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STABILITY CONDITIONS OF A RECURSIVE ROUTING ALGORITHM

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Abstract: In this paper, we prove formally the conditions under which recursive algorithms like the proposed study scheme can be considered stable. Such algorithms are of great use especially in network routers. Their main job is to select which packet is needed to be forwarded from the input ports to the output ports of the device. The stability property ensures that the algorithm will not drop any packets even if the incoming normalized traffic load is maximum.

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1. Introduction

Network delay is a very common problem that many people experience on a daily basis. This delay is caused, in most cases, by the main network relay, the router. At times of excess traffic, many such relays are unable to perform at a satisfactory level, and therefore they are either dropping the incoming packets or delaying them.

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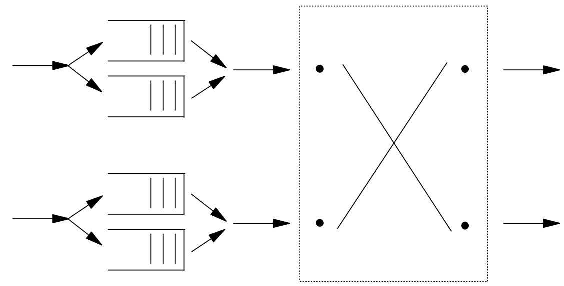

Figure 1: Input queued (IQ) switch

Inside the device, the module that causes the problem is the scheduler module, which is responsible for selecting which packet will be delivered next to the output ports. The scheduler has to select between a number of packets equal to the number of ports of the device. These decisions have to be quick and, at the same time, effective, especially when arriving packets come from time-critical applications, like multimedia and stock market data.

This paper introduces a new approach for scheduling routers, namely the study scheme. The fundamental architecture that the method is using, which involves separations of queues and splitting packets into fixed sized cells, is discussed in Section 2. In Section 3 we give the notations and the main definitions of our study. Then, the stability property is discussed, first for a general queuing system (Section 4) and then for the queuing system produced by the study scheme (Section 5). This system is proven to be stable when it satisfies four basic conditions.

2. Cell Switches with Input Queues

In Figure 1 there is a schematic representation of a cell switch based in input queues (IQ). Since this is a cell switch, it can function only using packets of fixed length, for example ATM cells. Following the ATM nomenclature, the word cell will be used from now on to denote a fixed size packet. Using cells implies that this switch has to make its forwarding decisions on constant time intervals. These intervals are called time slots, or, for simplicity slots. At most one cell can be removed from each input and at most one cell can be transferred to each output in every slot.

We assume that the switch has K "line-cards". This means that in total there exist $2K$ ports, half of them being input and the other half being output ports. We also assume that the speed of these lines is the same for all ports.

The incoming cells to the switch are stored in a queue before they are delivered to their proper output. Every input has a distinct queue for every output. Overall, in the switch, there exist K^2 queues. These queues are assumed to be able to fit a finite number of cells, the same for every queue. If a cell finds the queue full, it is dropped from the switch with no further implications. This architecture overcomes the well-known head-of-the-line blocking [4] problem and is called Virtual Output Queuing (VOQ) or Destination Queuing [1], [9].

3. Theoretical Analysis

In the literature, there exist a number of analytical approaches for studying IQ switches. Usually, an IQ switch is modelled as a controlled queueing system and therefore it can be studied using stochastic modelling techniques.

The major component of a queuing system of this kind is its stability. A queueing system is stable, when there is no queue growing to infinity, assuming that the arrival process is "admissible".

Definition 1. We call $A_{\chi\psi}$ the arrival process from port χ to port ψ and $\lambda_{\chi\psi}$ the average arrival rate. The aggregation of all process is $A = \{A_\chi, 1 \leq \chi \leq K\}$. An arrival process A is considered admissible when no port is overloaded, i.e.:

$$\sum_{\chi=1}^{K} \lambda_{\chi\psi} < 1, 1 \leq \psi \leq K\,,$$

$$\sum_{\psi=1}^{K} \lambda_{\chi\psi} < 1, 1 \leq \chi \leq K\,.$$

To prove that a queuing system is stable, a special function, called the "Lyapunov" function, is used. In [6], there is also an analysis for estimating delays using this method.

