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FRAMES AND BASES IN HILBERT MODULES OVER LOCALLY C^*-ALGEBRAS

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Abstract: In this paper we develop the module frame theory in Hilbert modules over unital locally C^*-algebras that possess orthogonal bases, such that the Hilbert spaces and Hilbert C^*-modules situations appear as a special case. We show that like orthonormal bases if $\{x_i\}_{i\in I}$ is a frame in Hilbert module M over a unital locally C^*-algebra A, then for any $x \in M$ the reconstruction formula $x = \sum_{i\in I} < x, S^{-1}x_i > x_i$ is valid, where S is a positive A-linear bounded adjointable operator on M. Moreover we consider the canonical dual frame in Hilbert A-module.

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1. Introduction

General module frame theory of finitely and countably generated Hilbert modules over *a* unital C^*-algebra introduced by M. Frank and D. R. Larson [4], [6],

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[7] as a natural generalization of the frame theory in Hilbert spaces. Furthermore the Hilbert modules over locally C^*-algebras discussed by A. Inoue [9] and N.C. Phillips [14] and Yu.I. Zhuraev and F. Sharipov [17]. Since topological $*$-algebras, in particular locally C^*-algebras is applied to relativistic quantum mechanics [3], in this article we consider the frames in Hilbert LC^*-modules, and extend some of the known results about bases to frames. We start with some information from [9], [14], [17] on locally C^*-algebras.

2. Preliminaries

Definition 2.1. A locally C^*-algebra (LC^*-algebra) is a complete Hausdorff topological $*$-algebra over $\mathbb{C}$ whose topology is determined by a separating family of continuous C^*-seminorms $P = \{P_\lambda\}_{\lambda\in\Lambda}$ satisfying the following conditions:

(a) $p_\lambda(ab) \leq p_\lambda(a)p_\lambda(b)$ for all $a, b \in A$ and $\lambda \in \Lambda$.

(b) $p_\lambda(a^*a) = (p_\lambda(a))^2$ for all $a \in A$ and $\lambda \in \Lambda$.

A family of C^*-seminorms is a family of seminorms P satisfying condition (b).

These objects are called locally C^*-algebras in [9], [17] and pro-C^*-algebras in [14]. If the topology of A is determined by only countably many C^*-seminorms, then it is called a σ-locally C^*-algebra. By Proposition 1.2 of [14], a topological $*$-algebra A is a locally C^*-algebra if and only if it is the inverse limit, of an inverse system of C^*-algebras and $*$-homomorphisms.

Examples 2.2. (1) Any C^*-algebra is a locally C^*-algebra

(2) A closed $*$-subalgebra of a locally C^*-algebra is again a locally C^*-algebra.

(3) Let X be a compactly generated space, then $C(X)$, the set of all continuous complex-valued functions on X with the topology of uniform convergence on compact subsets, is a locally C^*-algebra.

Let $\{a_i\}_{i\in I}$ be a net in A. We say that net $\{a_i\}_{i\in I}$ P-converges to 0 if and only if $P_\lambda(a_i) \longrightarrow 0$ for every continuous C^*-seminorm P_λ on A. Let A be a locally C^*-algebra. Then its unitization A^+ is the vector space $A \bigoplus \mathbb{C}$, topologized as the direct sum and with adjoint and multiplication defined as for the unitization of a C^*-algebra. Note that A^+ is a locally C^*-algebra.

Let A be a unital locally C^*-algebra and let $a \in A$. Then the spectrum $\operatorname{sp}(a)$ of $a \in A$ is the set $\{z \in \mathbb{C} : z1_A - a \notin Inv(A)\}$. If A is not unital, then the spectrum is taken with respect to A^+. An element $a \in A$ is called positive (and is written $a \geq 0$) if it is Hermitian and one of the following equivalent conditions is true:

(1) $\mathrm{sp}\,(a) \subseteq [0, \infty)$.

(2) $a = b^*b$ for some $b \in A$.

(3) $a = h^2$ for some Hermitian $h \in A$.

Furthermore the set of positive elements denoted by $P^+(A)$ is a P-closed convex cone in A and $P^+(A) \cap (-P^+(A)) = \{0\}$ If $a \in A$ is a positive element, then there exists a unique positive element $h \in A$ such that $a = h^2$. This element is called the square root of a, and is denoted by $a^{1/2} = \sqrt{a}$. For each two Hermitian elements $a, b \in A$, the inequality $a \geq b$ (or $b \leq a$) means that $a - b \geq 0$.

Lemma 2.3. (see [17], Lemma 1.1) *Let A be a unital LC^*-algebra. Then we have:*

(1) $P_\lambda(a) = P_\lambda(a^*)$ for all $a \in A$ and each $\lambda \in \Lambda$.

(2) $P_\lambda(1_A) = 1$ for all $\lambda \in \Lambda$.

(3) If $a, b \in P^+(A)$ and $a \leq b$, then $P_\lambda(a) \leq P_\lambda(b)$ for any $\lambda \in \Lambda$.

(4) If $a, b \in P^+(A)$ are invertible and $0 \leq a \leq b$ then $a^{-1} \geq b^{-1} \geq 0$.

(5) If $a, b, c \in A$ and a, b are Hermitian, $a \leq b$, then $c^*ac \leq c^*bc$.

(6) If $a, b \in P^+(A)$ and $a^2 \leq b^2$ then $0 \leq a \leq b$.

(7) If $a \in A$ and $P_\lambda(a) < 1$ for every $\lambda \in \Lambda$, then $1_A - a$ is invertible.

3. Hilbert Modules over LC^*-Algebras

We now define Hilbert modules over LC^*-algebras. The results are generalizations of the known results over C^*-algebras.

