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FRACTAL DIMENSIONS OF INFINITE PRODUCT SPACES

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Abstract: In this paper the fractal geometry of compact subsets of the infinite dimensional metric space $\mathbb{R}^\infty$ is studied. Our main interest is in products of self-similar or Cantor-like sets, and we show that there are many differences between infinite products of these sets in $\mathbb{R}^\infty$ and finite products in Euclidean space.

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1. Introduction

The fractal geometry of compact subsets of Euclidean space has been much studied since Mandelbrot first coined the term fractal. In recent years mathematicians have begun to investigate the fractal properties of compact subsets of more general metric spaces (see [3], [9] and the many references cited therein).

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An interesting example of an infinite dimensional, complete metric space, which is a natural generalization of Euclidean space, is $\mathbb{R}^\infty$ with the product topology and Frechet metric

$$d(x,y)=\sum_{n=1}^{\infty}2^{-n}\min\{1,|x_n-y_n|\}.$$

In this paper we will investigate the fractal geometry of compact subsets of $\mathbb{R}^\infty$. Our primary focus is on infinite Cartesian products of self-similar or Cantor-like sets, and we show that the fractal geometry of even these special sets is quite different from that of products in $\mathbb{R}^n$. For instance, it is well known that if E and F are self-similar sets in $\mathbb{R}^n$, then the box and Hausdorff dimensions of $E\times F$ coincide, and equal the sum of the dimensions of E and F. More generally, the box dimension of a finite product of compact subsets of $\mathbb{R}^n$ is dominated by the sum of the box dimensions of the sets. In contrast, we show that although the box and Hausdorff dimensions of an infinite product of uniform Cantor sets agree, these dimensions always equal one more than the sum of the dimensions of the individual Cantor sets. Furthermore, there are self-similar sets E_i with the lower and upper box dimensions of $\prod_{i=1}^{\infty}E_i$ different, and both strictly greater than the sum of the dimensions of E_i.

However, we also prove that given any self-similar sets $\{E_i\}_{i=1}^{\infty}$ there is another metric (depending on the sets) generating the product topology, whose truncation to $\mathbb{R}^n$ is Lipschitz equivalent to the usual Euclidean metric, and which has the property that the box and Hausdorff dimensions of $\prod_{i=1}^{\infty}E_i$ taken with respect to this metric coincide and equal the sum of the dimensions of E_i. For arbitrary, compact sets we can select the metric so that the box dimension of the product is 'almost' dominated by the sum of the box dimensions.

Another difference between $\mathbb{R}^\infty$ and $\mathbb{R}^n$ is that the order of the coordinates in the infinite product can affect the dimension, a consequence of the lack of symmetry in the Frechet metric. We illustrate this by constructing sets whose product has infinite dimension in one ordering and zero dimension in another.

A detailed analysis of the fractal dimensions of products of Cantor-like sets is given, both with the Frechet metric and topologically equivalent ones. For many such sets we establish that the Hausdorff and lower box dimensions of the product coincide. It is an open problem if the lower box dimension of a product of self-similar and/or Cantor-like sets must always be equal to the Hausdorff dimension.

2. Definitions and Elementary Properties

2.1. Fractal Dimensions

Suppose X is a metric space with metric d. The closed ball in X with centre x and radius r will be denoted by $B(x,r) \equiv \{y \in X : d(x,y) \leq r\}$. For $V \subseteq X$ we let $\operatorname{diam} V \equiv \sup\{d(x,y) : x, y \in V\}$ denote the diameter of V.

Hausdorff and box dimensions are defined for subsets of any metric space. We begin by recalling their definitions. The Hausdorff s-measure of a compact subset A of X is given by the formula

$$H^s(A) = \lim_{\delta \to 0^+} \left(\inf\{\sum_{i=1}^{\infty} (\operatorname{diam} V_i)^s : \operatorname{diam} V_i \leq \delta, \bigcup V_i \supseteq A\} \right)$$

and the Hausdorff dimension of A is

$$\dim_H A = \inf\{s : H^s(A) = 0\} = \sup\{s : H^s(A) = \infty\}.$$

Let $N_\delta(A)$ denote the minimum number of closed balls of radius δ which cover A. The upper and lower box dimensions of a subset A of X are given by

$$\overline{\dim}_B(A) = \limsup_{\delta \to 0} \frac{\log N_\delta(A)}{|\log \delta|}; \quad \underline{\dim}_B(A) = \liminf_{\delta \to 0} \frac{\log N_\delta(A)}{|\log \delta|}.$$

If $\overline{\dim}_B(A) = \underline{\dim}_B(A)$ then the box dimension of A, $\dim_B A$, is their common value. We always have the inequalities

$$\dim_H A \leq \underline{\dim}_B(A) \leq \overline{\dim}_B(A),$$

and often the three dimensions are equal. In this case we simply write $\dim A$.

For subsets of $\mathbb{R}^n$ there are many equivalent definitions of box dimension. For example, $N_\delta(A)$ could denote the smallest number of cubes with sides of length δ that cover A, the smallest number of sets of diameter at most δ that cover A, or the largest number of disjoint balls of radius δ, with centres in A (c.f. [4, Chapter 3]). Of course, the definition involving cubes will not be meaningful in arbitrary metric spaces, but the second and third choices for $N_\delta(A)$ continue to be equivalent.

An interesting example of a complete metric space which generalizes finite dimensional Euclidean space is $\mathbb{R}^\infty$, the product of countably many copies of $\mathbb{R}$, with the Frechet metric given by

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |x_n - y_n|\},$$

for $x = (x_n)$, $y = (y_n) \in \mathbb{R}^\infty$. An important compact subset of $\mathbb{R}^\infty$ is the Hilbert cube, $\prod_{i=1}^{\infty}[0,1]$. The metric restricted to the Hilbert cube (or subsets of it) is obviously given by

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n| .$$

Even though cubes are defined in $\mathbb{R}^\infty$ we still cannot replace $N_\delta(A)$ with the smallest number of cubes with sides of length δ that cover A. To see this simply note that the infinite product of the two element set $\{0,1\}$ has box dimension at most one since it can be covered by the 2^n balls of diameter 2^{-n}, centred at the points $\{a_1, \dots, a_n\}$, $a_i = 0,1$, but it has no finite covering by cubes of side lengths less than one. However, it is an easy exercise to show that for subsets of the Hilbert cube we can take $N_{2^{-n}}(A)$ to be the smallest number of sets of the form $A_1 \times A_2 \times \cdots \times A_n \times \prod_{i=n+1}^{\infty}[0,1]$ which cover A and satisfy $\operatorname{diam} A_i \leq 2^{-n}$ or even $\operatorname{diam} A_i \leq 2^{-n+i}$ (c.f. the proof of Theorem 4).

2.2. Cantor Sets

Many of the subsets $\mathbb{R}^\infty$ which we investigate in this paper are products of Cantor sets: subsets of $[0,1]$ which can be constructed inductively in a similar fashion to the classical middle third Cantor set. First, we remove from $[0,1]$ a non-empty, open, centred subinterval called the gap of step one, leaving two closed intervals of equal, positive length called the Cantor intervals of step one. The relation between the length of these intervals and the initial subinterval is called the ratio of dissection of step one. A similar operation is performed on each Cantor interval of step one, producing the four closed equal-length intervals of step two, two new gaps of step two and the ratio of dissection of step two. Repeating this construction inductively yields a decreasing sequence of closed sets whose intersection we will call a *Cantor set*. Any Cantor set is compact, uncountable, perfect and nowhere dense. If all the ratios of dissection of a Cantor set are equal then the Cantor set will be called *uniform*.

If we let r_i denote the ratio of dissection at step i in the construction of Cantor set C, then the intervals of step k have length $r_1 \cdots r_k$ and the gaps have length $(1-2r_k)r_1 \cdots r_{k-1}$. The formulas for the box and Hausdorff dimensions of a Cantor set can also be expressed in terms of the ratios of dissection (see [2]):

$$\dim_H C = \underline{\dim}_B C = \liminf_{n\to\infty} \frac{\log 2}{\left|\log (r_1 \cdots r_n)^{1/n}\right|}$$

and

$$\overline{\dim}_B C = \limsup_{n\to\infty} \frac{\log 2}{\left|\log (r_1 \cdots r_n)^{1/n}\right|}.$$

The dimension of a uniform Cantor set with ratio of dissection r is obviously $\log 2/ |\log r|$.

2.3. Self-Similar Sets

We also study the dimensions of products of self-similar sets, a self-similar set being the attractor of an iterated function system of similarities $\{F_1, \ldots, F_m\}$ for $m \geq 2$. The box and Hausdorff dimensions of any self-similar set E in $\mathbb{R}^n$ coincide, and if E satisfies the open set condition and has similarity factors $r_1, \ldots, r_m$, then $\dim E = s$, where s is the solution to $\sum_{i=1}^{m} r_i^s = 1$ (see [4, Chapter 9]). Clearly any self-similar set in $\mathbb{R}^n$, satisfying the open set condition, has positive Hausdorff dimension. Uniform Cantor sets are examples of self-similar sets (generated by two similarities) which satisfy the open set condition.