3.1. Notations

The notations used in the proofs are, as follows:

• Let K is the number of switch ports and Q is the number of queues (input and output). Clearly, $Q = K^2$.

• Let t and ν are two discrete time variables.

• Let $\Pi, P \in \mathbb{R}^n$, with $\Pi = [\pi_\chi]$ and $P = [\varrho_\chi]$. $\Pi \cdot P \triangleq \Pi \cdot P^T = \sum_\chi \pi_\chi \varrho_\chi$ is the scalar product of vectors Π and P. $\| \Pi \|$ is the Euclidean norm of Π, i.e. $\| \Pi \| = \sum \pi_\chi^2$.

• Let B_t be a vector showing the number of cells currently waiting in the system at time t: it has K^2 elements and the i-th element is the number of cells currently waiting in the i-th queue.

• Let A_t be a vector showing the arrivals at time t: its has K^2 elements, it is a binary vector and a 1 in the i-th element implies the arrival of a cell at the i-th queue at time t.

• Let Δ_t be a vector showing the departures at time t: its has K^2 elements, it is a binary vector and a 1 in the i-th element implies the departure of a cell from the i-th queue at time t. It also corresponds to a matching Δ between input and output ports.

• Let Ω be the set of all the departure vectors of size K: it holds that $|\Omega| = K!$.

• Let D^*_X be the Maximum Weight Matching (MWM) with state of the queue X: $D^*_X X = \max_{\Delta \in \Omega} \Delta X$.

• Let $B_{t+1} = [B_t + A_t - \Delta_t]^+$ be the evolution of the system. We assume that first the arrival take place and then cell depart from the queues.

• $E(Z)$ is the expected value of the random variable Z.

• $\Lambda \triangleq E[A_t] = E[A]$, with $\lambda_{\chi\psi}$ is the rate from input χ to output ψ, assuming stationary traffic.

4. Stability of Controlled Queueing Systems

This section contains some known results for the stability of controlled queueing systems.

Definition 2. A system of queues is stable, or achieves 100% throughput, if

$$\lim_{t\to\infty} \frac{B_t}{t} = \lim_{t\to\infty} \frac{1}{t} \sum_{\chi=0}^{t-1} (A_x - \Delta_x) = 0, \text{ with probability 1.}$$

Definition 3. A system of queues is weakly stable if, for every $\epsilon > 0$, there exists $C > 0$ such that:

$$\lim_{t\to\infty} \Pr\{||B_t|| > C\} < \epsilon.$$

Definition 4. A system of queues is strongly stable if:

$$\lim_{t\to\infty} \sup E||B_t|| < \infty.$$

We now show a theorem initially presented by Tweedie [12] and later modified by Tassiulas [10].

Theorem 1. *Suppose $\{B_t\}_{t=1}^{\infty}$ an aperiodic and irreducible Markov chain with countable state space X. Let $f(B)$ and $g(B)$ be real non-negative functions. Consider T a finite subset of X and T^c its complement. If*

$$g(B) \geq f(B), \qquad B \in T^c, \tag{1}$$

$$E[g(B_2)|B_1 = B] < \infty, \qquad B \in T, \tag{2}$$

$$E[g(B_2) - g(B_1)|B_1 = B] < -f(B), \qquad B \in T^c, \tag{3}$$

then the Markov chain is ergodic and

$$Ef(\overline{B}) < \infty,$$

where the random variable $\overline{B}$ has the steady state distribution of the Markov chain $\{B_t\}_{t=1}^{\infty}$.

We can assume that the stochastic process describing the evolution of the queuing system is an irreducible discrete time Markov chain (DTMC), whose state vector at time n is $Y_t = (B_t, K_t), Y_t \in \mathbb{N}^{Q''}, B_t \in \mathbb{N}^{Q}, K_t \in \mathbb{N}^{Q'}$ and $Q'' = Q + Q'$. Y_t is the combination of the queue length vector B_t and a vector of integer parameters K_t. Most systems of discrete-time queues of practical interest can be described with models that fall in the DTMC class. Therefore, the following general criterion for the strong stability of systems falling into this class is useful. We start with a definition.