Definition 3.1. Let A be a LC^*-algebra with respect to a family of continuous C^*-seminorms P, and let M be a complex vector space which is also a left A-module compatible with the complex algebra structure. Then M is a pre-Hilbert A-module if it is equipped with an A-valued inner product $< \cdot, \cdot >: M \times M \longrightarrow A$ with the following properties:

(1) $< x, x > \geq 0$ for any $x \in M$.

(2) $< x, x >= 0$ if and only if $x = 0$.

(3) $< x, y >=< y, x >^*$ for any $x, y \in M$.

(4) $< ax + by, z >= a < x, z > + b < y, z >$ for any $a, b \in A$ and $x, y, z \in M$.

While A be a unital LC^*-algebra and $1_A x = x$ $(x \in M)$ then M is called a unitary pre-Hilbert A-module.

By Lemma 3.1 of [17] for every $\lambda \in \Lambda$ and for all $x, y \in M$ the following Cauchy-Bunyakovskii inequality holds.

$$P_\lambda(< x, y >)^2 \leq P_\lambda(< x, x >)P_\lambda(< y, y >). \tag{1}$$

Moreover for any $\lambda \in \Lambda$ the function $\bar{P}_\lambda$, defined by

$$\bar{P}_\lambda(x) = P_\lambda(< x, x >)^{1/2} \quad (x \in M)\,, \tag{2}$$

is a seminorm on M, such that M is a locally convex space with respect to the separating family of seminorms $\bar{P} = \{\bar{P}_\lambda\}_{\lambda \in \Lambda}$. Furthermore for any $x \in M$ and each $\lambda \in \Lambda$ we have

$$\bar{P}_\lambda(x) = \sup_{\bar{P}_\lambda(y) \leq 1} P_\lambda(< x, y >)\,. \tag{3}$$

We say that M is a Hilbert A-module (Hilbert LC^*-module) if M is a complete locally convex space with respect to the family of seminorms $\bar{P}$. We refer to the pairing $\{M, < \cdot, \cdot >\}$ as a Hilbert A-module.

Examples 3.2. (1) Any P-closed left ideal I of a locally C^*-algebra A equipped with the inner product $< a, b >= ab^*$ is a Hilbert A-module.

(2) Let A be a LC^*-algebra with respect to the family of C^*-seminorms $P = \{p_\lambda\}_{\lambda \in \Lambda}$ and let $\{M_i\}_{i=1}^n$ be a finite set of Hilbert A-modules. We define their direct sum $\bigoplus_{i=1}^n M_i$ as the set of all n-tuples $x = \{x_i\}_{i=1}^n$ $(x_i \in M_i)$ equipped with coordinate-wise operations and with the A-valued inner product

$$< x, y >= \sum_{i=1}^n < x_i, y_i > \qquad x = \{x_i\}, y = \{y_i\} \text{ in } \bigoplus_{i=1}^n M_i \tag{4}$$

$\bigoplus_{i=1}^n M_i$ is a Hilbert A-module.

(3) If $\{M_i\}_{i \in \mathbb{N}}$ is a countable set of Hilbert A-modules then we define their direct sum $\bigoplus_{i \in \mathbb{N}} M_i$ as the set of all sequences $x = \{x_i\}_{i \in \mathbb{N}}$ $(x_i \in M_i)$ such that the series $\sum_{i \in \mathbb{N}} < x_i, x_i >$ is P-convergent in A, equipped with coordinate-wise operations and with the A-valued inner product

$$< x, y >= \sum_{i \in \mathbb{N}} < x_i, y_i > \quad \text{for} \quad x, y \in \bigoplus_{i \in \mathbb{N}} M_i \tag{5}$$

$\bigoplus_{i \in \mathbb{N}} M_i$ is a Hilbert A-module.

Proof. First we show that the inner product is well-defined. Since the series $\sum_{i \in \mathbb{N}} < x_i, x_i >$ and $\sum_{i \in \mathbb{N}} < y_i, y_i >$ are P-convergent in A, it follows that for any $\epsilon > 0$ and each $\lambda \in \Lambda$ there exists a number N such that for all $n \geq m \geq N$ the following estimate holds.

$$P_\lambda(\sum_{i=m}^{n} < x_i, x_i >) < \epsilon, \quad P_\lambda(\sum_{i=m}^{n} < y_i, y_i >) < \epsilon\,.$$

Now by Cauchy-Bunyakovskii inequality for all $n \geq m \geq N$ we have

$$\begin{aligned} &P_\lambda(\sum_{i=m}^{n} < x_i, y_i >) \\ &\qquad \leq P_\lambda(\sum_{i=m}^{n} < x_i, x_i >)^{1/2} P_\lambda(\sum_{i=m}^{n} < y_i, y_i >)^{1/2} < \epsilon\,. \end{aligned} \tag{6}$$

This proves that the inner product is well-defined. Moreover we show that A-module $\bigoplus_{i\in\mathbb{N}} M_i$ is a $\bar{P}$-complete space. Let $\{x^{(n)}\}_{n\in\mathbb{N}} \subseteq \bigoplus_{i\in\mathbb{N}} M_i$ and let $x^{(n)} = \{x_i^n\}_{i\in\mathbb{N}}$ be a $\bar{P}$-Cauchy sequence. Then for any $\epsilon > 0$ and each $\lambda \in \Lambda$, there exists a number N such that for all $n \geq m \geq N$

$$P_\lambda(\sum_{i\in\mathbb{N}} < x_i^n - x_i^m, x_i^n - x_i^m >) < \epsilon^2\,. \tag{7}$$

So,

$$\begin{aligned} \bar{P}_\lambda(x_i^n - x_i^m)^2 &= P_\lambda(< x_i^n - x_i^m, x_i^n - x_i^m >) \\ &\leq P_\lambda(\sum_{i\in\mathbb{N}} < x_i^n - x_i^m, x_i^n - x_i^m >) < \epsilon^2\,. \end{aligned}$$