2.4. Dimensions of Products

The fractal dimensions of finite products of metric spaces have been extensively studied (c.f. [1], [5], [7] and [10]). The following proposition summarizes some of the known facts. For proofs and further discussion the reader is referred to [4, Chapter 4].

Proposition 1. *If* E_1*and* E_2 *are Borel subsets of* $\mathbb{R}^n$, *then:*
(i) $\dim_H E_1 \times E_2 \geq \dim_H E_1 + \dim_H E_2$, *and*
(ii) $\overline{\dim}_B(E_1 \times E_2) \leq \overline{\dim}_B E_1 + \overline{\dim}_B E_2$.
(iii) *Equality holds in both cases if* $\dim_H E_i = \dim_B E_i$ *for* $i = 1, 2$, *and then also* $\dim_H E_1 \times E_2 = \dim_B E_1 \times E_2$.

In particular, if E_1 and E_2 are self-similar sets then

$$\dim_B E_1 \times E_2 = \dim_H E_1 \times E_2 = \dim E_1 + \dim E_2.$$

It is natural to ask if these inequalities are true for infinite Cartesian products $\prod_{i=1}^{\infty} E_i$ in $\mathbb{R}^\infty$. A monotonicity argument shows that (i) continues to hold. The following simple proposition shows that (ii) and (iii) fail to extend to infinite Cartesian products.

Proposition 2. *If C is a k element set, then*

$$\dim_H \prod_{i=1}^{\infty} C = \dim_B \prod_{i=1}^{\infty} C = \log k / \log 2$$

and yet $\sum_{i=1}^{\infty} \dim_B C = 0$.

Proof. Without loss of generality assume $C = \{a_1, \ldots, a_k\} \subseteq [0,1]$. Clearly the sets

$$\{(a_{i_1}, a_{i_2}, \ldots, a_{i_n}) \times \prod_{i=n+1}^{\infty} [0,1] : a_{i_j} \in C\}$$

cover $\prod_{i=1}^{\infty} C$ and have diameter at most 2^{-n}. As there are k^n of these sets,

$$\dim_H \prod_{i=1}^{\infty} C \le \overline{\dim}_B \prod_{i=1}^{\infty} C \le \frac{\log k}{\log 2}.$$

Now consider the probability measure on the finite set C with mass $1/k$ at each of the k elements, and let μ be the infinite product of these measures on $\prod_{i=1}^{\infty} C$.

Set $\delta = \min\{|a_i - a_j| : i \neq j\}$ and suppose $U \subseteq \prod_{i=1}^{\infty} C$ satisfies

$$\delta 2^{-(n+1)} \le \operatorname{diam} U < \delta 2^{-n}.$$

If $x, y \in U$ and $x_i \neq y_i$ for some $i = 1, \ldots, n$, then

$$d(x,y) \ge |x_i - y_i|\, 2^{-i} \ge \delta 2^{-n}.$$

This contradicts the assumption on the diameter of U. Consequently

$$U \subseteq (a_{i_1}, \ldots, a_{i_n}) \times \prod_{i=n+1}^{\infty} C,$$

for some $a_{i_1}, \ldots, a_{i_n} \in C$, and hence

$$\mu(U) \le k^{-n} \le c\,(\operatorname{diam} U)^{\log k / \log 2}.$$

By the mass distribution principle $\dim_H \prod_{i=1}^{\infty} C \ge \log k / \log 2$.

As C is finite, it is trivial that $\dim C = 0$. □

A monotonicity argument gives the following corollary.

Corollary 3. *Suppose* $C = \prod_{i=1}^{\infty} C_i$ *with* $\{a_1, \ldots, a_i\} \subseteq C_i$ *for each* i. *Then* $\dim_B \prod_{i=1}^{\infty} C_i = \dim_H \prod_{i=1}^{\infty} C_i = \infty$.

Example 2.1. For each $i = 1, 2, \ldots$ let E_i be a uniform Cantor set with ratio of dissection 2^{-k_i}. Because every Cantor set contains both 0 and 1, the proposition implies that the Hausdorff dimension of $\prod_{i=1}^{\infty} E_i$ is at least one. Since $\dim_H E_i = 1/k_i$, if we choose k_i such that $\sum_{i=1}^{\infty} 1/k_i < 1$ then clearly $\dim_H \prod_{i=1}^{\infty} E_i > \sum_{i=1}^{\infty} \dim E_i$.

Remark 2.1. We will prove in Section 3 that uniform Cantor sets E_i satisfy the relations

$$\dim_B \prod_{i=1}^{\infty} E_i = \dim_H \prod_{i=1}^{\infty} E_i = 1 + \sum_{i=1}^{\infty} \dim E_i.$$

Thus whenever $\sum_{i=1}^{\infty} \dim E_i < \infty$, then

$$\sum_{i=1}^{\infty} \dim E_i < \dim_H \prod_{i=1}^{\infty} E_i < \infty.$$

In contrast, we can arrange for (other) self-similar sets to have summable dimensions, but infinite dimension of the product.

Example 2.2. Consider the self-similar sets $\{E_i\}_{i=1}^{\infty} \subseteq [0, 1]$ generated by the iterated function systems $\{F_{i,1}, \ldots, F_{i,2^i}\}$, where $F_{i,j} = 2^{-2^i} x + (j-1)2^{-i}$. As $\dim E_i = i/2^i$, $\sum_{i=1}^{\infty} \dim E_i < \infty$. However, the points $(j-1)2^{-n}$ belong to E_i whenever $i \geq n$ and $j = 1, \ldots, 2^n$, and therefore the previous corollary yields $\dim_H \prod_{i=1}^{\infty} E_i = \infty$.

Remark 2.2. All of the examples which have been given in this section have the property that the upper box dimension of the infinite product equals the Hausdorff dimension. In Section 4 we will construct self-similar sets for which this is not true. We will also exhibit self-similar sets with $\dim_H \prod_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} \dim E_i < \infty$.

Remark 2.3. Another notion of dimension is packing dimension denoted $\dim_P$. For the definition see [3]. This is intermediate to Hausdorff and box dimension in the sense that for all sets E we have

$$\dim_H E \leq \dim_P E \leq \overline{\dim}_B E.$$

It is known (see [10]) that

$$\dim_P E \times F \leq \dim_P E + \dim_P F.$$

The same examples show that this inequality also fails to extend to infinite products.

3. Dimensions of Products of Cantor Sets

In this section we study infinite products of Cantor sets. For many Cantor sets, including uniform Cantor sets, the expected equality $\dim_H \prod_{i=1}^{\infty} E_i = \dim_B \prod_{i=1}^{\infty} E_i$ holds, and formulas for the dimensions will be given in terms of the ratios of dissection.

Theorem 4. *Let $E_i \subseteq [0,1]$, $i = 1, 2, \dots$ be Cantor sets with ratios of dissection $r_{i,k}$ at step k in the construction. For $n \in \mathbb{N}$, let $L(n,i)$ be the minimal integer such that the Cantor intervals of E_i of step $L(n,i)$ have length at most 2^{-n}.*

(i) *Then*

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i),$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i) - 1 \leq \overline{\dim}_B \prod_{i=1}^{\infty} E_i \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i)$$

and

$$\dim_H \prod_{i=1}^{\infty} E_i \geq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i) - 1.$$

(ii) *If either $\sup_{i,k} r_{i,k} < 1/2$ or for each $i = 1, 2, \dots$ the sequence of gap lengths,*

$$\{(1 - 2r_{i,k}) r_{i,1} \cdots r_{i,k-1}\}_k,$$

is decreasing, then

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i = \dim_H \prod_{i=1}^{\infty} E_i = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i)$$

and

$$\overline{\dim}_B \prod_{i=1}^{\infty} E_i = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i).$$

Proof. (i) Notice that if $I_i \subseteq E_i$ has diameter at most 2^{-n+i}, then

$$\begin{aligned}\operatorname{diam}(I_1 \times \cdots \times I_{n-1} \times \prod_{i=n}^{\infty} E_i) &\leq \sum_{i=1}^{n-1} 2^{-n+i} 2^{-i} + \sum_{i=n}^{\infty} 2^{-i} \\ &\leq (n+1)2^{-n}.\end{aligned}$$

Since the collection of all sets $I_{1,j_1} \times \cdots \times I_{n-1,j_{n-1}} \times \prod_{i=n}^{\infty} E_i$, where $\bigcup_j I_{i,j} \supseteq E_i$ is a cover of $\prod_{i=1}^{\infty} E_i$, we immediately obtain the bound

$$N_{(n+1)2^{-n}}(\prod_{i=1}^{\infty} E_i) \leq \prod_{i=1}^{n-1} N_{2^{-(n-i)}}(E_i).$$

But the Cantor intervals of E_i of step $L(n-i,i)$ provide a $2^{-(n-i)}$-covering of E_i, and there are $2^{L(n-i,i)}$ of these intervals, thus

$$N_{(n+1)2^{-n}}(\prod_{i=1}^{\infty} E_i) \leq \prod_{i=1}^{n-1} 2^{L(n-i,i)},$$

and therefore

$$\frac{\log N_{(n+1)2^{-n}}(\prod_{i=1}^{\infty} E_i)}{|\log(n+1)2^{-n}|} \leq \frac{\sum_{i=1}^{n-1} L(n-i,i)\log 2}{n\log 2 - \log(n+1)}.$$

Hence

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i = \liminf_{n\to\infty} \frac{\log N_{(n+1)2^{-n}}(\prod_{i=1}^{\infty} E_i)}{|\log(n+1)2^{-n}|} \leq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1} L(n-i,i)$$

and

$$\begin{aligned}\overline{\dim}_B \prod_{i=1}^{\infty} E_i &= \limsup_{n\to\infty} \frac{\log N_{(n+1)2^{-n}}(\prod_{i=1}^{\infty} E_i)}{|\log(n+1)2^{-n}|} \\ &\leq \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1} L(n-i,i)\,,\end{aligned}$$

as claimed.