Definition 5. An $M \times M$ matrix Q is copositive if $XQX^T \geq 0, \forall X \in \mathbb{R}^{+M}$

Theorem 2. *Given a system of queues with state vector $Y_t = (B_t, K_t)$, and a function $V(B_t) = B_t W B_t^T$ (called Lyapunov function), if there exists a symmetric copositive matrix $W \in \mathbb{R}^{\Omega\times\Omega}$ and two positive real numbers $\epsilon \in \mathbb{R}^+$ and $O \in \mathbb{R}^+$, such that:*

$$E[V(B_{t+1})|Y_t] < \infty, \tag{4}$$

$$E[V(B_{t+1}) - V(B_t)|Y_t] < -\epsilon||B_t||, \qquad \forall Y_t : ||B_t|| > O, \tag{5}$$

then the system of queues is strongly stable. In addition, all the polynomial moments of the queue lengths distributions are finite.

Proof. This is a re-phrasing of the results presented in [5] and can be proven using Theorem 1. Consider $g(B, K) = V(B), f(B, K) = \epsilon||B||$ and $T = \{X : V(X) \leq O\}$. Note that for $B_t \in T^C$:

$$g(B_t, K_t) = B_t W B_t \geq \epsilon B_t B_t \geq \epsilon ||B_t|| = f(B_t, K_t)$$

Now using Theorem 1 we can state:

$$E[\epsilon||\overline{B}||] < \infty \Rightarrow E[||\overline{B}||] < \infty,$$

where $B_t \rightarrow \overline{B}$. □

Since the identity matrix I is a symmetric positive semidefinite matrix, therefore a copositive matrix, we can state the following corollary.

Corollary 1. *Given a system of queues with state vector $Y_t = (B_t, K_t)$, if there exists $\epsilon \in \mathbb{R}^+, O \in \mathbb{R}^+$ such that:*

$$E[B_{t+1}B_{t+1} \mid Y_t] < \infty, \tag{6}$$

$$E[B_{t+1}B_{t+1} - B_t B_t \mid Y_t] < -\epsilon \parallel B_t \parallel, \qquad \forall Y_t : \parallel B_t \parallel > O, \tag{7}$$

then the system of queues is strongly stable, and all the polynomial moments of the queue lengths distribution are finite.

Based on the previous theorems, it was proven in [11], [8] the following theorem which is the first significant result about the stability of IQ cell-switches.

Theorem 3. *In a switch with VOQ buffering scheme, fed by admissible i.i.d. Bernoulli traffic, if the scheduler computes the MWM at each time slot, that is:*

$$\Delta_t = D^*_{B_t} = \arg\max_{\Delta \in \Omega}\{\Delta B_t\},$$

then the system is strongly stable, i.e. it achieves 100% throughput.

5. Stability of the Study Scheme

In this section we define the properties of the study scheme and we prove formally that the study scheme is able to achieve 100% throughput under any admissible i.i.d. Bernoulli arrival process. The following two subsections state some preliminary results to the main result, presented in Section 5.3. We refer always to the notations introduced in Section 3.1.

5.1. Some Preliminary Results

We start with the basic definition:

Definition 6. An algorithm is following the study scheme when: $\Delta_t B_t \geq \Delta_{t-1} B_t$ for every t.

The following lemma is an easy (but important) application of Birkhoff's Theorem presented in combinatorics [7]. Its proof is well known in literature; we present it for completeness.

Lemma 1. (Birkhoff) *Let $\Lambda = [\lambda_{\chi\psi}] = E[A]$ be the rate vector and*

$$\lambda_Q = \max\{\sum_x \lambda_{\chi\psi}, \sum_\psi \lambda_{\chi\psi}\}.$$

Assume the traffic is admissible, i.e. $\lambda_Q < 1$. Given the queue length vector B, it exists $\epsilon > 0$ such that it holds:

$$(\Lambda - D_B^*)B < -\epsilon||B||.$$

Proof. Since the traffic is admissible, $\lambda_Q < 1$ and it exists a finite set $\{\Delta_\chi : \Delta_\chi \subseteq \Omega\}$ and a finite set $\{\alpha_\chi : 0 \leq \alpha_\chi \leq 1\}$, such that:

$$\Lambda < \frac{1}{\lambda_Q}\Lambda \leq \sum_x \alpha_\chi \Delta_\chi \,, \qquad \text{with} \qquad \sum_x \alpha_\chi = 1\,. \tag{8}$$