Therefore for each $i \in \mathbb{N}$ the sequence $\{x_i^n\}_{n\in\mathbb{N}} \subseteq M_i$ is a $\bar{P}$-Cauchy sequence in M_i and since M_i is complete, $\{x_i^n\}_{n\in\mathbb{N}}$ converges to some $x_i \in M_i$. Let $x = \{x_i\}_{i\in\mathbb{N}}$. We show that $\lim_{n\to m} x^{(n)} = x$. Let $\epsilon > 0$ and $\lambda \in \Lambda$. Since $\lim_{n\to\infty} x_i^n = x_i$, it follows from (7) that for every $n \geq N$ we have

$$P_\lambda(< x_i^n - x_i, x_i^n - x_i >) < \frac{\epsilon^2}{2^i}. \tag{8}$$

Therefore

$$\bar{P}_\lambda(x^{(n)} - x)^2 \leq \sum_{i\in\mathbb{N}} P_\lambda(< x_i^n - x_i, x_i^n - x_i >) < \sum_{i\in\mathbb{N}} \frac{\epsilon^2}{2^i} = \epsilon^2\,.$$

This proves that $\lim_{n\to\infty} x^{(n)} = x$. Moreover by easy calculations we conclude that the series $\sum_{i\in\mathbb{N}} < x_i, x_i >$ is P-convergent in A.

(4) Let A be a LC^*-algebra with respect to a family of C^*-seminorms P. Then $l_2(A)$ defined by

$$l_2(A) = \{x = \{a_i\}_{i\in\mathbb{N}} : \sum_{i\in\mathbb{N}} a_i a_i^* \text{ is } P-\text{convergent in } A\},$$

is a Hilbert A-module with respect to the pointwise operations and the inner product

$$< x, y >= \sum_{i\in\mathbb{N}} a_i b_i^* \quad \text{for} \quad x, y \in l_2(A). \tag{9}$$

The Hilbert LC^*-module $l_2(A)$ is called the standard Hilbert module over A.

Let A be a LC^*-algebra. Let M be a Hilbert A-module and $N \subseteq M$ be a $\bar{P}$-closed submodule. We define $N^\perp$ by the equation

$$N^\perp = \{y \in M :< x, y >= 0 \text{ for all } x \in N\}.$$

Then $N^\perp$ is a $\bar{P}$-closed submodule of the Hilbert A-module M, too. However the equality $M = N \bigoplus N^\perp$ is not fulfilled always. For example if $A = C[0,1]$ is the algebra of all continuous functions on the segment $[0,1]$, and $\Lambda = [0,1]$, then for any $\lambda \in \Lambda$ the relation

$$P_\lambda(f) = |f(\lambda)| \quad (f \in A) \tag{10}$$

defines a C^*-seminorm on A, and with respect to the family of these C^*-seminorms, A is a LC^*-algebra. Let $M = A$. Then M is a Hilbert A-module. If we consider in the Hilbert A-module M the submodule $N = C(0,1)$ of functions vanishing at the end points of the segment $[0,1]$. Then obviously $N^\perp = \{0\}$.

Let A be a LC^*-algebra and let M be a Hilbert A-module. Then M is called finitely generated if there exists a finite set $\{x_1, x_2, \ldots, x_n\}$ of elements of M such that every $x \in M$ can be expressed as an A-linear combination $x = \sum_{i=1}^n a_i x_i$ $(a_i \in A)$. In that case the subset $\{x_1, x_2, \ldots, x_n\} \subseteq M$ is called a set of generators of M.

M is called countably generated if there exists a countable set $\{x_i\}_{i\in I} \subseteq M$ such that M equals the $\bar{P}$-closure of A-linear hull of $\{x_i\}_{i\in I}$ in M. In that case the subset $\{x_i\}_{i\in I}$ is called a set of generators of M.

Let A be a unital LC^*-algebra and let M be a finitely or countably generated Hilbert A-module. We will use I, J, to denote a generic countable (or finite) index set. A sequence $\{x_i\}_{i\in I}$ of elements in M is said to be an orthonormal system if for every $i, j \in I$, $< x_i, x_j >= \delta_{ij}$. Moreover a set of generators $\{x_i\}_{i \in I} \subseteq M$ is called a Hilbert basis of M if:

(i) For every $i \in I$ $x_i \neq 0$.

(ii) If $\sum_{i \in I_0} a_i x_i = 0$ for every $I_0 \subseteq I$ and $\{a_i\}_{i \in I_0} \subseteq A$ then for all $i \in I_0$, $a_i x_i = 0$.

So the orthonormal sysem $\{x_i\}_{i \in I}$ in M is said to be an orthonormal Hilbert basis for M if the set $\{x_i\}_{i \in I}$ is a set of generators for M.

If A is a unital LC^*-algebra, then the standard Hilbert A-module $l_2(A)$ possesses the orthonormal Hilbert basis $\{e_i\}_{i \in I}$, where $e_i = (0, \ldots, 0, 1_A, 0, \ldots, 0)$ with the unit at the i-th place. Obviously, an arbitrary Hilbert A-module need not possess an orthonormal basis.