To derive the lower bound for the upper box dimension we will show that there are suitably many disjoint balls of radius δ and centred in $\prod_{i=1}^{\infty} E_i$. Denote by W_n the set of $(w_i) \in \prod_{i=1}^{\infty} E_i$ with $w_i = 0$ if $i \geq n$ and w_i a left endpoint of an interval of step $L(n-i,i)-1$ if $i<n$. The minimality requirement in the definition of $L(m,i)$ ensures that the length of a step $L(m,i)-1$ interval is at least 2^{-m}, thus if $v,w\in W_n$ with $v_i\neq w_i$, then

$$d(v,w)\geq |v_i-w_i|\,2^{-i}\geq 2^{-n}.$$

This means that the balls of radius $\delta<2^{-n-1}$ and centred at $v\in W_n$ are disjoint. Consequently,

$$N_\delta(\prod_{i=1}^{\infty}E_i)\geq |W_n|=\prod_{i=1}^{n-1}2^{L(n-i,i)-1}$$

and therefore

$$\overline{\dim}_B\prod_{i=1}^{\infty}E_i\geq \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}L(n-i,i)-1.$$

To find a lower bound for the Hausdorff dimension we apply the mass distribution principle. Consider the uniform Cantor measures μ_i on E_i and let μ be the infinite product measure $\prod_{i=1}^{\infty}\mu_i$ on $\prod_{i=1}^{\infty}E_i$. Let $U\subseteq\prod_{i=1}^{\infty}E_i$ be any Borel set with

$$2^{-n-1}<\operatorname{diam}U\leq 2^{-n}$$

and choose $a=(a_i)\in U$. If $x=(x_i)$ also belongs to U, then since

$$d(a,x)=\sum_{i=1}^{\infty}|a_i-x_i|\,2^{-i}\leq \operatorname{diam}U\leq 2^{-n}$$

it must be the case that $x_i\in B_i(a_i,2^{-n+i})$ (where the subscript i on the ball denotes a ball in coordinate i). Hence

$$U\subseteq B_1(a_1,2^{-n+1})\times\cdots\times B_n(a_n,1)\times\prod_{i=n+1}^{\infty}E_i$$

and therefore $\mu(U)\leq\prod_{i=1}^{n}\mu_i(B_i(a_i,2^{-n+i}))$. Of course $\mu_i(B_i(a_i,2^{-n+i}))\leq 1$. We will obtain better bounds of the measure of $B_i(a_i,2^{-n+i})$ by determining how many Cantor intervals of a suitable size the ball intersects.

We claim that any subset V of E_i with diameter at most $2^{-(m+1)}$ can intersect at most two Cantor intervals of step $L(m,i)$. The reason for this is because if V intersected three such intervals, then V would contain both a Cantor interval of step $L(m,i)$ and the adjacent gap created at that step in the construction, and thus would have length at least half the length of a Cantor interval of step $L(m,i)-1$. But the definition of $L(m,i)$ guarantees that the Cantor intervals of step $L(m,i)-1$ must have length greater than 2^{-m}, and so this is a contradiction. Thus $\mu_i(V) \le 2^{-L(m,i)+1}$.

Applying the claim to $B_i(a_i, 2^{-n+i})$, which has diameter $2^{-(n-i-1)}$, we derive the bound

$$\mu_i(B_i(a_i, 2^{-n+i})) \le \begin{cases} 2^{-L(n-i-2,i)+1} & \text{if } i < n-2\,, \\ 1 & \text{else,} \end{cases}$$

and consequently,

$$\begin{aligned} \mu(U) &\le \prod_{i=1}^{n} \mu_i(B_i(a_i, 2^{-n+i})) \le 2^{-\sum_{i=1}^{n-3}(L(n-i-2,i)-1)} \\ &\le (\operatorname{diam} U)^{\left(\sum_{i=1}^{n-3} L(n-i-2,i)-(n-3)\right)/(n+1)}. \end{aligned}$$

Let $t < \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1} L(n-i,i)$. For large enough n,

$$\frac{1}{n+1}\left(\sum_{i=1}^{n-3} L(n-i-2,i) - (n-3)\right) > t-1$$

and thus

$$\mu(U) \le (\operatorname{diam} U)^{t-1}.$$

The choice for t and the mass distribution principle imply

$$\dim_H \prod_{i=1}^{\infty} E_i \ge \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1} L(n-i,i) - 1.$$

(ii) Suppose first that all ratios are bounded above by $1/2-\tau$ for some $\tau > 0$. Then (in any of the Cantor sets E_i) the ratio of the lengths of the gaps of step $k+1$ to the length of the step k intervals is at least 2τ. Two adjacent Cantor intervals of step $L(m,i)$ are always separated by a gap of (some) step $L \le L(m,i)$, and since any Cantor interval of E_i of step $L-1$ has length exceeding 2^{-m}, the length of this gap of step L must be greater than $2^{-m}2\tau$.

One consequence of this fact is that if V is a subset of E_i with $\operatorname{diam} V \le 2^{-m}2\tau$, then V cannot contain any gap of step $L \le L(m,i)$, and therefore can intersect at most one Cantor interval of step $L(m,i)$. Thus $\mu_i(V) \le 2^{-L(m,i)}$.

Choose an integer J such that $2^J\tau \ge 1$. Since the balls $B_i(a_i, 2^{-n+i})$ have diameter less than $2^{-n+i+J}2\tau$ we obtain the bound

$$\mu_i(B_i(a_i, 2^{-n+i})) \le \begin{cases} 2^{-L(n-i-J,i)} & \text{if } i < n-J\,, \\ 1 & \text{else.} \end{cases}$$

Thus if $U \subseteq \prod_{i=1}^{\infty} E_i$ is any Borel set with

$$2^{-n-1} < \operatorname{diam} U \leq 2^{-n},$$

then since

$$U \subseteq B_1(a_1, 2^{-n+1}) \times \cdots \times B_n(a_n, 1) \times \prod_{i=n+1}^{\infty} E_i$$

we have

$$\begin{aligned} \mu(U) &\leq \prod_{i=1}^{n} \mu_i(B_i(a_i, 2^{-n+i})) \leq 2^{-\sum_{i=1}^{n-J-1} L(n-i-J,i)} \\ &\leq (\operatorname{diam} U)^{\sum_{i=1}^{n-J-1} L(n-i-J,i)/(n+1)}. \end{aligned}$$

The mass distribution principle and the first part of the theorem yields the inequalities

$$\dim_H \prod_{i=1}^{\infty} E_i \geq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i) \geq \underline{\dim}_B \prod_{i=1}^{\infty} E_i.$$

Since the Hausdorff dimension is always dominated by the lower box dimension, the two dimensions coincide and their common value is $\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i)$.

Now suppose the gap lengths decrease. If we can prove that $\mu_i(V) \leq 2^{-L(m,i)}$ whenever V is any subset of E_i with diameter less than $2^{-(m+2)}$, then the same type of arguments as used above will show that the Hausdorff and lower box dimensions of $\prod_{i=1}^{\infty} E_i$ agree and again equal $\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i)$.

So assume the converse, i.e. $\mu_i(V) > 2^{-L(m,i)}$ for some V of diameter less than $2^{-(m+2)}$. Then V must intersect at least two Cantor intervals of step $L(m,i)$ and therefore will contain the gap between them, a gap of some step $L \leq L(m,i)$. Moreover, we claim that V must also contain at least one step $L(m,i)+1$ interval, for otherwise $V \bigcap E_i$ would be contained in the union of two Cantor intervals of step $L(m,i)+1$, and hence $\mu_i(V) \leq 2^{-(L(m,i)+1)+1}$. Any Cantor interval of step $L(m,i)-1$ contains a gap of step $L(m,i)$, two gaps of step $L(m,i)+1$ and 4 intervals of step $L(m,i)+1$. Thus the assumption of decreasing gap lengths ensures that the length of a step $L(m,i)-1$ interval is at most $4 \operatorname{diam} V$. Given the definition of $L(m,i)$, this implies that the diameter of V is at least $2^{-(m+2)}$ which is a contradiction.

