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OPTIMAL PRODUCTION FOR PRODUCTS SOLD WITH WARRANTY

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Abstract: Recently, R. H. Yeh, W. T. Ho and S. T. Tseng [15] show that the production run length for sold with free minimal repair warranty has a unique optimal. Furthermore, they also derive the bounds for the location of optimal run length. In this article, we attempt to establish a generalized form about the above-mentioned topics and apply our results to generalize the corresponding conclusions which appear in [15].

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1. Introduction

Consider the optimal production run length for a deteriorating production system in which the products are sold with free minimal repair warranty. In general, a production system continuously deteriorates due to usage or age such as corrosion, fatigue, and cumulative wear. Because of inevitable dete-

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rioration, the operating condition of a production system is usually classified into one of two states-in-control or out-of-control. Without any maintenance action, the system will eventually shift to the out-of control state in which more nonconforming or substandard items are produced than in the in-control state. Usually, a nonconforming item incurs more post-sale servicing cost than a conforming item, especially when the product is sold with warranty. To reduce the warranty cost, one should shorten the production run length to allow for frequent restoration, thereby retaining the system in the in-control state. However, a short production run length results in more setup cost and restoration cost. Therefore, there is a need to determine an optimal production run length such that the expected total cost per item is minimized.

Traditionally, an optimal production run length can be obtained by analyzing economic manufacturing quantity (EMQ) models [3]. In the traditional EMQ models, there is a basic assumption about the production system-the production process is perfect and stationary. Under this assumption, the production system does not deteriorate and continuously produces conforming items. However, this assumption may be invalid in practice. To relax this assumption, imperfect (deteriorating) production processes have been employed to generalize the traditional EMQ models [4]-[7], [12]-[14].

For an imperfect production system, most researchers either focus on finding an optimal maintenance policy to restore the system back to the in-control state [5], [6], [14], or on determining an optimal lot size to reduce the rework cost of defective items [4], [7], [12], [13]. Rosenblatt and Lee [13] further study the effects of a deteriorating production process on the optimal production run length. Without considering the restoration cost, they show that the optimal production run length is shorter than of the traditional EMQ model.

Most of the works mentioned earlier concentrate on minimizing the cost incurred by the product before it is sold but not after. For products sold with warranty, the resulting post-sale warranty cost is closely related to the quality of items produced by a deteriorating production system. Hence, it is important to incorporate the warranty cost in the EMQ model to reflect the practical situation. A detailed discussion and review of various issues related to warranty policies can be found in [1], [8]-[11]. Djamaludin et al. [2] deal with the lot-size problem by taking the warranty cost into account. In their model, the production process is modeled by a two-state discrete-time Markov chain and the product quality is characterized by two failure distributions. Without considering inventory holding cost, they propose a cost model to derive an optimal lot size such that the expected total cost per item is minimized.

Recently, R. H. Yeh, W. T. Ho and S. T. Tseng [15] employ a two-state continuous-time Markov chain to describe the deteriorating process of a production system. Taking both the restoration cost and the inventory holding cost into account. They show that there exists a unique optimal production run length which minimizes the expected total cost per item for products under free minimal repair warranty.

In this article, we attempt to establish a generalized form about the above-mentioned topics and apply our results to generalize the corresponding conclusions which appear in [15].

2. Main Results

Theorem 2.1. *Consider*

$$D(t) \equiv A + Bt + \frac{C}{t} + \frac{\eta(1-e^{-\lambda t})}{t} \qquad \textit{in} \qquad (0,\infty),$$

where $A, \eta \in \mathbf{R}$ *and* $\lambda, B, C > 0$. *Then, there is a unique optimal* $t_0 \in (0,\infty)$ *of* $D(t)$ *which minimizes* $D(t)$. *Furthermore, we also obtain that* $D'(t)(t-t_0) > 0$ *in* $(0,\infty)$, *i.e.* D *is convex in* $(0,\infty)$.

