.
\end{aligned} \tag{20}$$

Applying Lemma 2.1 to (20) and using formula (11), we get

$$\begin{aligned}\mathrm{Cov}\,\{f^{(N)}_{X_{a_1}X_{b_1}}(\lambda_1), f^{(N)}_{X_{a_2}X_{b_2}}(\lambda_2)\} &= K^{-2}\\ \times \Sigma_{k_1=1}^{K}\Sigma_{k_2=1}^{K}(H^{(N;k_1)}_{a_1b_1}(0)H^{(N;k_2)}_{a_2b_2}(0))^{-1}\{H^{(N;k)}_{a_1a_2}(\lambda_1-\lambda_2)\\ H^{*(N;k)}_{b_1b_2}(\lambda_1-\lambda_2)\times f_{X_{a_1}X_{a_2}}(\lambda_1)f_{X_{b_1}X_{b_2}}(-\lambda_1)+H^{(N;k)}_{a_1b_2}(\lambda_1+\lambda_2)\\ \times H^{*(N;k)}_{b_1a_2}(\lambda_1+\lambda_2)f_{X_{a_1}X_{b_2}}(\lambda_1)f_{X_{b_1}X_{a_2}}(-\lambda_1)\}\\ + N^{-2}K^{-2}\,\Sigma_{k_1=1}^{K}\Sigma_{k_2=1}^{K}R^{(N;k)}(\lambda_1,\lambda_2)+O(N^{-1}K^{-2}),\quad (21)\end{aligned}$$

where there is a non-negative constant ϕ such that

$$\left|R^{(N;k)}(\lambda_1,\lambda_2)\right| \le \phi\{\left|H^{(N;k)}_{a_1a_2}(\lambda_1-\lambda_2)\right| + \left|H^{*(N;k)}_{b_1b_2}(\lambda_1-\lambda_2)\right|$$

$$+\left|H^{(N;k)}_{a_1b_2}(\lambda_1+\lambda_2)\right| + \left|H^{*(N;k)}_{b_1a_2}(\lambda_1+\lambda_2)\right|\},\quad k=(k_1,k_2).$$

For $\lambda_1 \pm \lambda_2 = 0$, equation (21) has the form

$$\begin{aligned}&\mathrm{Cov}\,\{f^{(N)}_{X_{a_1}X_{b_1}}(\lambda_1), f^{(N)}_{X_{a_2}X_{b_2}}(\lambda_2)\}\\ &= K^{-2}\,\Sigma_{k_1=1}^{K}\Sigma_{k_2=1}^{K}(H^{(k_1)}_{a_1b_1}(0)H^{(k_2)}_{a_2b_2}(0))^{-1}\{H^{(k)}_{a_1a_2}(0)H^{(k)}_{b_1b_2}(0)f_{X_{a_1}X_{a_2}}(\lambda_1)\\ &\times f_{X_{b_1}X_{b_2}}(-\lambda_1)+H^{(k)}_{a_1b_2}(0)H^{(k)}_{b_1a_2}(0)f_{X_{a_1}X_{b_2}}(\lambda_1)f_{X_{b_1}X_{a_2}}(-\lambda_1)\}\\ &+O(N^{-1}K^{-2})\,. \quad (22)\end{aligned}$$

Also, from equations (11) and (21) we obtain

$$\lim_{N\to\infty}\mathrm{Cov}\,\{f^{(N)}_{X_{a_1}X_{b_1}}(\lambda_1), f^{(N)}_{X_{a_2}X_{b_2}}(\lambda_2)\} = 0\,,$$

for $\lambda_1 \pm \lambda_2 \neq 0$, that is, $f^{(N)}_{X_aX_b}(\cdot)$, a, $b = 1, 2, ..., r$, are asymptotically independent. From equation (22) we have

$$\begin{aligned}\mathrm{Var}\{f^{(N)}_{X_aX_b}(\lambda)\} &\approx K^{-2}\Sigma_{k_1=1}^{K}\Sigma_{k_2=1}^{K}(H^{(k_1)}_{ab}(0)H^{(k_2)}_{ab}(0))^{-1}\\ &\times\{H^{(k)}_{aa}(0)H^{(k)}_{bb}(0)f_{X_aX_a}(\lambda)f_{X_bX_b}(\lambda)+H^{(k)}_{ab}(0)H^{(k)}_{ba}(0)f^2_{X_aX_b}(\lambda)\}\quad (23)\end{aligned}$$

as $N \to \infty$.