To complete the proof it remains only to determine the upper box dimension if either the ratios are bounded away from 1/2 or the gap lengths decrease.

Again we use the method of counting numbers of disjoint balls with centres in the set. We let W_n' denote the set of $(w_i) \in \prod_{i=1}^{\infty} E_i$ with $w_i = 0$ if $i \geq n$ and w_i a left endpoint of an interval of step $L(n-i,i)$ if $i < n$.

Observe that any two left endpoints of step $L(m,i)$ intervals are separated by (at least) an interval of step $L(m,i)$ plus a gap of some step $L \leq L(m,i)$. As we remarked earlier in the proof, if the ratios are bounded by $1/2 - \tau$ for some $\tau > 0$, then any gap of step $L \leq L(m,i)$ will have length at least $2^{-m}2\tau$. Hence, under this assumption, we can conclude that if $v, w \in W_n'$ and $v \neq w$, then $d(v,w)$ is at least $2^{-n}2\tau$.

Otherwise, notice that since an interval of step $L(m,i) - 1$ is the union of two step $L(m,i)$ intervals and a gap of step $L(m,i)$, the combined lengths of an interval and gap of step $L(m,i)$ is at least $2^{-m}/2$. Thus if we assume the gaps are decreasing in length, then any two left endpoints of step $L(m,i)$ intervals will be separated by at least 2^{-m-1}. So this hypothesis allows one to conclude that if $v, w \in W_n'$ and $v \neq w$, then $d(v,w)$ is at least 2^{-n-1}.

Taking $\delta < 2^{-n+1}\tau$ in the first case and $\delta < 2^{-n-1}$ in the second case, it follows that the balls of radius $\delta/2$ and centred at points in W_n' are disjoint, and therefore $N_\delta \left(\prod_{i=1}^{\infty} E_i\right) \geq |W_n'| = \prod_{i=1}^{n-1} 2^{L(n-i,i)}$. Together with (i) this gives

$$\overline{\dim}_B \prod_{i=1}^{\infty} E_i = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i). \qquad \square$$

Some easy corollaries follow from this theorem.

Corollary 5. *If $E_i \subseteq [0,1]$ are Cantor sets and $\underline{\dim}_B \prod_{i=1}^{\infty} E_i = \infty$, then*

$$\dim_H \prod_{i=1}^{\infty} E_i = \infty.$$

Proof. In fact, the theorem implies that

$$\dim_H \prod_{i=1}^{\infty} E_i \geq \underline{\dim}_B \prod_{i=1}^{\infty} E_i - 1\,. \qquad \square$$

Of course, the converse is true for all sets.

Corollary 6. *If $E_i \subseteq [0,1]$ are uniform Cantor sets, then*

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i = \dim_H \prod_{i=1}^{\infty} E_i$$

Proof. Without loss of generality we may assume the ratios of dissection tend to zero, for otherwise $\dim_H \prod_{i=1}^{\infty} E_i \geq \sum_{i=1}^{\infty} \dim_H E_i = \infty$ and the result is clear. Thus the ratios are bounded away from 1/2 and the theorem can be invoked. □

The next theorem improves upon this corollary. We will show that for products of uniform Cantor sets the upper and lower box dimensions agree, and we will derive a simple formula for the common value. In contrast to the finite product case, the dimension of the infinite product of Cantor sets is never equal to the sum of the dimensions of the Cantor sets.

Theorem 7. *If $E_i \subseteq [0,1]$ are uniform Cantor sets, then*

$$\dim_B \prod_{i=1}^{\infty} E_i = \dim_H \prod_{i=1}^{\infty} E_i = 1 + \sum_{i=1}^{\infty} \dim E_i.$$

Proof. Let $r_i = 2^{-k_i}$ denote the ratio of dissection of E_i. Then $\dim_H E_i = 1/k_i$ and, as noted above, there is no loss of generality in assuming that $\sum_{i=1}^{\infty} 1/k_i$ converges. Thus the theorem implies that

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i = \dim_H \prod_{i=1}^{\infty} E_i = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i)$$

(where $L(n,i)$ is as defined in the statement of the previous theorem) and

$$\overline{\dim}_B \prod_{i=1}^{\infty} E_i = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i).$$

Let $S(n) = \{i : k_i \leq n\}$ and suppose there is a subsequence $\{n_j\}$ with $|S(n_j)|/n_j \geq \tau > 0$. By choosing a further subsequence if necessary (not renamed) we may assume that $|S(n_j)| - |S(n_{j-1})| \geq |S(n_j)|/2$. Thus

$$\sum_{i=1}^{\infty} \frac{1}{k_i} = \sum_{j=1}^{\infty} \sum_{i\in S(n_j)\setminus S(n_{j-1})} \frac{1}{k_i} \geq \sum_{j=1}^{\infty} \frac{|S(n_j)|}{2n_j} = \infty\,,$$

which is a contradiction since $\{1/k_i\}$ is a summable sequence. Therefore $|S(n)|/n \to 0$.

Since a Cantor interval of E_i of step L has length $2^{-k_i L}$, it follows that for $i < n$ $L(n-i,i)$ is the least integer greater or equal to $(n-i)/k_i$. Thus

$$\frac{1}{n} \sum_{i=1}^{n-1} L(n-i,i) \leq \frac{1}{n} \sum_{i=1}^{n-1} \left(\frac{n-i}{k_i}\right) + 1.$$

Moreover, when $k_i > n$ (i.e., $i \notin S(n)$) then $L(n,i) = 1$, hence

$$\frac{1}{n}\sum_{i=1}^{n-1} L(n-i,i) \geq \frac{1}{n}\left(\sum_{i=1;i\in S(n)}^{n-1}\left(\frac{n-i}{k_i}\right) + \sum_{i=1;i\notin S(n)}^{n-1} 1\right)$$
$$= \frac{1}{n}\sum_{i=1;i\in S(n)}^{n-1}\left(\frac{n-i}{k_i}\right) + \frac{n-1-|S(n)\bigcap\{1,\ldots,n-1\}|}{n}.$$

Certainly,

$$\frac{1}{n}\sum_{i=1;i\in S(n)}^{n-1}\left(\frac{n-i}{k_i}\right) \leq \sum_{i=1}^{\infty}\frac{1}{k_i}.$$

Furthermore, given any positive integer m and $0 < \alpha < 1$ there exists N such that for all $n \geq N$

$$\{i \in S(n) : i \leq \alpha n\} \supseteq \{1,\ldots,m\},$$

and therefore

$$\frac{1}{n}\sum_{i=1;i\in S(n)}^{n-1}\left(\frac{n-i}{k_i}\right) \geq (1-\alpha)\sum_{i=1}^{m}\frac{1}{k_i}.$$

Because m and $\alpha \in (0,1)$ were arbitrary this yields

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1;i\in S(n)}^{n}\left(\frac{n-i}{k_i}\right) = \sum_{i=1}^{\infty}\frac{1}{k_i}.$$

But we have already seen that $|S(n)|/n \to 0$, hence these calculations show that the Hausdorff and box dimensions of $\prod_{i=1}^{\infty} E_i$ equal

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1} L(n-i,i) = \sum_{i=1}^{\infty}\frac{1}{k_i} + 1 = \sum_{i=1}^{\infty}\dim E_i + 1. \qquad \square$$

It is a classical result that almost every projection of a fractal subset of $\mathbb{R}^n$ onto a sufficiently large subspace preserves Hausdorff dimension (c.f. [7], [8]). In the Banach space setting this is false; however, it is still true that almost every continuous projection maps sets of positive dimension onto sets of positive dimension [6]. In contrast, our techniques can be used to show that there are subsets of $\mathbb{R}^\infty$ of positive, finite dimension, which are mapped by every continuous linear operator onto zero dimensional sets.

Example 3.1. For $i = 1, 2, \ldots$ let E_i be the Cantor set in $[0,1]$ with ratios of dissection $2^{-2^{k+i}}$ at step k, and therefore upper box dimension zero. An application of Theorem 4 shows that $\dim_H \prod_{i=1}^{\infty} E_i = \dim_B \prod_{i=1}^{\infty} E_i = 1$ (and, in particular, is finite and positive). Since the vectors with only finitely many non-zero coordinates are dense in $\mathbb{R}^{\infty}$ with the product topology, it follows that any continuous linear operator α from $\mathbb{R}^{\infty}$ to $\mathbb{R}^N$ must be given by $\alpha(x) = \sum_{i=1}^{m} \alpha_i x_i$ for some collection of vectors $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^N$. Thus $\alpha(\prod_{i=1}^{\infty} E_i) = \alpha(\prod_{i=1}^{m} E_i)$ (where the action of α on the finite product space is understood in the natural way). Since α is a Lipschitz map this means that

$$\dim_H \alpha(\prod_{i=1}^{\infty} E_i) = \dim_H \prod_{i=1}^{m} E_i \leq \overline{\dim}_B \prod_{i=1}^{m} E_i \leq \sum_{i=1}^{m} \overline{\dim}_B E_i.$$

Thus, regardless of the choice of N, the image of $\prod_{i=1}^{\infty} E_i$ has Hausdorff dimension zero.