Proof. Since

$$D(t) \equiv A + Bt + \frac{C}{t} + \frac{\eta(1-e^{-\lambda t})}{t} \qquad \text{in} \qquad (0,\infty),$$

and so

$$D'(t) = B - \frac{C}{t^2} + \frac{\eta[(\lambda t+1)e^{-\lambda t}-1]}{t^2} \qquad \text{in} \qquad (0,\infty).$$

Let

$$g(t) \equiv Bt^2 - C + \eta[(\lambda t+1)e^{-\lambda t}-1] \qquad \text{in} \qquad (0,\infty).$$

We see that $g(t)$ and $D'(t)$ have the same sign in $(0,\infty)$.

Moreover, we obtain that

$$g'(t) = (2B - \eta\lambda^2 e^{-\lambda t})t \qquad \text{in} \qquad (0,\infty).$$

Now, we separate the rest proof into the following cases.

Case (1). Suppose that $\eta \leq 0$. It is easy to see that $g(t)$ is strictly increasing in $(0,\infty)$. It follows from

$$\lim_{t\to 0^+} g(t) = -C < 0 \qquad \text{and} \qquad \lim_{t\to\infty} g(t) = +\infty > 0,$$

that there exists a unique $t_0 \in (0,\infty)$ such that

$$g(t_0) = 0 = D'(t_0) \quad \text{and} \quad D'(t)(t-t_0) > 0 \quad \text{in} \quad (0,\infty).$$

Case (2). Suppose that $\eta > 0$. A simple calculation shows that

$$g'(t) > 0 \quad \text{in} \quad (\omega,\infty) \quad \text{and} \quad g'(t) < 0 \quad \text{in} \quad (0,\omega),$$

where $\omega \equiv \frac{ln(\frac{\eta\lambda^2}{2B})}{\lambda}$. It follows from

$$\lim_{t\to 0^+} g(t) = -C < 0 \quad \text{and} \quad \lim_{t\to\infty} g(t) = +\infty > 0,$$

that there exists a unique $t_0 \in (0,\infty)$ such that

$$g(t_0) = 0 = D'(t_0) \quad \text{and} \quad D'(t)(t-t_0) > 0 \quad \text{in} \quad (0,\infty).$$

By Cases (1) - (2), we complete the proof. □

In the next theorem, we will give bounds for t_0 of $D(t)$.

Theorem 2.2. *Suppose that $t_1 \equiv \sqrt{\frac{C}{B}}$ is the positive root of the equation $Bt^2 - C = 0$. Then, we have the following two results:*

(R_1) If $\eta \le 0$ then $0 < t_0 \le t_1$;

(R_2) If $\eta \ge 0$ then $t_0 \ge t_1$.

Proof. A simple calculation shows that

$$g(t_1) = Bt_1^2 - C + \eta[(\lambda t_1 + 1)e^{-\lambda t_1} - 1] \ge Bt_1^2 - C = 0 = g(t_0) \quad \text{if} \quad \eta \le 0$$

and

$$g(t_1) = Bt_1^2 - C + \eta[(\lambda t_1 + 1)e^{-\lambda t_1} - 1] \le Bt_1^2 - C = 0 = g(t_0) \quad \text{if} \quad \eta \ge 0.$$

Since $g(t)$ is strictly increasing in (t_0,∞) and $g(t) < 0$ in $(0,t_0)$, we obtain the desired results.

3. Applications for the Optimal Production Run Length with Warranty

Consider that the deterioration of a production system at any point in time can be classified into one of two states - in-control and out-of-control. It is assumed that the elapsed time, X, of the system in the in-control state follows an exponential distribution with finite mean $\frac{1}{\lambda}$. Once the system shifts to the out-of-control state, it stays there until the end of a production run. After the completion of a production run, the system is setup with cost $k > 0$ and is inspected to reveal the state of the system. If the system is out-of-control, then it is brought back to the in-control state with an additional restoration cost $r > 0$ for the next production run.

Suppose that the production rate of the system and the demand rate of the product are p and d, respectively, where $p > d > 0$. The manufacturing cost of an item is c_0 and the inventory holding cost for carrying a product is $h > 0$ per unit time. It is assumed that all the items produced are operational and can be classified as being either conforming or nonconforming depending on whether its performance meets the products' specifications or not.