For any LC^*-pre-Hilbert module M, the following polarization equality is obviously satisfied

$$4 < x, y >= \sum_{k=0}^{3} i^k < x + i^k y, x + i^k y > \quad \text{for all } x, y \in M. \tag{11}$$

Proposition 3.3. *Let M be a Hilbert LC^*-module over a unital LC^*-algebra A and let $\{x_i\}_{i \in I}$ be an orthogonal system in M such that $< x_i, x_i > \leq 1_A$ $(i \in I)$.*

(i) If $I_0 \subseteq I$ is any finite set of indices, then each $x \in M$ satisfies.

$$\sum_{i \in I_0} < x, x_i >< x_i, x > \leq < x, x > .$$

(ii) If I_1 is any finite set of indices such that $I_0 \subseteq I_1$ then

$$\langle x - \sum_{i \in I_1} < x, x_i > x_i, x - \sum_{i \in I_1} < x, x_i > x_i \rangle \\ \leq \langle x - \sum_{i \in I_0} < x, x_i > x_i, x - \sum_{i \in I_0} < x, x_i > x_i \rangle .$$

Proof. (i) Let $x \in M$, then with a simple calculation we have

$$\langle x - \sum_{i \in I_0} < x, x_i > x_i, x - \sum_{i \in I_0} < x, x_i > x_i \rangle + \sum_{i \in I_0} < x, x_i > \\ \times (1_A - < x_i, x_i >) < x_i, x >=< x, x > - \sum_{i \in I_0} < x, x_i >< x_i, x > .$$

Since all terms $< x, x_i > (1_A - < x_i, x_i >) < x_i, x >$ $(i \in I_0)$ are positive elements in A, so (i) holds.

(ii) If I_1 is a finite set such that $I_0 \subseteq I_1$ then we have

$$\langle x - \sum_{i\in I_0} < x, x_i > x_i, x - \sum_{i\in I_0} < x, x_i > x_i \rangle - \langle x - \sum_{i\in I_1} < x, x_i > x_i, x - \sum_{i\in I_1} < x, x_i > x_i \rangle = \sum_{i\in I_1 - I_0} < x, x_i > (2 \cdot 1_A - < x_i, x_i >) < x_i, x > .$$

Since the sum at the right hand side is a sum of positive summands then (ii) holds. □

Now we can generalize Lemma 1 and Theorem 1 of [1] to Hilbert LC^*-modules.

Proposition 3.4. *Let A be a LC^*-algebra with respect to the family of C^*-seminorms $P = \{p_\lambda\}_{\lambda\in\Lambda}$ and let M be a Hilbert A-module. Suppose $\{x_i\}_{i\in I}$ is an orthogonal system in M such that the values $\{< x_i, x_i >: i \in I\}$ are projections in A. Then every $x \in M$ satisfies*

$$\bar{P}_\lambda(x - \sum_{i\in I_0} < x, x_i > x_i) \leq \bar{P}_\lambda(x - \sum_{i\in I_0} a_i x_i)\,, \tag{12}$$

where $I_0 \subseteq I$ is a finite set of indices and coefficients $\{a_i\}_{i\in I_0} \subseteq A$ and $\lambda \in \Lambda$.

Proof. Since $< x_i, x_i > x_i = x_i$ $(i \in I)$ we have

$$\langle x - \sum_{i\in I_0} < x, x_i > x_i, x - \sum_{i\in I_0} < x, x_i > x_i \rangle + \sum_{i\in I_0} (a_i - < x, x_i >) \times < x_i, x_i > (a_i^* - < x_i, x_i >) = \langle x - \sum_{i\in I_0} a_i x_i, x - \sum_{i\in I_0} a_i x_i \rangle$$

and since all terms at the above equality are positive elements in A then

$$0 \leq < x - \sum_{i\in I_0} < x, x_i > x_i, x - \sum_{i\in I_0} < x, x_i > x_i > \leq < x - \sum_{i\in I_0} a_i x_i, x - \sum_{i\in I_0} a_i x_i > .$$

This implies that

$$\bar{P}_\lambda(x - \sum_{i\in I_0} < x, x_i > x_i) \leq \bar{P}_\lambda(x - \sum_{i\in I_0} a_i x_i).$$

□

Theorem 3.5. *Let M be a countably or finitely generated Hilbert LC^*-module over a LC^*-algebra A and let $\{x_i\}_{i\in I}$ be an orthogonal system in M such that the values $\{< x_i, x_i >: i \in I\}$ are non-zero projections in A. Then the following statements are mutually equivalent:*

(i) $\{x_i\}_{i\in I}$ is an orthogonal basis for M.

(ii) $x = \sum_{i\in I} < x, x_i > x_i$ for every $x \in M$ and the series is $\bar{P}$-convergent in M.

(iii) $< x, x >= \sum_{i\in I} < x, x_i >< x_i, x >$ for any $x \in M$ and the series is P-convergent in A.

(iv) $< x, y >= \sum_{i\in I} < x, x_i >< x_i, x >$ for all $x, y \in M$.

Proof. (i)$\Longrightarrow$(ii). Let $x \in M$ and $\epsilon > 0$, $\lambda \in \Lambda$ be arbitrary. Since the set $\{x_i\}_{i\in I}$ is a basis in M, there exists a finite set $I_0 \subseteq I$ and the coefficients $\{a_i\}_{i\in I_0} \subseteq A$ such that

$$\bar{P}_\lambda(x - \sum_{i\in I_0} a_i x_i) < \epsilon\,.$$

By Proposition 3.4 we have

$$\bar{P}_\lambda(x - \sum_{i\in I_0} < x, x_i > x_i) < \epsilon\,.$$

If $I_1 \subseteq I$ is a finite set such that $I_0 \subseteq I_1$, then by Proposition 3.3 and Lemma 2.3 we have

$$\bar{P}_\lambda(x - \sum_{i\in I_1} < x, x_i > x_i) \leq \bar{P}_\lambda(x - \sum_{i\in I_0} < x, x_i > x_i) < \epsilon\,.$$

This proves (ii).

(ii)$\Longrightarrow$(i) In view of definition the only thing remaining to prove is that if $I_0 \subseteq I$ and $\{a_i\}_{i\in I_0} \subseteq A$ and $\sum_{i\in I_0} a_i x_i = 0$ then $a_i x_i = 0$ $(i \in I_0)$. Let $k \in I_0$ be arbitrary. Then

$$0 =< \sum_{i\in I_0} a_i x_i, a_k x_k >=< a_k x_k, a_k x_k > \;.$$

It follows that $a_k x_k = 0$.