4. Dimensions of Products of Self-Similar Sets

Theorem 7 cannot be extended to products of self-similar sets (even replacing 1 by some other constant) since we have already seen (Example 2.2) that there are self-similar sets $E_i \subseteq [0,1]$ with $\dim_H \prod_{i=1}^{\infty} E_i = \infty$ and $\sum_{i=1}^{\infty} \dim E_i < \infty$.

One could ask, however, if the box and Hausdorff dimensions of an infinite product of self-similar sets coincide, as is the case for finite products. We show next that this is not true since the upper and lower box dimensions need not agree. We have not been able to determine if the lower box dimension must always equal the Hausdorff dimension.

Proposition 8. *There are self-similar sets E_i satisfying the open set condition with*

$$\overline{\dim}_B \prod_{i=1}^{\infty} E_i > \underline{\dim}_B \prod_{i=1}^{\infty} E_i \geq \dim_H \prod_{i=1}^{\infty} E_i > \sum_{i=1}^{\infty} \dim E_i.$$

Proof. Set $n_1 = 2$ and inductively define an increasing sequence of even integers $\{n_k\}$ satisfying $n_k \geq k n_{k-1}$ and $k n_k 8^{-n_k/2} \leq 2^{-k}$. For any integer $i \in (n_k/2, n_k]$ let $E_i \subseteq \mathbb{R}$ be the self-similar set generated by the iterated function system $\{f_{i,1}, \ldots, f_{i,2^k}\}$, where $f_{i,j} = 2^{-8^i} x + (j-1)2^{-k}$. For $i \notin (n_k/2, n_k]$ let E_i be the self-similar set generated by the two functions $x \longmapsto 2^{-8^i} x$ and $x \longmapsto 2^{-8^i} x + 1/2$.

As all the sets E_i contain 0 and 1/2, Proposition 2 implies that $\dim_H \prod_{i=1}^\infty E_i \geq 1$. The open set condition is satisfied for all E_i, thus $\dim E_i = k8^{-i}$ if $i \in (n_k/2, n_k]$ and equals 8^{-i} otherwise. Together with the definition of $\{n_k\}$ this gives

$$\sum_{i=1}^\infty \dim E_i = \sum_{i \in \bigcup_k (n_k/2, n_k]} \frac{k}{8^i} + \sum_{i \notin \bigcup_k (n_k/2, n_k]} \frac{1}{8^i}$$
$$\leq \frac{1}{2}\sum_{k=1}^\infty k n_k 8^{-n_k/2} + \sum_{i=1}^\infty 8^{-i} < 1 \leq \dim_H \prod_{i=1}^\infty E_i.$$

Thus it suffices to establish that the lower box dimension of $\prod_{i=1}^\infty E_i$ is finite, while the upper box dimension is infinite.

Let $F_i = \{0, 2^{-k}, \ldots, 1-2^{-k}\}$ for $i \in (n_k/2, n_k]$ and $F_i = \{0, 1/2\}$ otherwise; note that $F_i \subseteq E_i$. We will actually prove that even $\overline{\dim}_B \prod_{i=1}^\infty F_i$ is infinite by the familiar method of counting disjoint balls whose centres are in $\prod_{i=1}^\infty F_i$. To do this, for each k let W_k denote the set of sequences $w = (w_i) \in \prod_{i=1}^\infty F_i$ with $w_i = 0$ for all $i \notin (n_k/2, n_k]$. Obviously, if $w, v \in W_k$ and $w \neq v$, then $d(v,w) \geq 2^{-k}2^{-i}$ for some $i \in (n_k/2, n_k]$, and hence $d(v,w) \geq 2^{-k-n_k}$. Since there are $2^{kn_k/2}$ such sequences in W_k it follows that

$$\overline{\dim}_B \prod_{i=1}^\infty E_i \geq \overline{\dim}_B \prod_{i=1}^\infty F_i \geq \limsup_{k\to\infty} \frac{kn_k/2}{(k+n_k)} = \infty.$$

To obtain an upper bound for the lower box dimension we consider 2^{-m+1} -coverings of $\prod_{i=1}^\infty E_i$. The same arguments as used before show that

$$N_{2^{-m+1}}(\prod_{i=1}^\infty E_i) \leq \prod_{i=1}^m N_{2^{-m}}(E_i).$$

Let $L(m,i)$ be the minimal integer such that $2^{-8^i L(m,i)} \leq 2^{-m}$; the set E_i can be covered by $2^{kL(m,i)}$ intervals of length at most 2^{-m} if $i \in (n_k/2, n_k]$ and $2^{L(m,i)}$ intervals otherwise. Choosing $m = n_{N+1}/2$ gives

$$N_{2^{-m+1}}(\prod_{i=1}^\infty E_i) \leq \prod_{k=1}^N \prod_{i \in (n_k/2, n_k]} 2^{kL(m,i)} \prod_{i \notin \bigcup_k (n_k/2, n_k]; i \leq m} 2^{L(m,i)}.$$

As $L(m,i) \le m/8^i + 1$ this yields the bound

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i \le \liminf_{m\to\infty} \frac{1}{m} \left(\sum_{k=1}^{N} \sum_{i\in(n_k/2,n_k]} k\left(\frac{m}{8^i}+1\right) + \sum_{i\notin\bigcup_k (n_k/2,n_k];i\le m} \left(\frac{m}{8^i}+1\right) \right)$$

Clearly,

$$\frac{1}{m}\sum_{k=1}^{N} \sum_{i\in(n_k/2,n_k]} k\left(\frac{m}{8^i}+1\right) \le \sum_{i\in\bigcup_k (n_k/2,n_k]} \frac{k}{8^i} + \sum_{k=1}^{N} \frac{kn_k}{2m}$$

and

$$\frac{1}{m} \sum_{i\notin\bigcup_k (n_k/2,n_k];i\le m} \left(\frac{m}{8^i}+1\right) \le \sum_{i\notin\bigcup_k (n_k/2,n_k]} \frac{1}{8^i} + 1,$$

thus

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i \le \left(\sum_{i\in\bigcup_k (n_k/2,n_k]} \frac{k}{8^i} + \sum_{i\notin\bigcup_k (n_k/2,n_k]} \frac{1}{8^i} \right) + \liminf_{m\to\infty} \sum_{j=1}^{N} \frac{kn_k}{2m} + 1.$$

Recall that the bracketed part of the expression above is equal to $\sum_{i=1}^{\infty} \dim E_i$. Also, the lacunarity of the sequence $\{kn_k\}$ ensures that

$$\sum_{k=1}^{N} \frac{kn_k}{2m} \le \frac{2Nn_N}{n_{N+1}} \le 2.$$

Thus we obtain

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i \le \sum_{i=1}^{\infty} \dim E_i + 3,$$

and hence $\underline{\dim}_B \prod_{i=1}^{\infty} E_i$ is finite as claimed. □

Corollary 9. *There are finite sets F_i with*

$$0 < \underline{\dim}_B \prod_{i=1}^{\infty} F_i < \overline{\dim}_B \prod_{i=1}^{\infty} F_i = \infty.$$

Proof. Just take the finite sets F_i introduced in the proof of the proposition. There it was shown that $\overline{\dim}_B \prod_{i=1}^{\infty} F_i = \infty$. As each F_i contains $\{0, 1/2\}$, $\underline{\dim}_B \prod_{i=1}^{\infty} F_i \geq 1$, and because F_i is contained in the set E_i of the proposition, $\underline{\dim}_B \prod_{i=1}^{\infty} F_i \leq \underline{\dim}_B \prod_{i=1}^{\infty} E_i < \infty$. □

We also observe that by relaxing the requirement that a uniform Cantor set must contain $0, 1$ we can construct self-similar sets with

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} \dim E_i < \infty.$$

Example 4.1. Let E_i be the self-similar sets generated by the iterated function systems

$$\{2^{-2^i} x, 2^{-2^i} x + 2^{-2^i} - 2^{-2^{i+1}}\}.$$

Each set E_i has a Cantor-like construction, with initial interval $[0, 2^{-2^i}]$ and ratio of dissection 2^{-2^i}. Thus $\dim E_i = 1/2^i$.