After production, the item has two types of failures, which can occur at any age. Type I failure, defined as minor failure and is corrected by a minimal repair at cost c_m to the manufacture. Type II failure, defined as catastrophic failure and is replaced with a new item at cost c_r to the manufacture. Let $\alpha(t)$ and $\beta(t)$ denote the probability of Type II failure associated with a conforming and a nonconforming item, respectively. Since item can only be detected after a period of time in use, all the items produced are released for sale with a free minimal repair or replacement warranty. Under this free warranty, failures occur within the warranty period result in valid warranty claims and are rectified by minimal repair or replacement instantaneously at no cost to the buyers. After a minimal repair, the hazard rate of an item remains the same as that just before failure.

Let $h_1(t)$ and $h_2(t)$ denote the hazard rate associated with a conforming and a nonconforming item, respectively. We further assumed that $h_1(t) < h_2(t)$ for $t \geq 0$ which implies a nonconforming item is more likely to fail than a conforming item. Due to manufacturing variability, an item is nonconforming with probability θ_1 (or θ_2) when the production process is in-control (or out-of-control), where $\theta_1 < \theta_2$.

For the deteriorating production system mentioned above, the expected total cost incurred includes manufacturing cost, inventory holding cost, setup cost, restoration cost, and warranty cost.

Given that the production run length is t, the number of items produced is pt and the time duration of a production cycle is $\frac{pt}{d}$. The maximum inventory level is $(p-d)t$ and the expected total holding cost in a production cycle is $\frac{1}{2}(p-d)t(\frac{pt}{d})h$. Hence, the expected holding cost per item becomes $\frac{1}{2}(p-d)t(\frac{1}{d})h$.

In order to abbreviate our discussion; we adopt the following notations in this section. Let

$M(t) \equiv c_0 + \frac{(p-d)ht}{2d} + \frac{k}{pt} + \frac{r(1-e^{-\lambda t})}{pt}$: the expected pre-sale cost per item;
$N(t)$: the number of nonconforming items in a production run with length;
$E(N(t)) = \theta_2 pt - p(\theta_2 - \theta_1)\frac{1-e^{-\lambda t}}{\lambda}$: the expected value of $N(t)$;
$q(t) \equiv \frac{E(N(t))}{pt} = \theta_2 - (\theta_2 - \theta_1)\frac{1-e^{-\lambda t}}{\lambda t}$: the fraction of nonconforming items;

$$\begin{aligned}
W(t)& \\
&\equiv c_m[(1-q(t))\int_0^W (1-\alpha(\tau))h_1(\tau)d\tau + q(t)\int_0^W (1-\beta(\tau))h_2(\tau)d\tau] \\
&\quad + c_r[(1-q(t))\int_0^W \alpha(\tau)h_1(\tau)d\tau + q(t)\int_0^W \beta(\tau)h_2(\tau)d\tau] \\
&= (1-q(t))[c_m\int_0^W (1-\alpha(\tau))h_1(\tau)d\tau + c_r\int_0^W \alpha(\tau)h_1(\tau)d\tau] \\
&\quad + q(t)[c_m\int_0^W (1-\beta(\tau))h_2(\tau)d\tau + c_r\int_0^W \beta(\tau)h_2(\tau)d\tau]\,,
\end{aligned}$$

the expected post-sale warranty cost per item, where $\alpha, \beta \in C([0,\infty); [0,1])$ represent the probability of Type II failure (catastrophic failure) for conforming and nonconforming items, respectively;

$L(t) \equiv M(t) + W(t)$: the expected total cost per item.