For definition, $D_B^* B \geq \Delta_\chi B$, then

$$\Lambda B - D_B^* B \leq \lambda_Q (\sum_x \alpha_\chi \Delta_\chi) B - D_B^* B$$

$$\leq \lambda_Q \sum_x \alpha_\chi D_B^* B - D_B^* B = (\lambda_Q - 1) D_B^* B \leq -(1-\lambda_Q)||B||\,. \tag{9}$$

Choose now $\epsilon = \frac{1-\lambda_Q}{2}$ and note that $-(1-\lambda_Q)||B|| < -\epsilon||B||$. □

Lemma 2. *Given a matching $\Delta' \in \Omega$, a set of matchings $\{\Delta_\chi, 0 \leq \chi \leq \zeta - 1\} \subset \Omega$ and a set of arrival vectors $\{A_\chi, 0 \leq \chi \leq \zeta - 1\}$, for any finite $\zeta \in \mathbb{N}$ it holds:*

$$|\sum_{\chi=0}^{\zeta-1} (A_\chi - \Delta_\chi)\Delta'| \leq \zeta Q\,. \tag{10}$$

Proof. By Cauchy inequality, assuming $K \geq 2$, we have:

$$(A-\Delta)\Delta^{'} \leq ||A-\Delta||||\Delta^{'}|| \leq \sqrt{2K}\sqrt{K} = \sqrt{2}K < K^2 = Q\,. \tag{11}$$

Summing ζ terms less than M, we obtain the lemma. □

Lemma 3. *For any finite $\zeta \in \mathbb{N}$, it holds:*

$$|D^*_{B_t}B_t - D^*_{B_{t-\zeta}}B_{t-\zeta}| \leq 2\zeta Q\,. \tag{12}$$

Proof. From one time slot to another, the weight of the maximum weight matching can increase or decrease by, at most $2K$. Hence,

$$|D^*_{B_t}B_t - D^*_{B_{t-\zeta}}B_{t-\zeta}| \leq |\sum_{\psi=0}^{\zeta-1} D^*_{B_{t-\psi}}B_{t-\psi} - D^*_{B_{t-\psi-1}}B_{t-\psi-1}|$$

$$\leq \sum_{\psi=0}^{\zeta-1} 2K \leq 2\zeta K \leq 2\zeta K^2 = 2\zeta Q\,. \tag{13}$$

5.2. Stability of Epoch-Based Algorithms

A study on epoch-based algorithms will help us prove the stability of the study scheme. All algorithms of the study scheme will be shown to be equivalent to epoch-based algorithms, and therefore they inherit their properties.

An epoch-based algorithm ψ makes use of epochs. When the ν-th epoch starts, at time t_ν, the current state of the queuing system is saved and then the algorithm tries to progressively find the maximum weight matching of this state. When it finally comes up with the MWM, at time $\sigma_\nu \geq t_\nu$, the algorithm saves the new state of the queuing system and continues in the same fashion starting from time slot $\sigma_{\nu+1}$.

A clarification of the above follows.

Definition 7. Stopping times series $\{t_\nu\}$ and $\{\sigma_\nu\}$ are defined as follows:

$$\begin{gathered} t_0 = 0, \qquad \sigma_0 = 0, \\ \sigma_\nu = \inf\{t \geq t_\nu : \Delta_{\sigma_\nu}B_{t_\nu} = D^*_{B_{t_\nu}}B_{t_\nu}\}, \\ t_{\nu+1} = \sigma_\nu + 1, \qquad \text{for } \nu \geq 0, \\ \zeta_\nu = \sigma_\nu - t_\nu, \qquad \text{for } \nu \geq 0, \end{gathered}$$

Figure 2 shows an example of this definition of stopping times.

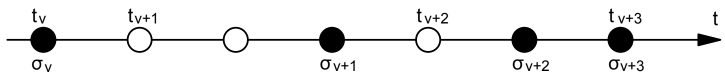

Figure 2: Example of definition of stopping times: the dot is full when the weight of the current matching is maximum, otherwise it is empty

Definition 8. The ν-th epoch is defined as the interval $[t_\nu, \sigma_\nu]$, for $\nu \geq 1$, and its duration is ζ_ν time slots.