We consider $\delta \equiv 2^{1-2^k}$-coverings of $\prod_{i=1}^{\infty} E_i$. Notice that for $i < k$, the Cantor-like intervals of E_i of step $2^{k-i} - 1$ have length 2^{-2^k}, while for $i \geq k$ a single interval of length 2^{-2^k} covers E_i. Thus

$$\log N_\delta (\prod_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{k} (2^{k-i} - 1) \log 2,$$

and therefore

$$\underline{\dim}_B \prod_{i=1}^{\infty} E_i \leq \liminf_{k \to \infty} \frac{\sum_{i=1}^{k} (2^{k-i} - 1)}{2^k} = \sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{i=1}^{\infty} \dim E_i.$$

Since it is always true that $\underline{\dim}_B \prod_{i=1}^{\infty} E_i \geq \sum_{i=1}^{\infty} \dim E_i$, we have equality.

5. Topologically Equivalent Metrics

More generally, one could consider $\mathbb{R}^\infty$ with metric

$$d^\phi(x, y) = \sum_{i=1}^{\infty} \phi(i) \min\{1, |x_i - y_i|\},$$

where $\{\phi(i)\}$ is any summable sequence of positive numbers. The choice with weight $\phi(i) = 2^{-i}$ is the standard Frechet metric we studied in the previous sections, however all these metrics give the product topology on $\mathbb{R}^\infty$. Of course, one could similarly consider $\mathbb{R}^n$ with weighted metrics $d(x,y)=\sum_{i=1}^n \phi(i)\left|x_i-y_i\right|$, but in Euclidean space such choices for $\{\phi(i)\}_{i=1}^n$ give Lipschitz equivalent metrics.

To emphasize the dependence of the dimension upon the weight function ϕ we will write $\dim^\phi$ in this section. Note that monotonicity implies that $\dim_H^\phi \prod_{i=1}^\infty E_i \geq \dim_H^\phi \prod_{i=1}^n E_i$, and the latter dimension is equal to $\dim_H \prod_{i=1}^n E_i$ by the Lipschitz equivalence of $d^\phi|_{\mathbb{R}^n}$ to the usual metric. But $\dim_H \prod_{i=1}^n E_i \geq \sum_{i=1}^n \dim_H E_i$, thus one relationship which holds for all weights ϕ is

$$\underline{\dim}_B^\phi \prod_{i=1}^\infty E_i \geq \dim_H^\phi \prod_{i=1}^\infty E_i \geq \sum_{i=1}^\infty \dim_H E_i.$$

5.1. Box Dimension of Products

Our first result for general weights can be viewed as an infinite dimensional analogue of Proposition 1 (ii).

Proposition 10. *Given any sets $E_i \subset [0,1]$ and $\varepsilon>0$, there is a weight function ϕ such that*

$$\overline{\dim}_B^\phi \prod_{i=1}^\infty E_i \leq \sum_{i=1}^\infty \overline{\dim}_B E_i + \varepsilon.$$

Remark 5.1. It follows that if $\overline{\dim}_B E_i = 0$ for all i, then for a suitable choice of weight function ϕ, $\overline{\dim}_B^\phi \prod_{i=1}^\infty E_i$ is arbitrarily small. If $\sum_{i=1}^\infty \overline{\dim}_B E_i < \infty$, then ϕ can be chosen with $\overline{\dim}_B^\phi \prod_{i=1}^\infty E_i < \infty$.

Proof. Choose $\varepsilon_i>0$ such that $\sum_{i=1}^\infty \varepsilon_i<\varepsilon$ and let $\overline{\dim}_B E_i \equiv d_i$. The definition of the upper box dimension ensures that $\log N_{2^{-n}}(E_i) \leq (d_i+\varepsilon_i)n \log 2$ if n is sufficiently large, say $n \geq M_i$. If necessary, redefine $\{M_i\}$ so that it is a strictly increasing sequence of integers and let ϕ be the weight function $\phi(i)=4^{-M_i}$.

Given $n \geq 1$ choose $J \equiv J(n)$ such that $M_{J-1}<n \leq M_J$ and suppose that intervals $I_i \subseteq E_i$ have diameter less than 2^{-n} for $i=1,\ldots,J-1$. Then the ϕ-diameter of $I_1 \times \cdots \times I_{J-1} \times \prod_{i=J}^\infty E_i$ is bounded by

$$\sum_{i=1}^{J-1} 2^{-n} 4^{-M_i}+\sum_{i=J}^\infty 4^{-M_i} \leq 2^{-n}/3+4^{-M_J+1}/3 \leq 2^{-n}.$$

When $i \leq J-1$, then $n>M_i$ and therefore $N_{2^{-n}}(E_i) \leq 2^{n(d_i+\varepsilon_i)}$. Thus $\prod_{i=1}^\infty E_i$ is covered by $\prod_{i=1}^{J-1} N_{2^{-n}}(E_i) \leq 2^{n \sum_{i=1}^{J-1}(d_i+\varepsilon_i)}$ sets of ϕ -diameter less than 2^{-n}

and hence

$$\overline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i \leq \limsup_{n\to\infty} \frac{\log 2^{n\sum_i (d_i+\varepsilon_i)}}{n\log 2} \leq \sum_{i=1}^{\infty} \overline{\dim}_B E_i + \varepsilon. \qquad \square$$

For products of self-similar sets this result can be improved; note in particular the contrast between the proposition below and Theorem 7 and Proposition 8.

Proposition 11. *Let $E_i \subseteq [0,1]$ be self-similar sets satisfying the open set condition. Then there is a weight function ϕ such that*

$$\dim_B^{\phi} \prod_{i=1}^{\infty} E_i = \dim_H^{\phi} \prod_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} \dim E_i.$$

Proof. Suppose E_i is generated by the iterated function system $\{F_{ij}\}_{j=1}^{N_i}$ and choose a non-empty, bounded, open set V_i such that the iterated sets, $F_{ij}(V_i)$, $j = 1, \ldots, N_i$, are pairwise disjoint and contained in V_i. Let t_i be the minimum contraction factor of the functions $\{F_{ij}\}_{j=1}^{N_i}$. For each $n \geq |\log t_1|$ let m_n be the maximal integer such that $\sum_{i=1}^{m_n} |\log t_i| \leq n$. Define the weight ϕ so that $\phi(m_n) \leq 2^{-2^n}$, $\phi(i)\operatorname{diam} V_i \leq 2^{-i}$ and $\phi(k+1) \leq \phi(k)/2$.

In the proof of Theorem 9.3 of [4] it is shown that if $\dim E_i = s_i$, then given any $r < 1$, E_i may be covered by $[t_i^{-s_i} r^{-s_i}]$ sets of diameter at most $r\mathrm{diam}V_i$. We will use this fact to bound the upper box dimension of $\prod_{i=1}^{\infty} E_i$.

Let $\delta > 0$ and choose n such that $\phi(m_{n+1}) < \delta \leq \phi(m_n)$. Applying the above remark with $r = \delta$, it follows that $\prod_{i=1}^{\infty} E_i$ can be covered by $\prod_{i=1}^{m_{n+1}} \left[t_i^{-s_i}\delta^{-s_i}\right]$ sets of diameter at most

$$\sum_{i=1}^{m_{n+1}} \phi(i)\delta \operatorname{diam} V_i + \sum_{i=m_{n+1}+1}^{\infty} \phi(i) \leq \delta + \phi(m_{n+1}) \leq 2\delta.$$

Thus

$$\frac{\log N_{2\delta}\left(\prod_{i=1}^{\infty} E_i\right)}{|\log 2\delta|} \leq \frac{\sum_{i=1}^{m_{n+1}} \left(s_i\,|\log t_i| + s_i\,|\log \delta)|\right)}{|\log 2\delta|}.$$

Of course, $s_i \leq 1$ as $E_i \subseteq \mathbb{R}$, hence we obtain the inequalities

$$\begin{aligned} \frac{\log N_{2\delta}\left(\prod_{i=1}^{\infty} E_i\right)}{|\log 2\delta|} &\leq \frac{\sum_{i=1}^{m_{n+1}} |\log t_i|}{|\log 2\phi(m_n)|} + \frac{\sum_{i=1}^{m_{n+1}} s_i}{1 - \log 2/\,|\log \phi(m_n)|} \\ &\leq \frac{n+1}{(2^n-1)\log 2} + \frac{\sum_{i=1}^{m_{n+1}} s_i}{1 - 2^{-n}\log 2}. \end{aligned}$$

Letting $\delta \to 0$ gives

$$\limsup_{\delta\to 0} \frac{\log N_{2\delta}(\prod_{i=1}^{\infty} E_i)}{|\log 2\delta|} \leq \sum_{i=1}^{\infty} s_i = \sum_{i=1}^{\infty} \dim E_i.$$

Therefore $\overline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i \leq \sum_{i=1}^{\infty} \dim E_i$. But since all weights yield

$$\sum_{i=1}^{\infty} \dim E_i \leq \dim_H^{\phi} \prod_{i=1}^{\infty} E_i \leq \overline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i$$

we have equality throughout. □

5.2. Uniform Cantor Sets

One can also prove results similar to Theorem 4 and Theorem 7 for arbitrary weights. We state the results for products of uniform Cantor sets as these are the most interesting.

