International Journal of Pure and Applied Mathematics
Volume 14 No. 1 2004, 59-63

ALGEBRAIC COMPACTNESS OF $\prod M_\alpha / \bigoplus M_\alpha$

Radoslav M. Dimitrić
Mathematics/GACD
Texas A&M University
P.O. Box 1675, Galveston, TX 77553, USA
e-mail: dimitric@tamug.edu

Abstract: In this note, we are working within the category R**Mod** of (unitary, left) R-modules, where R is a **countable** ring. It is well known (see e.g. Kiełpiński and Simson [5], Theorem 2.2) that the latter condition implies that the (left) pure global dimension of R is at most 1. Given an infinite index set A, and a family $M_\alpha \in R\mathbf{Mod}$, $\alpha \in A$ we are concerned with the conditions as to when the R-module

$$\prod / \coprod = \prod_{\alpha \in A} M_\alpha / \bigoplus_{\alpha \in A} M_\alpha$$

is or is not algebraically compact. There are a number of special results regarding this question and this note is meant to be an addition to and a generalization of the set of these results. Whether the module in the title is algebraically compact or not depends on the numbers of algebraically compact and non-compact modules among the components M_α.

AMS Subject Classification: 16D10, 16D80, 13C13
Key Words: algebraically compact, product mod direct sum of modules, reduced product of modules, pure global dimension 1, countable rings

Received: April 5, 2004

*

Given an (infinite) cardinal κ, an R-module M is *κ-compact*, if, every system of $\leq \kappa$ linear equations over M (with unknowns x_j and almost all $r_{ij} = 0$):

$$\sum_{j\in J} r_{ij}x_j = m_i \in M, \quad i \in I, \quad r_{ij} \in R, \quad |I|, |J| \leq \kappa\,, \tag{1}$$

has a solution in M whenever all its finite subsystems have solutions (in M). A module is (*algebraically*) *compact* if it is κ-compact, for every cardinal κ. It is well-known that if $M \in R\mathbf{Mod}$ is κ-compact, for some $\kappa \geq |R|$, then M is algebraically compact. Algebraic compactness of M is equivalent to pure injectivity and this in turn is equivalent to $\mathrm{Pext}^1_R(X, M) = 0$, for every $X \in R\mathbf{Mod}$.

Recall that $\prod / \coprod$ is a special case of a more general construction of the reduced product $\prod M_\alpha/\mathcal{F}$, where $\mathcal{F}$ is the cofinite filter on A. Given a subset $B \subseteq A$, then $\mathcal{F} \cap B$ and $\mathcal{F} \cap (A \setminus B)$ are cofinite filters on B and on $A \setminus B$ respectively, if $\mathcal{F}$ is the cofinite filter on A. One can now easily prove the following isomorphism (alternatively use Theorem 1.10 in [2]):

$$\prod_{\alpha\in A} M_\alpha / \bigoplus_{\alpha\in A} M_\alpha \cong \prod_{\alpha\in B} M_\alpha / \bigoplus_{\alpha\in B} M_\alpha \times \prod_{\alpha\in A\setminus B} M_\alpha / \bigoplus_{\alpha\in A\setminus B} M_\alpha. \tag{2}$$

The proof of the following result is straightforward, since it uses a powerful classical result of Mycielski.

Proposition 1. *For every countable index set B,*

$$\prod / \coprod = \prod_{\alpha\in B} M_\alpha / \bigoplus_{\alpha\in B} M_\alpha$$

is an algebraically compact R-module.

Proof. Since B is countable, there is a countable family of cofinite subsets of B with empty intersection. By a classical result of Mycielski [6], Theorem 1), $\prod / \coprod$ is $\aleph_0$-compact. This is equivalent to its algebraic compactness, since the rings we consider here are countable. □

Note that this result need not hold true, if R is uncountable. For instance, if K is a field and $R = K[[X, Y]]$ is the two-variable power series algebra, then $R^{\mathbb{N}}/R^{(\mathbb{N})}$ is not algebraically compact (see [4], Theorem 8.42).