Definition 9. Phase ν-th, for $\nu \geq 1$, is defined as the interval $Z_\nu = \{\chi : \chi \geq 0, t_\nu + \chi < \sigma_\nu\}$. Note that $| Z_\nu |= \zeta_\nu$ and it can happen $Z_\nu = 0$ when $\zeta_\nu = 0$. During this phase, the algorithm searches for the MWM of the state of the queuing system at time t_ν. Note also that for $\psi \in Z_\nu$, it holds $\Delta_{t_\nu+\psi} B_{t_\nu} < D^*_{B_{t_\nu}} B_{t_\nu}$.

Theorem 4. *An algorithm ψ is stable, under any admissible i.i.d. Bernoulli arrival process, if it has the following properties (we assume $\alpha \geq 0$):*

$$\exists b \in \mathbb{R}^+ : b < \infty \qquad \text{and} \qquad E[\zeta_\nu^\mu] < b, \quad \text{for} \quad \mu = 1, 2, 3, \tag{14}$$
$$\Delta_{t_\nu+k} B_{t_\nu} \geq \Delta_{t_\nu} B_{t_\nu} - \alpha k^2, \quad \text{for} \quad k \in Z_\nu, \tag{15}$$
$$\Delta_{t_\nu} B_{t_\nu} \geq \Delta_{t_\nu-1} B_{t_\nu} = \Delta_{\sigma_\nu-1} B_{t_\nu}, \tag{16}$$
$$\Delta_{\sigma_\nu} B_{t_\nu} = D^*_{B_{t_\nu}} B_{t_\nu}. \tag{17}$$

It is obvious that all epoch-based algorithms satisfy properties (15), (16), and (17). What the theorem states is that, if for an algorithm the duration of an epoch starts with the first three moments finite, then the algorithm is stable.

Proof. Note that the sequence $\{t_\nu\}$ is proper because of property (14). The system evolution satisfies the following equation:

$$B_{t_{\nu+1}} = B_{t_\nu} + \sum_{i=0}^{\zeta_\nu - 1} (A_{t_\nu+i} - \Delta_{t_\nu+i}).$$

It holds that $E[A_t A_t] < \infty$. We are going to use the Lyapunov function $V(B_{t_\nu}) = B_{t_\nu} B_{t_\nu}$, in order to show that there exists $\epsilon > 0$, such that:

$$\lim_{\| B_{t_\nu} \| \to \infty} \frac{E[V(B_{t_{\nu+1}}) \mid B_{t_\nu}] - V(B_{t_\nu})}{\| B_{t_\nu} \|} < -\epsilon\,. \tag{18}$$

We continue the calculation of (18) as follows:

$$E[V(B_{t_{\nu+1}}) \mid B_{t_\nu}] - V(B_{t_\nu})$$
$$= E[(B_{t_\nu} + \sum_{i=0}^{\zeta_\nu - 1}(A_{t_\nu+i} - \Delta_{t_\nu+i}))(B_{t_\nu} + \sum_{i=0}^{\zeta_\nu - 1}(A_{t_\nu+i} - \Delta_{t_\nu+i}))]$$
$$-V(B_{t_\nu}) = 2E[\sum_{i=0}^{\zeta_\nu - 1}(A_{t_\nu+i} - \Delta_{t_\nu+i})B_{t_\nu} + \zeta_\nu \Delta_{\sigma_\nu} B_{t_\nu} - \zeta_\nu \Delta_{\sigma_\nu} B_{t_\nu}]$$
$$= 2E[\zeta_\nu](E[A] - \Delta_{\sigma_\nu})B_{t_\nu} + 2E[\Delta_\delta]B_{t_\nu}. \tag{19}$$

We have used Wald's Lemma [3] and defined: $\Delta_\delta = \zeta_\nu \Delta_{\sigma_\nu} - \sum_{i=0}^{\zeta_\nu - 1} \Delta_{t_\nu+i}$. We will now show that $\Delta_\delta = o(\| B_{t_\nu} \|)$, that is:

$$\lim_{\|B_{t_\nu}\| \to \infty} \frac{E[\Delta_\delta]B_{t_\nu}}{\| B_{t_\nu} \|} = 0. \tag{20}$$

From property (15), $\Delta_{t_\nu+i}B_{t_\nu} \geq \Delta_{t_\nu}B_{t_\nu} - ai^2, i \in Z_\nu$, we can state:

$$\sum_{i=0}^{\zeta_\nu - 1} \Delta_{t_\nu+i}B_{t_\nu} \geq \sum_{i=0}^{\zeta_\nu - 1} \Delta_{t_\nu}B_{t_\nu} - \alpha \sum_{i=0}^{\zeta_{\nu-1}} i^2 = \zeta_\nu \Delta_{t_\nu} B_{t_\nu} - g(\zeta_\nu)\,, \tag{21}$$

where $g(\zeta_\nu) = \alpha(\zeta_\nu - 1)\zeta_\nu(2\zeta_\nu + 3)/6 = O(\zeta_\nu^3)$.