Notation. Throughout this subsection, $\phi = \{\phi(i)\}$ will be a summable, decreasing sequence of positive numbers and $A(n)$ will denote the least integer such that $\sum_{i=A(n)+1}^{\infty} \phi(i) \leq 2^{-n}$. Obviously, we always have $|\{i : \phi(i) > 2^{-n}\}| \leq A(n)$, and when ϕ is the Frechet metric, then $A(n) = n = |\{i : \phi(i) > 2^{-n}\}|$.

Lemma 12. $\limsup_{n\to\infty} |\{i : \phi(i) > 2^{-n}\}| /n$
$= \limsup_{n\to\infty} A(n)/n.$

Proof. If $\limsup_{n\to\infty} |\{i : \phi(i) > 2^{-n}\}| /n = \infty$ then we are obviously done. So assume

$$\limsup_{n\to\infty} \left|\{i : \phi(i) > 2^{-n}\}\right| /n = C < \infty,$$

and fix $\varepsilon > 0$. Because ϕ is decreasing,

$$\phi(n) \leq 2^{1-(n-1)/(C+\varepsilon)}$$

when n is sufficiently large. A routine calculation shows that

$$\limsup_{n\to\infty} A(n)/n \leq C + \varepsilon.$$ □

Proposition 13. *Suppose $E_i \subseteq [0,1]$, $i = 1, 2, \ldots$ are uniform Cantor sets. Then:*

(i) $\overline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} \dim E_i + \limsup_{n\to\infty} |\{i : \phi(i) > 2^{-n}\}| /n$,

(ii) $\dim_H^{\phi} \prod_{i=1}^{\infty} E_i \geq \sum_{i=1}^{\infty} \dim E_i + \liminf_{n\to\infty} |\{i : \phi(i) > 2^{-n}\}| /n$ *and*

(iii) $\underline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i \leq \sum_{i=1}^{\infty} \dim E_i + \liminf_{n\to\infty} A(n)/n.$

Proof. The arguments are similar to those used in the proofs of Theorems 4 and 7. We will assume the ratio of dissection of E_i is 2^{-k_i} and define $L(n,i)$ as in 4. For convenience, write $\phi(i) = 2^{-f(i)}$.

First we prove (ii). For this we may clearly assume $\sum_{i=1}^{\infty} \dim E_i < \infty$ since $\dim_H^{\phi} \prod_{i=1}^{\infty} E_i \geq \sum_{i=1}^{\infty} \dim E_i$. Hence there is some $\tau > 0$ satisfying $2^{-k_i} \leq 1/2 - \tau$ for all i. Choose a positive integer J such that $2^J \tau \geq 1$.

Assume $U \subseteq \prod_{i=1}^{\infty} E_i$ and $\operatorname{diam} U \in (2^{-(n+1)}, 2^{-n}]$. Routine arguments show that

$$U \subseteq \prod_{i=1}^{\infty} B_i(a_i, 2^{-n+f(i)}),$$

where (a_i) is a (fixed) point in U (notice $B_i(a_i, 2^{-n+f(i)}) \supseteq E_i$ if $f(i) \geq n$). If $f(i) < n - J$, then the gaps separating Cantor intervals of E_i of step $L(n - f(i) - J, i)$ have length at least $2\tau 2^{-n+f(i)+J}$. This dominates the diameter of $B_i(a_i, 2^{-n+f(i)})$, thus $B_i(a_i, 2^{-n+f(i)})$ can intersect only one Cantor interval of step $L(n - f(i) - J, i)$. Hence if μ_i denotes the uniform Cantor measure on E_i and μ the infinite product measure on $\prod_{i=1}^{\infty} E_i$, then

$$\mu_i(B_i(a_i, 2^{-n+f(i)})) \leq \begin{cases} 2^{-L(n-f(i)-J,i)} & \text{if } f(i) < n - J\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

and therefore

$$\mu(U) \leq (\operatorname{diam} U)^{\sum_i L(n-f(i)-J,i)/(n+1)},$$

where the sum is taken over those i such that $f(i) < n - J$. An application of the mass distribution principle shows that

$$\dim_H^{\phi} \prod_{i=1}^{\infty} E_i \geq \liminf_{n\to\infty} \frac{1}{n} \sum_{i:f(i)<n-J} L(n - f(i) - J, i).$$

Since $\sum_{i=1}^{\infty} \dim E_i < \infty$, one can apply arguments very similar to those used in Theorem 7 to show that

$$\begin{aligned} & \liminf_{n\to\infty} \frac{1}{n} \sum_{i:f(i)<n-J} L(n - f(i) - J, i) \\ = \; & \liminf_{n\to\infty} \sum_{i:f(i)<n-J, k_i \leq n} \frac{n - f(i) - J}{n k_i} + \frac{|\{i : f(i) < n - J\}|}{n} \\ = \; & \sum_{i=1}^{\infty} \dim E_i + \liminf_{n\to\infty} \left|\{i : \phi(i) > 2^{-n}\}\right| / n, \end{aligned}$$

completing the proof of (ii).

For (iii) we consider, as usual, coverings of $\prod_{i=1}^{\infty} E_i$ by Cantor intervals. If $\operatorname{diam} I_j \leq 2^{-n}$ then the definition of $A(n)$ gives that

$$\begin{aligned}\operatorname{diam} I_1 \times \cdots \times I_{A(n)} \times \prod_{i=A(n)+1}^{\infty} E_i &\leq \sum_{i=1}^{A(n)} 2^{-n}\phi(i) + \sum_{i=A(n)+1}^{\infty} \phi(i) \\ &\leq 2^{-n}(\sum_{i=1}^{\infty} \phi(i) + 1) \equiv 2^{-n}(S+1).\end{aligned}$$

Thus $N_{(S+1)2^{-n}}(\prod_{i=1}^{\infty} E_i) \leq \prod_{i=1}^{A(n)} 2^{L(n,i)}$ and therefore

$$\begin{aligned}\underline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i &\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{A(n)} L(n,i) \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{A(n)} \left(\frac{n}{k_i} + 1\right) \\ &= \sum_{i=1}^{\infty} \dim E_i + \liminf_{n\to\infty} A(n)/n.\end{aligned}$$

Proving

$$\overline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i \leq \sum_{i=1}^{\infty} \dim E_i + \limsup_{n\to\infty} A(n)/n$$

is similar, thus in order to complete the proof of (i) it suffices to verify that

$$\overline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i \geq \sum_{i=1}^{\infty} \dim E_i + \limsup_{n\to\infty} \left|\{i : \phi(i) > 2^{-n}\}\right| /n.$$

To show this we will again use the method of counting disjoint balls and we may assume that $\sum_{i=1}^{\infty} \dim E_i < \infty$. Note that the definition of $L(m,i)$ and choice of J ensures that both the Cantor intervals of step $L(m,i) - 1$ and the gaps between those intervals have length at least 2^{-m-J+1}. Fix n and let W be the set of sequences $(w_i) \in \prod_{i=1}^{\infty} E_i$ such that w_i is an endpoint of a Cantor interval of E_i of step $L(n - f(i) - J, i) - 1$ if $f(i) \leq n - J$ and otherwise $w_i = 0$. The lengths of these intervals and the gaps between them are sufficiently long that if $v, w \in W$ and $v \neq w$, then $d(v,w) \geq 2^{-n+1}$. Thus

$$N_{2^{-n}}(\prod_{i=1}^{\infty} E_i) \geq |W| = \prod_{i:f(i)<n-J} 2^{L(n-f(i)-J,i)},$$

and reasoning as in Theorem 7 yields

$$\begin{aligned}\overline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i &\geq \limsup_{n\to\infty} \frac{1}{n} \sum_{i:f(i)<n-J} L(n-f(i)-J,i) \\ &= \sum_{i=1}^{\infty} \dim E_i + \limsup_{n\to\infty} \left|\{i : \phi(i) > 2^{-n}\}\right|/n\end{aligned}$$

as claimed. □

There are some immediate consequences of the proposition.

Corollary 14. (i) *If* $\lim_{n\to\infty} |\{i : \phi(i) > 2^{-n}\}|/n$ *exists, then*

$$\dim_H^{\phi} \prod_{i=1}^{\infty} E_i = \dim_B^{\phi} \prod_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} \dim E_i + \lim_{n\to\infty} \left|\{i : \phi(i) > 2^{-n}\}\right|/n.$$

(ii) *If* $\liminf_{n\to\infty} |\{i : \phi(i) > 2^{-n}\}|/n = \liminf_{n\to\infty} A(n)/n$, *then*

$$\dim_H^{\phi} \prod_{i=1}^{\infty} E_i = \underline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} \dim E_i + \liminf_{n\to\infty} \left|\{i : \phi(i) > 2^{-n}\}\right|/n.$$

(iii) *If* $\limsup_{n\to\infty} A(n)/n \neq \liminf_{n\to\infty} A(n)/n$ *and* $\sum_{i=1}^{\infty} \dim E_i < \infty$, *then* $\overline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i \neq \underline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i$.