Lemma 2. *Assume that pure global dimension of R is ≤ 1. If $\mathbf{E}: \quad 0 \longrightarrow A \stackrel{*}{\longrightarrow} B \longrightarrow C \longrightarrow 0$ is a pure exact sequence and B is pure injective, then C is likewise pure injective (algebraically compact).*

Proof. Given an arbitrary $X \in R\mathbf{Mod}$, the segment of the $\mathrm{Pext}^1_R(X, \mathbf{E})$ exact sequence we are interested in is as follows: $\ldots \longrightarrow \mathrm{Pext}^1_R(X, B) \longrightarrow \mathrm{Pext}^1_R(X, C) \longrightarrow \mathrm{Pext}^2_R(X, A) \longrightarrow \ldots$ Since *puregld* $R \leq 1$ we have $\mathrm{Pext}^2_R(X, A) = 0$. Since B is pure injective, we have $\mathrm{Pext}^1_R(X, B) = 0$. These facts now force $\mathrm{Pext}^1_R(X, C) = 0$, i.e. C is pure injective. ☐

Proposition 3. *Let puregld $R \leq 1$ and let A be an arbitrary (infinite) index set; if every M_α, $\alpha \in A$ is algebraically compact, then $\prod_{\alpha\in A} M_\alpha / \bigoplus_{\alpha\in A} M_\alpha$ is algebraically compact.*

Proof. It is well known that $\coprod = \bigoplus_{\alpha\in A} M_\alpha$ is a pure submodule of $\prod = \prod_{\alpha\in A} M_\alpha$ and that $\prod$ is algebraically compact iff all the components M_α are algebraically compact. Appeal to Lemma 2 completes the proof. ☐

Theorem 4. *Given any index set A, let $B \subseteq A$ be (at most) a countable set and $\forall\alpha \in B$, M_α is not algebraically compact, while $\forall\alpha \in A \setminus B$, M_α is algebraically compact. Then*

$$\prod / \coprod = \prod_{\alpha\in A} M_\alpha / \bigoplus_{\alpha\in A} M_\alpha$$

is algebraically compact.

Proof. By Proposition 1, the R-module $\prod_{\alpha\in B} M_\alpha / \bigoplus_{\alpha\in B} M_\alpha$ is algebraically compact. By Proposition 3, $\prod_{\alpha\in A\setminus B} M_\alpha / \bigoplus_{\alpha\in A\setminus B} M_\alpha$ is likewise algebraically compact. Now use isomorphism (2) to conclude that $\prod / \coprod$ is algebraically compact. ☐

Our main concern is the converse of Theorem 4: If $\prod / \coprod$ is algebraically compact, can we conclude that at most countably many M_α's are not algebraically compact?

Every linear system (1) has a short-hand representation $\mu \cdot \mathbf{x} = \mathbf{m}$, where $\mu = (r_{ij})_{i\in I, j\in J}$ is the corresponding row-finite matrix (call it the system matrix) and $\mathbf{x} = (x_j)_{j\in J}$, $\mathbf{m} = (m_i)_{i\in I}$ are the corresponding column vectors. The rows of matrix μ (which are the left hand sides of equations (1)) may be viewed as elements of the free R-module $\oplus_{j\in J} R x_j$. The cardinality of these R-modules is $|R| 2^{|J|}$. Thus the cardinality of the set of different matrices μ representing (left-hand-sides) of (1) is at most $(|R|2^{|J|})^{|I|} = |R|^{|I|} 2^{|J||I|}$. For purposes of algebraic compactness, it suffices to consider only $|I| = |J| = \max(|R|, \aleph_0)$, thus the latter cardinality is at most $\max(2^{|R|}, 2^{\aleph_0})$; for countable rings this bound is $2^{\aleph_0}$. This is an important fact that we use in the proof of the next result.

Proposition 5. *Let $|A| > \max(2^{|R|}, 2^{\aleph_0})$ and $\forall\alpha \in A$, M_α is not algebraically compact. Then $\prod / \coprod$ is not algebraically compact.*

Proof. For every M_α, $\alpha \in A$, there is a system of equations of type (1)

$$S_\alpha: \sum_{j\in J} r_{ij}^\alpha x_j^\alpha = m_i^\alpha \in M_\alpha,\ i \in I,\ r_{ij} \in R,\ |I| = |J| = \max(|R|, \aleph_0)\,, \tag{3}$$

with the corresponding row finite system matrices $\mu_\alpha = (r_{ij}^\alpha)_{i\in I, j\in J}$ and the property that every finite subsystem is solvable, without the whole system being solvable. By the observation on the number of different system matrices μ_α, the number of different left hand sides of systems S_α is $\max(2^{|R|}, 2^{\aleph_0})$. By the assumption on the cardinality of A, we conclude that there are $|A|$ many systems S_α with identical left hand sides. Without loss of generality we assume this is correct for all $\alpha \in A$, thus we consider systems (3) where the coefficients $r_{ij}^\alpha = r_{ij}$ do not vary by coordinates $\alpha \in A$. This coefficient uniformity enables a passage to the induced system in $\prod M_\alpha / \oplus M_\alpha$:

$$S: \quad \sum_{j\in J} r_{ij} \overline{(x_j^\alpha)_{\alpha\in A}} = \overline{(m_i^\alpha)_{\alpha\in A}}, \quad i \in I \tag{4}$$