(ii)$\Longrightarrow$(iii) Let $x \in M$ and $I_0 \subseteq I$ be a finite set of indices then by Proposition 3.3 we have

$$\langle x - \sum_{i\in I_0} < x, x_i > x_i, x - \sum_{i\in I_0} < x, x_i > x_i\rangle$$

$$= < x, x > - \sum_{i \in I_0} < x, x_i > < x_i, x > \ .$$

This shows that (ii)$\Longleftrightarrow$(iii).

(iii)$\Longleftrightarrow$(iv) is evident by polarization equality. $\square$

4. Frames in Hilbert LC^*-Modules

Let A be a unital LC^*-algebra with respect to the family of continuous C^*-seminorms $P = \{P_\lambda\}_{\lambda \in \Lambda}$ and let M be a finitely or countably generated Hilbert A-module. Suppose $\mathbb{N}$ denote the natural numbers. We will use I, J to denote a generic countable (or finite) index set.

A sequence $\{x_i\}_{i \in I}$ of elements in M is said to be a frame if there exist real constants $C, D > 0$ such that for every $x \in M$ the series $\sum_{i \in I} < x, x_i > < x_i, x >$ is P-convergent in A and the following inequality is valid.

$$C < x, x > \leq \sum_{i \in I} < x, x_i > < x_i, x > \leq D < x, x > \ . \tag{13}$$

The optimal constants (maximal for C and minimal for D) are called the frame bounds. The frame $\{x_i\}_{i \in I}$ is said to be a tight frame if $C = D$ and said to be normalized if $C = D = 1$. Since for every $x \in M$ the series $\sum_{i \in I} < x, x_i > < x_i, x >$ is P-convergent, the sequence $\{x_i\}_{i \in I}$ is called a standard frame.

A sequence $\{x_i\}_{i \in I}$ is called a Bessel sequence (with a Bessel bound D) if there exists a real constant $D > 0$ such that

$$\sum_{i \in I} < x, x_i > < x_i, x > \leq D < x, x > , \tag{14}$$

for any $x \in M$. It is obvious from Theorem 3.5 that every orthogonal basis $\{x_i\}_{i \in I}$ of M such that the values $< x_i, x_i >$, for $i \in I$, are non-zero projections in A is a normalized tight frame in M.

An A-linear functional $f : M \longrightarrow A$ is called a bounded functional on the module M if for each $\lambda \in \Lambda$ there exists a constant C_λ such that

$$P_\lambda(f(x)) \leq C_\lambda \bar{P}_\lambda(x) \quad (x \in M) \, . \tag{15}$$

We denote by M^* the set of all bounded A-linear functionals on M. In the general case, the analogue of the Riesz Representation Theorem for bounded A-linear functionals $f : M \longrightarrow A$ is not valid for M. Moreover M^*-topology of M is called the weak topology of M. Since every $f \in M^*$ is $\bar{P}$-continuous and since M^*-topology is the weakest topology on M with this property, then M^*-topology is weaker than $\bar{P}$-topology on M. Consequently, if $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ is a net in M such that $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ is $\bar{P}$-convergent to $u \in M$ then for each $v \in M$ the net $\{< u_\alpha, v >\}_{\alpha\in\mathcal{A}}$ is P-convergent to $< u, v >$ in A.

Let $\{M, < \cdot, \cdot >_M\}$ and $\{N, < \cdot, \cdot >_N\}$ be two Hilbert LC^*-modules over a unital LC^*-algebra A, then A-linear operator $T : M \longrightarrow N$ is called bounded operator if for each $\lambda \in \Lambda$ there exists a constant C_λ such that

$$P_\lambda(< Tx, Tx >_N) \leq C_\lambda P_\lambda(< x, x >_M) \quad (x \in M). \tag{16}$$

We denote by $\mathrm{Hom}_A(M, N)$ the set of all bounded A-linear operators from M into N. If $M = N$ then we denote by $\mathrm{End}_A(M)$ the set of all bounded A-linear operators on M.

An operator $T \in \mathrm{Hom}_A(M, N)$ is called adjointable if there exists an operator $T^* \in \mathrm{Hom}_A(N, M)$ such that

$$< Tx, y >_N = < x, T^*y >, \tag{17}$$

for all $x \in M$ and $y \in N$. We denote all of these operators by $\mathrm{Hom}^*_A(N, M)$. If $M = N$, then we denote it by $\mathrm{End}^*_A(M)$. By Theorem 4.2 of [14] the set $\mathrm{End}^*_A(M)$ is a locally C^*-algebra with respect to the family of C^*-seminorms $\hat{P} = \{\hat{P}_\lambda\}_{\lambda\in\Lambda}$, where the map $\hat{P}_\lambda$ $(\lambda \in \Lambda)$ defined by

$$\hat{P}_\lambda(T) = \sup_{\bar{P}_\lambda(x)\leq 1} \bar{P}_\lambda(Tx) \quad (T \in \mathrm{End}^*_A(M)). \tag{18}$$

An A-linear operator $T \in \mathrm{Hom}^*_A(M, N)$ is called an isometry if $T^*T = 1$ and $T \in \mathrm{Hom}^*_A(M, N)$ is called co-isometry if $TT^* = I$. Two Hilbert A-modules $\{M, < \cdot, \cdot >_M\}$ and $\{N, < \cdot, \cdot >_N\}$ are unitarily isomorphic if there exists a bijective A-linear operator $T \in \mathrm{Hom}^*_A(M, N)$ such that $< Tx.Ty >_N = < x, y >_M$ for any $x, y \in M$.

A sequence $\{x_i\}_{i\in I}$ is said to be a Riesz basis if $\{x_i\}_{i\in I}$ is a frame and is also a basis for M.