Formula (23) shows that $\mathrm{Var}\{f^{(N)}_{X_aX_b}(\lambda)\} \to 0$ as $K \to \infty$, hence the multitaper spectrum estimator $f^{(N)}_{X_aX_b}(\lambda)$ is consistent.

3.3. Asymptotic Distribution of $f^{(N)}_{\mathbf{XX}}(\lambda)$

Making use of equation (18), we deduce that

$$I^{(N;k)}_{\mathbf{YY}}(\lambda) = (2\pi H^{(N;k)}(0))^{-1} \left| J^{(N;k)}_{\mathbf{Y}}(\lambda) \right|^2$$

$$= (2\pi H^{(N;k)}(0))^{-1} \{[\mathrm{Re}\, J^{(N;k)}_{\mathbf{Y}}(\lambda)]^2 + [\mathrm{Im}\, J^{(N;k)}_{\mathbf{Y}}(\lambda)]^2\},$$

$$-\pi \leq \lambda \leq \pi, \tag{24}$$

where $\mathrm{Re}\, J^{(N;k)}_{\mathbf{Y}}(\lambda) = [J^{(N;k)}_{\mathbf{Y}}(\lambda) + J^{*(N;k)}_{\mathbf{Y}}(\lambda)]/2$, $\mathrm{Im}\, J^{(N;k)}_{\mathbf{Y}}(\lambda) = [J^{(N;k)}_{\mathbf{Y}}(\lambda) - J^{*(N;k)}_{\mathbf{Y}}(\lambda)]/2i$ and $H^{(N;k)}(0) = \Sigma^{N}_{\tau=1}(h^{(N;k)}(\tau))^2$, $k = 1, 2, ..., K$. Theorem 2.1 implies

$$[\mathrm{Re}\, J^{(N;k)}_{\mathbf{Y}}(\lambda)]^2/\pi(p+\in q)^2 H^{(N;k)}(0) f_{\mathbf{XX}}(\lambda) \overset{d}{=} \chi^2_1\,, \tag{25}$$

$$[\mathrm{Im}\, J^{(N;k)}_{\mathbf{Y}}(\lambda)]^2/\pi(p+\in q)^2 H^{(N;k)}(0) f_{\mathbf{XX}}(\lambda) \overset{d}{=} \chi^2_1,$$

$\lambda \neq 0, \pm\pi$, where the ‘d’ above the equal sign means ‘is distributed as’, $\in$ is a non-zero constant and $p + q = 1$. The symbol $\chi^2_{\upsilon_0}$ denotes the chi-squared distribution with υ_0 degrees of freedom. Using equations (17), (24) and (25) we get $2Kf^{(N)}_{\mathbf{XX}}(\lambda)/f_{\mathbf{XX}}(\lambda) \overset{d}{=} \chi^2_{2K}$ for $\lambda \neq 0, \pm\pi$. Also, we can show that $Kf^{(N)}_{\mathbf{XX}}(\lambda)/f_{\mathbf{XX}}(\lambda) \overset{d}{=} \chi^2_K$ for $\lambda = 0, \pm\pi$. Hence,

$$f^{(N)}_{\mathbf{XX}}(\lambda) \overset{d}{=} [f_{\mathbf{XX}}(\lambda)/\upsilon_0]\chi^2_{\upsilon_0}\,, \tag{26}$$

where

$$\upsilon_0 = \left\{ \begin{array}{lll} 2K, & \text{for} & \lambda \neq 0, \ \pm\pi; \\ K, & \text{for} & \lambda = 0, \ \pm\pi. \end{array} \right\}. \tag{27}$$

This implies that $\mathrm{Var}\{f^{(N)}_{\mathbf{XX}}(\lambda)\} = 2f^2_{\mathbf{XX}}(\lambda)/\upsilon_0$.