Now, setting

$$\begin{aligned}
A \equiv c_0 &+ (1-\theta_2)[c_m\int_0^W (1-\alpha(\tau))h_1(\tau)d\tau + c_r\int_0^W \alpha(\tau)h_1(\tau)d\tau] \\
&+ \theta_2[c_m\int_0^W (1-\beta(\tau))h_2(\tau)d\tau + c_r\int_0^W \beta(\tau)h_2(\tau)d\tau],
\end{aligned}$$

$$\begin{aligned}
\eta \equiv \frac{r}{p} &+ \frac{(\theta_2-\theta_1)}{\lambda}[c_m\int_0^W (1-\alpha(\tau))h_1(\tau)d\tau + c_r\int_0^W \alpha(\tau)h_1(\tau)d\tau] \\
&- \frac{(\theta_2-\theta_1)}{\lambda}[c_m\int_0^W (1-\beta(\tau))h_2(\tau)d\tau + c_r\int_0^W \beta(\tau))h_2(\tau)d\tau]\,,
\end{aligned}$$

$$B \equiv \frac{(p-d)h}{2d} \quad \text{and} \quad C \equiv \frac{k}{p}.$$

Using Theorem 2.1 and Theorem 2.2, we obtain the following corollaries which generalize the results appear in R.H. Yeh, W.T. Ho and S.T. Tseng [15].

Corollary 3.1. *The expected total cost per item function*

$$\begin{aligned} L(t) &\equiv M(t)+W(t) \\ &= c_0+\frac{(p-d)ht}{2d}+\frac{k}{pt}+\frac{r(1-e^{-\lambda t})}{pt} \\ &+(1-q(t))[c_m\int_0^W(1-\alpha(\tau))h_1(\tau)d\tau+c_r\int_0^W\alpha(\tau)h_1(\tau)d\tau] \\ &+q(t)[c_m\int_0^W(1-\beta(\tau))h_2(\tau)d\tau+c_r\int_0^W\beta(\tau)h_2(\tau)d\tau] \end{aligned}$$

is convex with respect to t in $(0,\infty)$ and there is a unique optimal $t^*\in(0,\infty)$ of $L(t)$ which minimizes $L(t)$.

Proof. It follows from

$$\begin{aligned} L(t)\equiv M(t)+W(t) &= c_0+\frac{(p-d)ht}{2d}+\frac{k}{pt}+\frac{r(1-e^{-\lambda t})}{pt} \\ &+(1-q(t))[c_m\int_0^W(1-\alpha(\tau))h_1(\tau)d\tau+c_r\int_0^W\alpha(\tau)h_1(\tau)d\tau] \\ &+q(t)[c_m\int_0^W(1-\beta(\tau))h_2(\tau)d\tau+c_r\int_0^W\beta(\tau)h_2(\tau)d\tau] \\ &= c_0+(1-\theta_2)[c_m\int_0^W(1-\alpha(\tau))h_1(\tau)d\tau+c_r\int_0^W\alpha(\tau)h_1(\tau)d\tau] \\ &+\theta_2[c_m\int_0^W(1-\beta(\tau))h_2(\tau)d\tau+c_r\int_0^W\beta(\tau)h_2(\tau)d\tau] \\ &+\frac{(p-d)h}{2d}t+\frac{k}{p}\frac{1}{t}+\Bigg\{\frac{r}{p}+\frac{(\theta_2-\theta_1)}{\lambda} \\ &\times[c_m\int_0^W(1-\alpha(\tau))h_1(\tau)d\tau+c_r\int_0^W\alpha(\tau)h_1(\tau)d\tau]\Bigg\}\frac{1-e^{-\lambda t}}{t} \\ &-\Bigg\{\frac{(\theta_2-\theta_1)}{\lambda}[c_m\int_0^W(1-\beta(\tau))h_2(\tau)d\tau+c_r\int_0^W\beta(\tau)h_2(\tau)d\tau]\Bigg\} \\ &\times\frac{1-e^{-\lambda t}}{t}=A+Bt+\frac{C}{t}+\frac{\eta(1-e^{-\lambda t})}{t}=D(t)\,, \end{aligned}$$

and Theorem 2.1, we obtain the desired result. □

Corollary 3.2. *Suppose that $t_1^*\equiv\sqrt{\frac{2dk}{p(p-d)h}}$, then we have the following :*

(R_1^*) If $\eta\le 0$ then $0<t^*\le t_1^*$;

(R_2^*) *If* $\eta \geq 0$ *then* $t^* \geq t_1^*$.

Proof. Following Theorem 2.2, we easy obtain the desired result. □

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