Proposotion 4.1. (see [17], Proposition 3.2) *For an operator $T : M \longrightarrow M$ the following conditions are equivalent:*

(i) T is a positive element of LC^*-algebra $\mathrm{End}^*_A(M)$.

(ii) For any element $x \in M$ the inequality $< Tx, x > \geq 0$ holds, i.e. this element is positive in the algebra A.

Proposition 4.2. *Let M be a finitely or countably generated Hilbert LC^*-module over a unital LC^*-algebra A, suppose that $\{x_i\}_{i\in I}$ is a standard frame with the frame bounds C, D respectively, then:*

(i) $< x_k, x_k > \leq D \cdot 1_A$ *for any* $k \in I$.

(ii) For every $x \in M$ *the sequence* $\{< x, x_i >\}_{i \in I}$ *is* P*-convergent to* 0 *in* A.

Proof. (i) Let $k \in I$ be arbitrary, then we have

$$< x_k . x_k >^2 \leq \sum_{i \in I} < x_k, x_i >< x_i, x_k > \leq D < x_k, x_k > .$$

So, $< x_k, x_k >^2 \leq D < x_k, x_k >$ and therefore $< x_k, x_k > \leq D1_A$.

(ii) Let $x \in M$ be arbitrary. Since the series $\sum_{i \in I} < x, x_i >< x_i, x >$ is P-convergent in A hence for any $\lambda \in \Lambda$ we have

$$\lim_{i \to \infty} P_\lambda(< x, x_i >)^2 = \lim_{i \to \infty} P_\lambda(< x, x_i >< x_i, x >) = 0. \qquad \square$$

Theorem 4.3. (Frame Operator) *Let* A *be a unital* LC^**-algebra, and let* M *be a finitely or countably generated Hilbert* A*-module. Let* $\{x_i\}_{i \in I}$ *be a sequence in* M*. Then the following statements are equivalent:*

(i) $\{x_i\}_{i \in I}$ *is a standard frame with bounds* C, D *in* M.

(ii) There exists a positive operator $S \in \mathrm{End}^*_A(M)$ *defined by*

$$Sx = \sum_{i \in I} < x, x_i > x_i \quad (x \in M) \tag{19}$$

such that $C.I \leq S \leq D.I$*, where* I *is the identity operator on* M.

Proof. (i)$\Longrightarrow$(ii) Let $K \subset J$. Then

$$\begin{aligned}
\bar{P}_\lambda(\sum_{i \in K} < x, x_i > x_i)^2 &= \sup_{\bar{P}_\lambda(y) \leq 1} P_\lambda(< \sum_{i \in K} < x, x_i > x_i, y >)^2 \\
&\leq \sup_{\bar{P}_\lambda(y) \leq 1} P_\lambda(< \sum_{i \in K} < x, x_i >< x_i, x >) P_\lambda(< \sum_{i \in K} < y, x_i >< x_i, y >) \\
&\leq D P_\lambda(< \sum_{i \in K} < x, x_i >< x_i, x >) P_\lambda(y) \\
&\leq D P_\lambda(< \sum_{i \in K} < x, x_i >< x_i, x >).
\end{aligned}$$

So $\{\sum_{i=1}^n < x, x_i > x_i\}$ is a Cauchy sequence in M and since M is $\bar{p}$-complete, the series $\sum_i < x, x_i > x_i$ is $\bar{p}$-convergent in M. This means that S is well defined. The above proof for $K = J$ shows that

$$\bar{P}_\lambda(Sx)^2 = P_\lambda(< Sx, Sx >) \leq D P_\lambda(\sum_{i \in J} < x, x_i >< x_i, x >)$$

$$\leq D^2 P_\lambda(< x, x >) = D^2 \bar{P}_\lambda(x)^2 \,,$$

i.e., S is bounded. Plainly for every $x \in M$, $< Sx, x >= \sum_j < x, x_i >< x_i, x >$ and so $C < x, x > \leq < Sx, x > \leq D < x, x >$. Now by Proposition 4.1, S is positive.

(ii)$\Longrightarrow$(i) Since the convergence in (19) is $\bar{P}$-convergence and $< Sx, x >= \sum < x, x_i >< x_i, x >$ this series is $\bar{P}$-convergent and from $CI \leq S \leq DI$ we have the result. □

Corollary 4.4. (Reconstruction Formula) *Let A be a unital LC^*-algebra, and let M be a finitely or countably generated Hilbert A-module, suppose that $\{x_i\}_{i\in I}$ is a standard frame with bounds C, D respectively for M. Then:*

(i) The frame operator $S : M \longrightarrow M$ is an invertible operator and $D^{-1}I \leq S^{-1} \leq C^{-1}I$.

(ii) The sequence $\{S^{-1}x_i\}_{i\in I}$ is a standard frame with the bounds D^{-1}, C^{-1} respectively.

(iii) The following equality

$$x = \sum_{i\in I} < x, S^{-1}x_i > x_i = \sum_{i\in I} < x, x_i > S^{-1}x_i \tag{20}$$

is valid for every $x \in M$ and S is the unique operator with this property.