Making use of equation (26), the n-th moment, about zero, of $f^{(N)}_{\mathbf{XX}}(\lambda)$ is

$$\acute{\mu}_n = E\{[f^{(N)}_{\mathbf{XX}}(\lambda)]^n\} = (2f_{\mathbf{XX}}(\lambda)/\upsilon_0)^n \Gamma(\upsilon_0/2 + n)/\Gamma(\upsilon_0/2), \tag{28}$$

where $\Gamma(\cdot)$ is the Gamma function and $n = 1, 2, ...; -\pi \leq \lambda \leq \pi$.

3.4. Confidence Interval of $f_{\mathbf{XX}}(\lambda)$

The asymptotic $100(1-\alpha)\%$ confidence interval for $f_{\mathbf{XX}}(\lambda)$ is obtained as follows:

$$\chi^2_{\upsilon_0,\alpha/2} \leq \chi^2_{\upsilon_0} \leq \chi^2_{\upsilon_0,1-\alpha/2},$$

where $\chi^2_{\upsilon_0,1-\alpha/2}$ and $\chi^2_{\upsilon_0,\alpha/2}$ can be determined by the tables of the chi-squared distribution (Koopmans [10]). Thus, from equation (26) we get

$$\upsilon_0 f^{(N)}_{\mathbf{XX}}(\lambda)/\chi^2_{\upsilon_0,1-\alpha/2} \leq f_{\mathbf{XX}}(\lambda) \leq \upsilon_0 f^{(N)}_{\mathbf{XX}}(\lambda)/\chi^2_{\upsilon_0,\alpha/2},$$

where υ_0 is defined by (27).

3.5. Coefficients of Skewness and Kurtosis of the Distribution of $f^{(N)}_{\mathbf{XX}}(\lambda)$

The coefficient of skewness of the distribution (Freund [5]) of $f^{(N)}_{\mathbf{XX}}(\lambda)$ is measured by $\alpha_3 = \mu_3/\sigma^3$,where $\mu_3 = \acute{\mu}_3 - 3\mu\acute{\mu}_2 + 2\mu^3$, $\mu = \acute{\mu}_1$, and $\sigma = [\acute{\mu}_2 - \mu^2]^{1/2}$. From equation (28) we get $\alpha_3 = \sqrt{8/\upsilon_0}$.

The coefficient of kurtosis of the distribution of $f^{(N)}_{\mathbf{XX}}(\lambda)$ is $\alpha_4 = \mu_4/\sigma^4$, where $\mu_4 = \acute{\mu}_4 - 4\mu\acute{\mu}_3 + 6\mu^2\acute{\mu}_2 - 3\mu^4$. Therefore, $\alpha_4 = 3 + 12/\upsilon_0$.

Clearly, $\alpha_3 \to 0$ and $\alpha_4 \to 3$ as $K \to \infty$, which are the coefficients of skewness and kurtosis of the normal distribution.

4. Spectral Distribution Estimator of $X_a(t)$ and its Statistical Properties

Assume that the strictly stationary time series $\mathbf{X}(\mathbf{t})$, $\mathbf{t} = 0, \pm 1, ...$, has the spectral density matrix $f_{\mathbf{XX}}(\lambda)$, then its spectral distribution matrix is given by $F_{\mathbf{XX}}(\lambda) = \int_0^\lambda f_{\mathbf{XX}}(\alpha)d\alpha$, $0 \leq \lambda \leq \pi$. If the time series $\mathbf{X}(\mathbf{t})$ has some randomly distorted observations, then we can estimate the components of $F_{\mathbf{XX}}(\lambda) = \{F_{X_aX_b}(\lambda) = \int_0^\lambda f_{X_aX_b}(\alpha)d\alpha\}^r_{a,b=1}$ by $F^{(N)}_{X_aX_b}(\lambda) = \int_0^\lambda f^{(N)}_{X_aX_b}(\alpha)d\alpha$, $0 \leq \lambda \leq \pi$, where $f^{(N)}_{X_aX_b}(\cdot)$ is given by(17). One can prove the following lemma from relation of the Dirac delta function to Fourier transforms (Stoica and Moses [12]).