Hence:

$$\Delta_\delta B_{t_\nu} \leq \zeta_\nu(\Delta_{\sigma_\nu} - \Delta_{t_\nu})B_{t_\nu} + g(\zeta_\nu)\,. \tag{22}$$

From property (16):

$$\Delta_{t_\nu}B_{t_\nu} \geq \Delta_{t_\nu - 1}B_{t_\nu} = \Delta_{t_\nu - 1}B_{t_{\nu-1}} + \Delta_{t_\nu - 1}\sum_{i=0}^{\zeta_{\nu-1}-1}(A_{t_{\nu-1}+i} - \Delta_{t_{\nu-1}+i})$$
$$\geq D^*_{B_{t_{\nu-1}}} B_{t_{\nu-1}} - \zeta_{\nu-1}Q, \tag{23}$$

where the last inequality is produced from using Lemma 2 and property $\Delta_{t_\nu - 1} = D^*_{B_{t_{\nu-1}}}$. Recall that $\Delta_{\sigma_\nu} = D^*_{B_{t_\nu}}$. Combining equations (22), (23) and Lemma 3:

$$\Delta_\delta B_{t_\nu} \leq \zeta_\nu(D^*_{B_{t_\nu}}B_{t_\nu} - D^*_{B_{t_{\nu-1}}}B_{t_{\nu-1}}) + \zeta_{\nu-1}Q + g(\zeta_\nu)$$
$$\leq \zeta_\nu(2\zeta_{\nu-1}Q) + \zeta_{\nu-1}Q + g(\zeta_\nu) = g'(\zeta_\nu)Q, \tag{24}$$

where $E[g^{'}(\zeta_\nu)] = E[O(\zeta_\nu^3)] = O(b)$, since $\{\zeta_\nu\}$ are independent of $\{t_\nu\}$ and from property (14).

Now we can finally show (20):

$$\lim_{\| B_{t_\nu} \| \to \infty} \frac{E[\Delta_\delta] B_{t_\nu}}{\| B_{t_\nu} \|} = \frac{E[g^{'}(\zeta_\nu)]Q}{\| B_{t_\nu} \|} = \frac{O(b)Q}{\| B_{t_\nu} \|} = 0, \tag{25}$$

if property (14) holds.

By definition of σ_ν, $\Delta_{\sigma_\nu} B_{t_\nu} = D^*_{B_{t_\nu}} B_{t_\nu}$, hence by combining (19), (25) and Lemma 1, we obtain:

$$\lim_{\| B_{t_\nu} \| \to \infty} \frac{E[V(B_{t_{\nu+1}}) \mid B_{t_\nu}] - V(B_{t_\nu})}{\| B_{t_\nu} \|} < \frac{-2\epsilon E[\zeta_\nu] \| B_{t_\nu} \|}{\| B_{t_\nu} \|} < -2\epsilon E[\zeta_\nu], \tag{26}$$

which is what we were looking for. □

5.3. Stability of an Algorithm Following the Study Scheme

Theorem 4 will help us prove the main result about the study scheme. Theorem 5 will define the conditions that an algorithm has to satisfy in order to be stable.

Definition 10. An algorithm works off-line if the evolution of the queuing system is given from the equation $X_t = X_{t-1} = \cdots = X_0$. This means that the arrivals and departures do not affect the system's state, while the departures Δ_t are computed at every time slot.

The algorithm continues working in this state, as long as the departure vector is not equal to the MWM of the queuing system at time $t = 0$.