Proof. (i) By Theorem 4.3 we have $C.I \leq S \leq DI$. Thus

$$0 \leq I - D^{-1}S \leq I - CD^{-1}I \,.$$

By using items (3) and (7) of Lemma 2.3, we obtain

$$\hat{P}_\lambda(I - D^{-1}S) \leq \hat{P}_\lambda(I - CD^{-1}I) = 1 - CD^{-1} < 1 \,,$$

for all $\lambda \in \Lambda$. Consequently S is an invertible operator on M. Let $x \in M$ be arbitrary. Since

$$< S^{-1}x, x >=< S^{-1}x, S(S^{-1}x) >= \sum_{i\in I} < S^{-1}x, x_i >< x_i, S^{-1}x > \;.$$

Similarly

$$< x, S^{-1}x >=< S(S^{-1}x), S^{-1}x >= \sum_{i\in I} < S^{-1}x, x_i >< x_i, S^{-1}x > ,$$

for all $x \in M$, and since the summands at the right hand side are positive, by Proposition 4.1, it follows that $(S^{-1})^* = S^{-1}$ and S^{-1} is a positive operator of LC^*-algebra $\mathrm{End}^*_A(M)$. Since

$$C < S^{-1}x, S^{-1}x > \leq < S^{-1}x, x > \leq D < S^{-1}x, S^{-1}x >,$$

we conclude that

$$D^{-1} < S^{-1}x, x > \leq < S^{-1}x, S^{-1}x > \leq C^{-1} < S^{-1}x, x >,$$

for all $x \in M$. So $D^{-1}S^{-1} \leq (S^{-1})^2 \leq C^{-1}S^{-1}$, which implies that $D^{-1}I \leq S^{-1} \leq C^{-1}I$.

(iii) For every $x \in M$, we have

$$x = S(S^{-1}x) = \sum_{i\in I} < S^{-1}x, x_i > x_i = \sum_{i\in I} < x, S^{-1}x_i > x_i$$

and

$$x = S^{-1}(Sx) = \sum_{i\in I} < x, x_i > S^{-1}x_i \, .$$

Suppose that $T \in \mathrm{End}_A^*(M)$ is an invertible positive operator such that $x = \sum_{i\in I} < x, T^{-1}x_i > x_i$ for every $x \in M$. Then

$$x = \sum_{i\in I} < x, T^{-1}x_i > x_i = \sum < (T^{-1})^* x, x_i > x_i = S(T^{-1})^* x \, ,$$

which implies that $S(T^{-1})^* = I$. By taking adjoints on both sides, we get $T^{-1}S = I$, and hence $T = S$.

(ii) Since

$$S^{-1}x = S^{-1}(\sum_{i\in I} < x, S^{-1}x_i > x_i) = \sum_{i\in I} < x, S^{-1}x_i > S^{-1}x_i \, ,$$

for all $x \in M$, then (ii) follows. □

Corollary 4.5. (Reconstruction Formula) *Let $\{x_i\}_{i\in I}$ be a standard tight frame with bound C in a finitely or countably generated Hilbert A-module M over a unital LC^*-algebra A. Then:*

(i) $S = CI$ and the equality $x = C^{-1}\sum_{i\in I} < x, x_i > x_i$ is valid for all $x \in M$.

(ii) A sequence $\{x_i\}_{i\in I}$ is a standard normalized tight frame for M if and only if the following equality

$$x = \sum_{i\in I} < x, x_i > x_i \tag{21}$$

holds for every $x \in M$.

Proof. This follows immediately from Theorem 4.3 and Corollary 4.4. □

Corollary 4.6. *Let $\{x_i\}_{i\in I}$ be a standard frame in a finitely or countably generated Hilbert A-module M over a unital LC^*-algebra A. Then the sequence $\{S^{-1/2}x_i\}_{i\in I}$ is a standard normalized tight frame for M.*

Proof. Since

$$x = S^{-1/2}SS^{-\frac{1}{2}}x = S^{-\frac{1}{2}}\big(\sum_{i\in I} < S^{-\frac{1}{2}}x, x_i, x_i > x_i\big)$$
$$= S^{-1/2}\big(\sum_{i\in I} < x, S^{-\frac{1}{2}}x_i > x_i\big) = \sum_{i\in I} < x, S^{-1/2}x_i > S^{-1/2}x_i\,,$$

for all $x \in M$, therefore from Corollary 4.5 the result follows. □

Proposition 4.7. (Frame Transform) *Let A be a unital LC^*-algebra, and M be finitely or countably generated Hilbert A-module. Suppose that $\{x_i\}_{i\in I}$ is a standard frame in M. Then:*

(i) There exists a finitely or countably generated Hilbert A-module N and a bounded A-linear adjointable operator $\theta : M \longrightarrow N$ such that $S = \theta^*\theta$.

(ii) Let N be any finitely or countably generated Hilbert A-module and let $\theta : M \longrightarrow N$ be a bounded A-linear adjointable invertible operator with the property that $\{\theta x_i\}_{i\in I}$ is a standard normalized tight frame in N, then $S^{-1} = \theta^*\theta$.

Proof. (i) Let

$$N = l_2(A, I) = \{y = \{a_i\}_{i\in I} : \sum_{i\in I} a_i a_i^* \text{ is } P-\text{convergent in } A\}\,, \tag{22}$$

and let $\theta : M \longrightarrow N$ be the usual frame transform defined by

$$\theta(x) = \sum_{i\in I} < x, x_i > e_i \quad (x \in M)\,. \tag{23}$$

Then N is a finitely or countably generated standard Hilbert A-module with an orthonormal basis $\{e_i\}_{i\in I}$, where $e_i = (0, \ldots, 0, 1_A, 0, \ldots, 0)$ with the unit at the i-th place. Since $\{x_i\}_{i\in I}$ is a standard frame for M, and since

$$< \theta(x), \theta(x) >= \sum_{i\in I} < x, x_i >< x_i, x >\,, \tag{24}$$

for every $x \in M$, with the convergence being in the P-convergent sense. Consequently θ is well-defined. Let C, D be the frame bounds for $\{x_i\}_{i\in I}$ respectively. Then

$$C < x, x >\leq< \theta(x), \theta(x) >\leq D < x, x >\,,$$

for all $x \in M$. Furthermore by Lemma 2.3 the inequality

$$CP_\lambda(< x, x >) \leq P_\lambda(< \theta(x), \theta(x) >) \leq DP_\lambda(< x, x >) \tag{25}$$