Lemma 4.1. *Under the properties of* $h_a^{(N;k)}(t)$, *we have*

(a) $\int_{-\pi}^{\pi} H^{(N;k)}_{a_1a_2}(\alpha_1 \pm \alpha_2) H^{*(N;k)}_{b_1b_2}(\alpha_1 \pm \alpha_2) d\alpha_2 = 2\pi\ H^{(N;k)}_{a_1a_2b_1b_2}(0)$;

(b) $\int_{-\pi}^{\pi} H^{(N;k)}_{a_1a_2}(\alpha_1 \pm \alpha_2) d\alpha_2 = 2\pi\ h^{(k_1)}_{a_1}(0)\ h^{(k_2)}_{a_2}(0)$,

where $a_s, b_s = 1, 2, ..., r$; $k_s = 1, 2, ..., K$; $s = 1, 2$.

4.1. Expected Value of $\boldsymbol{F^{(N)}_{X_a X_b}(\lambda)}$

Equation (19) implies $E\{F^{(N)}_{X_a X_b}(\lambda)\} = F_{X_a X_b}(\lambda) + O(K^{-1}N^{-1})$. Hence,

$$E\{F^{(N)}_{X_a X_b}(\lambda)\} \to F_{X_a X_b}(\lambda)\,, \tag{29}$$

as $N \to \infty$ or $K \to \infty$.

4.2. Covariance of $\boldsymbol{F^{(N)}_{X_a X_b}(\lambda)}$

Making use of Lemma 4.1 and formula (21), we get

$$\begin{aligned} &\operatorname{Cov}\{F^{(N)}_{X_{a_1} X_{b_1}}(\lambda_1), F^{(N)}_{X_{a_2} X_{b_2}}(\lambda_2)\} \\ &= 2\pi(NK^2)^{-1}\Sigma_{k_1=1}^{K}\Sigma_{k_2=1}^{K}(H^{(k_1)}_{a_1 b_1}(0)H^{(k_2)}_{a_2 b_2}(0))^{-1}H^{(k)}_{a_1 a_2 b_1 b_2}(0) \\ &\times \int_0^{\lambda_1}\{f_{X_{a_1} X_{a_2}}(\alpha_1) f_{X_{b_1} X_{b_2}}(-\alpha_1) + f_{X_{a_1} X_{b_2}}(\alpha_1) f_{X_{b_1} X_{a_2}}(-\alpha_1)\}d\alpha_1 \\ &+O(N^{-1}K^{-2}), \quad k = (k_1, k_2). \end{aligned} \tag{30}$$

From the boundness of $f_{X_a X_b}(\cdot)$, equation (30) yields

$$\operatorname{Var}\{F^{(N)}_{X_a X_b}(\lambda)\} \to 0\,, \tag{31}$$

as $N \to \infty$ or $K \to \infty$.

Using equations (29) and (31), we conclude that $F^{(N)}_{X_a X_b}(\lambda)$ is asymptotically unbiased and consistent estimator of the spectral distribution $F_{X_a X_b}(\lambda)$.

5. Concluding Remarks

In this paper we have discussed an approach to analyze a multidimensional strictly stationary time series with some randomly distorted observations in the frequency domain via multitapering technique. We find that as $p_{a_s} \to 1$, $a_s = 1, 2, ..., r$; $s = 1, 2, ..., n$, we get the previous results in the spectra analysis of strictly stationary time series as mentioned before, where all observations are available. Also, for $\in_a = 0$ and $K = 1$ we get the results that obtained before in the case of time series with randomly missed observations. The multitaper spectrum estimator $f^{(N)}_{\mathbf{XX}}(\lambda)$ and spectral distribution estimator $F^{(N)}_{\mathbf{XX}}(\lambda)$ are consistent estimators of the spectral density $f_{\mathbf{XX}}(\lambda)$ and spectral distribution $F_{\mathbf{XX}}(\lambda)$,respectively, as $K \to \infty$. The coefficients of skewness and kurtosis of the distribution of the estimator $f^{(N)}_{\mathbf{XX}}(\lambda)$ tend to zero as $K \to \infty$.

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