Definition 11. Given a scheduling algorithm working off-line and a state of the queues $B_0 \in Z_+^Q$, with $\| B_0 \| < \infty$ and any initial matching Δ_0, the tracking time $T(B_0)$ is defined as follows:

$$T(B_0) = \inf\{t \geq 0 : \Delta_t B_0 = D^*_{B_0} B_0\}.$$

In general, the tracking time is a random variable which depends also on Δ_0. Now we can show our main result.

Theorem 5. *Consider an algorithm following the study scheme working off-line. If for any $B_0 \in Z_+^Q$, with $\| B_0 \| < \infty$, and for any initial matching Δ_0, $T(B_0)$ is such that: $E[T(B_0)^\mu] < \infty$, $\mu = 1, 2, 3$, then the algorithm ψ is stable, i.e. achieves 100% throughput under any admissible traffic pattern.*

Proof. The proof is based completely on Theorem 4. The way is to show that all its assumptions are satisfied. Property (15) of Theorem 4 means that the epoch approach behaves like working off-line during an epoch, as the epoch-based algorithm freezes the state of the system at the beginning of the epoch and all the $\Delta_{t_\nu+i}$ are computed, for $i \in Z_\nu$, independent from the states of the queues $B_{t_\nu+i}$ and the arrivals $A_{t_\nu+i}$. Assumption (15) is satisfied, by using the following Lemma 4, and setting $\alpha = 2Q$. According to Theorem 4, $\zeta_\nu = T(B_\nu)$, so assumption (14) holds. The algorithm by definition has the propertyis (16). Since the algorithm finds the MWM after $T(B_\nu)$ also assumption (17) holds. $\square$

Lemma 4. *For an algorithm following the study scheme it holds:*

$$\Delta_{t+k} B_t \geq \Delta_t B_t - 2k^2 Q, \quad \textit{with } 0 < k < \infty.$$

This lemma is referred (for $k = 1$) by Tassiulas in Property (16) of his paper [10].

Proof. B_{t+k}, for $k \in \mathbb{N}$, can be written as:

$$B_{t+k} = B_t + \sum_{i=0}^{k-1} (A_{t+i} - \Delta_{t+i}) \,. \tag{27}$$

Now we have:

$$\begin{aligned}
\Delta_{t+k} B_t - \Delta_{t+k-1} B_t &= (\Delta_{t+k} - \Delta_{t+k-1})(B_{t+k} - \sum_{i=0}^{k-1} (A_{t+i} - \Delta_{t+i})) \\
&= (\Delta_{t+k} - \Delta_{t+k-1}) B_{t+k} - (\Delta_{t+k} - \Delta_{t+k-1}) \sum_{i=0}^{k-1} (A_{t+i} - \Delta_{t+i}) \\
&\geq -(\Delta_{t+k} - \Delta_{t+k-1}) \sum_{i=0}^{k-1} (A_{t+i} - \Delta_{t+i}) \geq -2kQ \,,
\end{aligned} \tag{28}$$

where the last line is due to the property ($\Delta_{t+k} B_{t+k} \geq \Delta_{t+k-1} B_{t+k}$) and to Lemma 2. Now we can expand recursively:

$$\Delta_{t+k} B_t \geq \Delta_{t+k-1} B_t - 2kQ \geq (\Delta_{t+k-1} B_t - 2kQ) - 2kQ \geq \ldots \tag{29}$$

$$\geq \Delta_{t+k} B_t \geq \Delta_{t+k-j} B_t - j(2kQ), \quad \text{for} \quad j \in [0, k] \,. \tag{30}$$

We set $j = k$, and get:

$$\Delta_{t+k} B_t \geq \Delta_t B_t - 2k^2 M \,. \tag{31}$$

6. Conclusions

In this paper, we have shown the conditions under which an algorithm following the study scheme can be considered to be stable. The algorithms were implicitly shown to be equivalent to algorithms based on epochs, while they are working off-line. Then, based on a theorem used for the epoch algorithms, we have shown that their properties are inherited by the study scheme algorithms.

We also have to state that the results obtained in this paper were tested by simulations. A router simulator has been developed to study the performance of the considered algorithms. The simulator has been written in the special simulation language *MODSIM III* and is described in detail in [2].

All simulations have shown that the study scheme algorithms that satisfied the conditions posed in this paper were achieving 100% throughput, and therefore they were not loosing any packets even in severe traffic scenarios. These outcomes strengthen the results of this study even more.

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