holds for every $x \in M$ and every $\lambda \in \Lambda$. So θ is an A-linear bounded operator. Moreover the image of θ is $\bar{P}$-closed because M is $\bar{P}$-closed by assumption. To calculate the values of the adjoint operator θ^* of θ consider the equality

$$< x, \theta^*(x) >=< \theta(x), e_k >= \sum_{i \in I} < x, x_i >< e_i, e_k >=< x, x_k >, \tag{26}$$

which is satisfied for every $x \in M$ and every $k \in I$. Thus $\theta^*(e_k) = x_k$ for every $k \in I$. Consequently θ^* is at least defined for the elements of the selected orthonormal basis $\{e_i\}_{i \in I}$. Now we show that θ^* is bounded. By the Cauchy-Bunyakovskii inequality and (3) we have

$$\begin{aligned} &P_\lambda(< \theta^*(y), \theta^*(y) >) \\ &\quad = \sup_{\bar{P}_\lambda(x) \leq 1} P_\lambda(< \theta^*(y), x >)^2 = \sup_{\bar{P}_\lambda(x) \leq 1} P_\lambda(< y, \theta(x) >)^2 \\ &\quad \leq \sup_{\bar{P}_\lambda(x) \leq 1} P_\lambda(< y, y >) P_\lambda(< \theta(x), \theta(x) >) \leq D.P_\lambda(< y, y >), \end{aligned}$$

for all y in domain of θ^* and for every $\lambda \in \Lambda$. Since the operator θ^* has to be A-linear by definition we can extend this operator to the $\bar{P}$-dense subset of all finite A-linear combinations of the elements of the selected basis $\{e_i\}_{i \in I}$ of N. Define $\theta^* : N \longrightarrow M$ by

$$\theta^*(y) = \sum_{i \in I} a_i x_i, \tag{27}$$

where $y = \{a_i\}_{i \in I}$ is an element of the domain of θ^*. Since the sequence $\{x_i\}_{i \in I}$ is a set of generators of M, θ^* is well-defined. If follows that θ is a bounded A-linear adjointable operator. Finally for any $x \in M$ the following equality is valied.

$$\begin{aligned} \theta^*\theta(x) = \theta^*(\sum_{i \in I} < x, x_i > e_i) &= \sum_{i \in I} < x, x_i > \theta^*(e_i) \\ &= \sum_{i \in I} < x, x_i > x_i = Sx. \end{aligned}$$

(ii) The existence of such an operator follows from Corollary 4.6. Let $y_i = \theta x_i$ $(i \in I)$. Then

$$\sum_{i\in I} < x, \theta^*\theta x_i > x_i = \sum_{i\in I} < \theta x, \theta x_i > x_i$$
$$= \sum_{i\in I} < \theta x, y_i > \theta^{-1} y_i = \theta^{-1}(\sum_{i\in I} < \theta x, y_i > y_i) = \theta^{-1}\theta x = x\,,$$

for all $x \in M$. Now by the uniqueness of frame operator we get $S^{-1} = \theta^*\theta$. □

Proposition 4.8. *Let A be a unital LC^*-algebra. M, N be finitely or countably generated Hilbert A-modules.*

(i) If $\{x_i\}_{i\in I}$ is a standard (normalized tight) frame for M and let $T : M \longrightarrow N$ be a co-isometry, then $\{Tx_i\}_{i\in I}$ is a standard (normalized tight) frame for N.

(ii) Let $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ be standard normalized tight frames for M and N respectively and let $T : M \longrightarrow N$ be an adjointable bounded A-linear operator such that $Tx_i = y_i$ for $i \in I$. Then T is a co-isometry. If T is invertible, then it is a unitary.

(iii) Let $\{x_i\}_{i\in I}$ be a standard (normalized tight) frame for M and let $T : M \longrightarrow N$ be a partial isometry. Then $\{Tx_i\}_{i\in I}$ is a standard (normalized tight) frame for $T(M)$.

Proof. (i) Let C and D be the frame bounds for the standard frame $\{x_i\}_{i\in I}$. Then for every $x \in M$ and every $y \in N$, we have

$$\sum_{i\in I} < y, Tx_i >< Tx_i, y >= \sum_{i\in I} < T^*y, x_i >< x_i, T^*y > \ .$$

Since T is a co-isometry, thus we obtain the following inequality

$$C < y, y >= C < T^*y, T^*y >\leq \sum_{i\in I} < y, Tx_i >< Tx_i, y >$$
$$\leq D < T^*y, T^*y >= D < y, y > \ .$$

(ii) This follows from:

$$< T^*y, T^*y >= \sum_{i\in I} < T^*y, x_i >< x_i, T^*y >$$
$$= \sum_{i\in I} < y, Tx_i >< Tx_i, y >= \sum_{i\in I} < y, y_i >< y_i, y >=< y, y >,$$

for all $y \in \mathbb{N}$.

(iii) For every $x \in M$ we have

$$\sum_{i\in I} < T^*Tx, x_i >< x_i, T^*Tx >= \sum_{i\in I} < Tx, Tx_i >< Tx_i, Tx > \ .$$

Since T is a partial isometry, thus we obtain the following inequality

$$C < Tx, Tx >= C < T^*Tx, T^*Tx > \leq \sum_{i\in I} < Tx, Tx_i >< Tx_i, Tx >$$
$$\leq D < T^*Tx, T^*Tx >= D < Tx, Tx > . \quad \square$$

Let $\{x_i\}_{i\in I}$ be a standard frame in a finitely or countably generated Hilbert A-module M over a unital LC^*-algebra A. Then the standard frame defined by $x_i^* = S^{-1}x_i$ $(i \in I)$ in Corollary 4.4 is called the canonical dual frame of $\{x_i\}_{i\in I}$ in M and (23) shows that the frame operator of $\{x_i^*\}$ is S^{-1}.

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