The following is an example of a weight ϕ and uniform Cantor sets where the upper and lower box dimensions of the product differ.

Example 5.1. Set $n_1 = 1$ and inductively define a rapidly increasing sequence $\{n_k\}$ satisfying $n_k > kn_{k-1}$ and

$$2(k+1)n_{k+1}2^{-n_{k+1}} \leq \min\{kn_k 2^{-n_k}, 2^{-(k+1)n_k}\}.$$

Set $\phi(i) = 2^{-n_k+1}$ if $i \in (n_k, (k+1)n_k]$ for some k, and $\phi(i) = 2^{-i}$ otherwise. Clearly $|\{i : \phi(i) > 2^{-n_k}\}|/n_k \geq k$, so

$$\limsup_{n\to\infty} \left|\{i : \phi(i) > 2^{-n}\}\right|/n = \infty$$

and therefore, for any choice of uniform Cantor sets E_i, $\overline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i = \infty$.

One can calculate that

$$\begin{aligned}\sum_{i=(j+1)n_j+1}^{\infty} \phi(i) &\leq \sum_{k=j+1}^{\infty} \sum_{i\in(n_k,(k+1)n_k]} \phi(i) + \sum_{i=(j+1)n_j+1}^{\infty} 2^{-i} \\ &\leq 2^{-n_{j+1}+2}(j+1)n_{j+1} + 2^{-(j+1)n_j} \leq 2^{-(j+1)n_j+2}.\end{aligned}$$

Thus $A((j+1)n_j-2) \leq (j+1)n_j+1$ and therefore $\liminf_{n\to\infty} A(n)/n \leq 1$. If the Cantor sets are chosen with $\sum_{i=1}^{\infty} \dim E_i < \infty$, our previous results show that $\underline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i \leq \sum_{i=1}^{\infty} \dim E_i + 1 < \infty$.

Proposition 15. *Suppose $E_i \subseteq [0,1]$, $i = 1, 2, \ldots$ are uniform Cantor sets and assume $(\log A(n))/n \to 0$. Then*

$$\dim_H^{\phi} \prod_{i=1}^{\infty} E_i = \underline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} \dim E_i + \liminf_{n\to\infty} \left|\{i : \phi(i) > 2^{-n}\}\right| / n.$$

Proof. Let $B(n)$ be the maximum i such that $2^{-n} < \phi(i) \equiv 2^{-f(i)}$; as ϕ is decreasing $B(n) = |\{i : \phi(i) > 2^{-n}\}| \leq A(n)$. If $\operatorname{diam} I_j \leq 2^{-n+f(j)}$, then the definitions of $A(n)$ and $B(n)$ imply that

$$\begin{aligned}
&\operatorname{diam} I_1 \times \cdots \times I_{B(n)} \times \prod_{i=B(n)+1}^{\infty} E_i \\
&\qquad \leq \sum_{i=1}^{B(n)} 2^{-n} + \sum_{i=B(n)+1}^{A(n)} 2^{-f(i)} + \sum_{i=A(n)+1}^{\infty} 2^{-f(i)} \\
&\qquad \leq 2^{-n}B(n) + 2^{-n}(A(n)-B(n)) + 2^{-n} = 2^{-n}(A(n)+1).
\end{aligned}$$

Thus

$$N_{2^{-n}(A(n)+1)}\left(\prod_{i=1}^{\infty} E_i\right) \leq \prod_{i=1}^{B(n)} 2^{L(n-f(i),i)}$$

(where, as usual, $L(n,i)$ is the least integer such that the Cantor intervals of E_i of step $L(n,i)$ have length at most 2^{-n}). Since $2^{-n}A(n) \to 0$,

$$\underline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i \leq \liminf_{n\to\infty} \frac{\sum_{i=1}^{B(n)} L(n-f(i),i)\log 2}{|\log 2^{-n}(A(n)+1)|},$$

and because $(\log A(n))/n \to 0$ this yields

$$\begin{aligned}
\underline{\dim}_B^{\phi} \prod_{i=1}^{\infty} E_i &\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{B(n)} L(n-f(i),i) \\
&\leq \sum_{i=1}^{\infty} \dim_H E_i + \liminf_{n\to\infty} \left|\{i : \phi(i) > 2^{-n}\}\right| / n \leq \dim_H^{\phi} \prod_{i=1}^{\infty} E_i
\end{aligned}$$

according to Proposition 13. □

Corollary 16. (i) *If* $\sup A(n)/n < \infty$ *then*

$$\underline{\dim}_B^\phi \prod_{i=1}^\infty E_i = \dim_H^\phi \prod_{i=1}^\infty E_i.$$

(ii) *If* $\overline{\dim}_B^\phi \prod_{i=1}^\infty E_i < \infty$ *then* $\underline{\dim}_B^\phi \prod_{i=1}^\infty E_i = \dim_H^\phi \prod_{i=1}^\infty E_i$.

Proof. (i) If $\sup A(n)/n < \infty$ then $\log A(n)/n \to 0$.

(ii) The earlier results imply that if $\overline{\dim}_B^\phi \prod_{i=1}^\infty E_i < \infty$, then $\sup A(n)/n < \infty$. □

5.3. Order of Coordinates

Permuting the values of the weight function is equivalent to changing the order of the factors in the product. In the Euclidean case this is irrelevant; again the infinite dimensional setting is different.

Proposition 17. *There are sets* $E_i \subset [0,1]$ *such that the Hausdorff dimension (with respect to the usual weight* $\phi(i) = 2^{-i}$*) of* $\prod_{i=1}^\infty E_i$ *is infinite, but the Hausdorff dimension of a suitable rearrangement of the coordinates,* $\prod_{i=1}^\infty E_{\pi(i)}$, *is zero.*

Proof. We will let $E_i = \{0\}$ for i odd, and let E_{2i} be a Cantor set with step k Cantor intervals of length 4^{-k} for $k \le 2i$ and of length $4^{-2^{ik}}$ if $k > 2i$. Take π to be the mapping which sends 2^k to $2k$ and maps $\mathbb{N} \setminus \{2^k\}$ onto the odd integers in the order preserving way.

To see that $\dim_H \prod_i E_i = \infty$ we use the mass distribution principle. Let $\mu = \prod \mu_i$, where μ_i is the point mass measure at zero if i is odd and μ_{2i} is the Cantor measure on E_{2i}. Fix a positive integer N and suppose $U \subseteq \prod E_i$ satisfies

$$8^{-2(2n+2)} \le \operatorname{diam} U < 8^{-4n}$$

for some $n \ge N$. If $x, y \in U$ and x_{2i}, y_{2i} belong to different Cantor intervals of step $2i$ in E_{2i} for some $i \le n$, then

$$d(x,y) \ge |x_{2i} - y_{2i}|\, 2^{-2i} \ge 8^{-2i} > \operatorname{diam} U.$$

Consequently,

$$U \subseteq I_1 \times \cdots \times I_{2n} \times \prod_{i=2n+1}^\infty E_i\,,$$

where $I_{2i+1} = \{0\}$ and I_{2i} is a (fixed) step $2i$ Cantor interval of E_{2i}. Thus

$$\mu(U) \leq \prod_{i=1}^{n} 2^{-2i} = 2^{-n(n+1)} \leq (\operatorname{diam} U)^{N/12}.$$

By the mass distribution principle $\dim_H \prod_{i=1}^{\infty} E_i \geq N/12$, and since N was arbitrary, $\dim_H \prod_{i=1}^{\infty} E_i = \infty$.

Next, we will check that $\underline{\dim}_B \prod_{i=1}^{\infty} E_{\pi(i)} = 0$ by considering coverings of $\prod E_{\pi(i)}$ of diameter at most $\delta \equiv 2^{-2^m}$. Clearly $N_\delta(\prod E_{\pi(i)}) \leq \prod_{i=1}^{2^m} N_\delta(E_{\pi(i)})$. Recall that $E_{\pi(i)} = \{0\}$ if $i \neq 2^n$ for any n and $E_{\pi(2^i)} = E_{2i}$. Thus $\prod_{i=1}^{2^m} N_\delta(E_{\pi(i)}) = \prod_{i=1}^{m} N_\delta(E_{2i})$. Since the Cantor intervals of step $2m+1$ provide a δ -covering of E_{2i} when $i \leq m$, $N_\delta(E_{2i}) \leq 2^{2m+1}$ and hence

$$N_\delta(\prod_{i=1}^{\infty} E_{\pi(i)}) \leq \prod_{i=1}^{m} 2^{2m+1} \leq 2^{m(2m+1)}.$$

This gives

$$\dim_H \prod_{i=1}^{\infty} E_{\pi(i)} \leq \underline{\dim}_B \prod_{i=1}^{\infty} E_{\pi(i)} \leq \liminf_{m \to \infty} \frac{m(2m+1)}{2^m} = 0. \qquad \square$$

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