(bars denote the classes mod $\oplus_{\alpha\in A} M_\alpha$). Every finite subsystem of S is equivalent to the set of coordinate finite subsystems of S_α, for all but finitely many $\alpha \in A$. These have solutions, which will be the coordinates of the solutions of the original finite subsystem of S. But S has no global solution, for if $x_j = \overline{(s_j^\alpha)_{\alpha\in A}}, j \in J$ were global solutions of S, then $x_j^\alpha = s_j^\alpha, j \in J$ would provide global solutions of S_α, for almost all $\alpha \in A$. This contradiction then completes the proof that $\prod M_\alpha / \oplus M_\alpha$ is not algebraically compact. □

As we have not succeeded in extending the latter result to all infinite $|A|$, we formulate the following

Conjecture. *If $|A|$ is an uncountable index set of cardinality $\leq 2^{|R|}$ and all $M_\alpha \in R\mathbf{Mod}$, $\alpha \in A$, are not algebraically compact, then $\prod / \coprod$ is not algebraically compact. If this is true then, for countable rings R, $\prod / \coprod$ is algebraically compact if and only if all but countably many $M_\alpha \in R\mathbf{Mod}$, $\alpha \in A$ are algebraically compact.*

Remarks. There are strong indications the conjecture is correct: Gerstner [3] proved that $\mathbb{Z}^A/\mathbb{Z}^{(A)}$ is algebraically compact, iff A is countable. A generalization follows for reduced powers of modules over countable rings: If $M \in R\mathbf{Mod}$ is not algebraically compact, then use Lemma 1.2 in [1] to conclude that if $M^A/M^{(A)}$ is algebraically compact then A must be countable. For Abelian groups, Rychkov [7] proved that $\prod / \coprod$ is algebraically compact if and only if A is countable. In fact, if $\mathcal{S}$ denotes a set of system matrices with the property that for every $M \in R\mathbf{Mod}$ that is not algebraically compact, there is a $\mu \in \mathcal{S}$ that is a system matrix for a system proving algebraic non-compactness of M, let $\mathfrak{n}$ denote minimal cardinality of all such systems. Close inspection of the proof of Proposition 1, ibid. seems to reveal that the RD-purity used there is not essential, namely that it may be replaced by purity (a condition always satisfied for Prüfer domains). In that case, if $|A| > \max(\mathfrak{n}, \aleph_0)$ and all M_α, $\alpha \in A$ are non-compact implies that $\prod / \coprod$ is non-compact.

References

[1] B. Franzen, Algebraic compactness of filter quotients, In: *Proceedings Abelian Group Theory, Oberwolfach, 1981*, Lecture Notes in Math., Springer-Verlag, **874** (1981), 228-241.

[2] T. E. Frayne and A. C. Morel and D. S Scott, Reduced direct products, *Fundamenta Mathematicae*, **51** (1962), 195-228.

[3] O. Gerstner, Algebraische kompaktheit bei Faktorgruppen von Gruppen ganzzahliger Abbildungen, *Manuscripta math.*, **11** (1974), 103-109.

[4] C. U. Jensen and H. Lenzing, *Model Theoretic Algebra with Particular Emphasis on Fields, Rings, Modules*, Gordon and Breach Science Publishers, New York (1989).

[5] R. Kiełpiński and D. Simson, On pure homological dimension, *Bulletin de L'Acad. Polon. Sci, Sé. Math.*, **23**, No. 1 (1975), 1-6.

[6] Jan Mycielski, Some compactifications of general algebras, *Colloquium Mathematicum*, **13**, No. 1 (1964), 1-9.

[7] S. V. Rychkov, On factor-group of the direct product of abelian groups modulo its direct sum, *Math. Notes*, **29**, No. 3-4 (1981), 252-257.